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Article

Age Analysis of Status Updating System with Probabilistic Packet Preemption

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Abstract: The age of information (AoI) metric was proposed to measure the freshness of messages obtained at the terminal node of a status updating system. In this paper, the AoI of discrete time status updating system with probabilistic packet preemption is investigated by analyzing the steady state of a three-dimensional discrete stochastic process. Assuming the queue used in system is $Ber/Geo/1/2^*/\eta$, which represents that the system size is 2 and the packet in buffer can be preempted by fresher packet with probability η. Instead of considering system's AoI separately, we use a three-dimensional state vector (n, m, l) to simultaneously track the real time changes of the AoI, the age of packet in server, and the age of packet waiting in buffer. We give the explicit expression of system's average AoI, and show that the average AoI of system without packet preemption is obtained by letting $\eta = 0$. When η is set to 1, the mean of AoI of system having $Ber/Geo/1/2^*$ queue is obtained as well. Combining the results we have obtained and comparing them with corresponding average continuous AoIs, we propose a possible relationship between average discrete AoI with $Ber/Geo/1/c$ queue and the average continuous AoI with $M/M/1/c$ queue. For each of two extreme cases where $\eta = 0$ and $\eta = 1$, we also determine the stationary distribution of AoI using the probability generation function (PGF) method. The relations between average AoI and the packet preemption probability η, as well as AoI's distribution curves in two extreme cases are illustrated by numerical simulations. Notice that the probabilistic packet preemption may occur, for example, in an energy harvest (EH) node of wireless sensor network, where the packet in buffer can be replaced only when the node collects enough energy. In particular, to exhibit the usefulness of our idea and methods and highlight the merits of considering discrete time systems, in this paper we give much more explanations showing that how the results about continuous AoI is derived by analyzing the corresponding discrete time system, and how the discrete age analysis is generalized to the system with multiple sources. In terms of packet service process, we also propose our idea to analyze system's AoI when the service time distribution is relaxed to be arbitrary.

Keywords: age of information; discrete time status updating system; probabilistic preemption; probability generation function; stationary distribution

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1. Introduction

The freshness of transmitted messages gets more and more attention in designing practical communication systems. Messages obtained by a controller in a real-time monitor system may be used to perform the traffic scheduling or resource allocation, and for such applications system's timeliness is crucial for the scheduler to make the right response and precise control. The age of information (AoI) metric was proposed in paper [1] as the time elapsed since the generation time of the last received packet in the destination, which has been used widely in recent years to measure the packet's freshness and characterize the timeliness of various communication networks. A simple introduction to the AoI theory can be found in work [2], and in paper [3], the authors made a detailed summary about the analytical results of age of information, along with employing the AoI optimization in many cyber-physical applications.

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1.1. Related work

For the status updating system with simple queue models, such as $M/M/1$, $M/D/1$, and $D/M/1$ queues, the expression of average AoI was obtained in papers [4–7]. In particular, work [7] considered the queue using Last-Come-First-Served (LCFS) discipline, and the newer packet from the source can preempt the packet currently in service. The influence of different packet management strategies on system's average AoI was investigated in [8,9], where only one or two packets can be stored in the system. Specifically, average AoI of system with three queues, that is $M/M/1/1$, $M/M/1/2$, and $M/M/1/2^*$ were determined. The difference between last two queues lies at whether the packet waiting in buffer can be substituted by following packets from the source. For two cases where system size is equal to 2, it was shown that updating the waiting packet with fresher one can always achieve lower average AoI, which is apparent because transmitting the packet having smaller age is helpful for improving the timeliness of the information transmission systems. Apart from these, the benefit of introducing a proper packet deadline, both deterministic and random to reduce the average age of information was proved in papers [10–12]. Controlling packet preemptions to improve the freshness of transmitted message was discussed in [13–15]. The authors of paper [16] shown that the average AoI can be significantly improved when adding a period of waiting time before the service of a new packet begins. Assuming there are two parallel servers in status updating system, the expressions of average AoI was determined in paper [17]. Notice that when more than one server was present, the updating packet can get to the destination through different paths. In these situations, since a packet generated behind may be transmitted to destination via a short-delay path, it is possible that this packet arrives at the receiver before the packets who were generated before it. Recently, many papers have been launched considering the AoI of status updating networks with simple structures, such as the status updating system with multiple sources [18–23], the system having more than one hop transmission [24–28], and the system in which the packet transmission was assisted by a relay [29–33]. For each of above systems, the average performance of AoI was characterized, even some properties about AoI's distribution were obtained in certain papers. For example, for the age on a line network of preemptive memoryless servers, in work [34] the author proved that the age at a node is identical in distribution to the sum of independent exponential service times, by calculating the Moment Generation Function (MGF) of the defined age vector. In works [35,36], the distribution of AoI was studied in a wireless networked control system with two-hop packet transmission. The authors devised the problem that minimizing the tail of the AoI distribution with respect to the sampling rate under First-Come First-Serve (FCFS) queuing discipline. In paper [37], for the phase-type (PH-type) interarrival time or packet service time, the authors numerically obtained the exact distribution of (peak) age of information for the system with $PH/PH/1/1$ and $M/PH/1/2$ queues. Within the paper, they used the sample path arguments and the theory of Markov Fluid Queues (MFQ). Except the works we mentioned above which focus on obtaining analytical results of AoI for the status updating systems with various queue models, even more papers are published in which the authors considered designing optimal systems under different timeliness requirements, such as in papers [38–47]. In such problems, usually the age of information is used as freshness metric and is studied as the optimization objective.

1.2. Discussion of existing methods

As far as we know, at least three methods have been proposed to analyze the AoI of continuous time status updating system. The first one is the method based on the graph of AoI stochastic process which was given in [2]. The time average AoI is obtained by calculating the area below the sample path of the AoI process. Using the common assumption that the age process is ergodic, this time average AoI converges to the AoI's mean as the observation time tends to infinity. It shows that the average AoI of a status updating system is determined by

$$\mathbb{E}[\Delta] = \frac{\mathbb{E}[YT] + \mathbb{E}[Y^2]/2}{\mathbb{E}[Y]} \tag{1}$$

when the packet arrival process and the distribution of service time are specified, in which the notation Y denotes the interarrival time between two succesive updating packets and T represents the packet *system time*. Secondly, in paper [6], the authors illustrated the usage of the *Stochastic Hybrid System* (SHS) approach to the analysis of system's stationary AoI. They employed a continuous state vector to track the real time age of the updating packets from the source, and described all the possible state vector transfers under system's random dynamics, such as if there comes a new packet, whether the packet service is completed. Then, the steady state of the multiple-dimensional continuous time Markov process is characterized by a group of differential equations, and the first few of AoI's moments can be obtained using the theory of SHS [48]. This method was used later to determine the average AoI of more general systems, including the system with multiple sources, packet preemption, even stochastic energy harvesting at certain system nodes. The last method was introduced in work [5], where the authors proposed a novel description of the AoI process and characterized its sample paths using a new point process. They proved that the stationary distribution of the AoI can be represented in terms of the distributions of system's delay and the peak AoI. From this point of view, lots of analytical formulas about AoI's stationary distribution were obtained (in the form of its *Laplace Stieljes Transform* (LST)) for single-server systems. Also we found that the same method has been used to consider the distribution of discrete time (peak) AoI in papers [49,50], where the z-transform of (peak) AoI's distribution was derived for the system with some discrete queues.

Although plenty of results have been obtained using the methods mentioned above, the interested readers may find that most of the results are heavily dependent on the assumptions that the packet arrivals form a Poisson process and the service time distribution is exponential, especially for the SHS method. The memoryless property of both interarrival time distribution and the distribution of packet service time dramatically simplifies the age analysis of considered status updating system. So far, the first method which is based on the graphical argument of AoI process is used only to calculate AoI's mean, but it seems that the theory of *Level Crossing* in [51] may be useful when considering the AoI's distribution from the sample paths themselves. Level crossing method has been used to derive the steady-state probability density function of queue wait in several variants of the $M/G/1$ queue. It is worth trying whether this theory can be used to find the stationary distribution of continuous AoI. Using the SHS method, similarly only the first few of AoI's moments can be calculated. In order to obtain the distribution property of system's AoI, one has to solve the system of differential equations which is extremely hard in general. At last, although in paper [5], the authors pointed out that the general formula they proposed holds sample-path-wise, regardless of the service discipline or the distributions of interarrival and packet service times. However, the results they obtained are not straight-forward, they only derived the LST of the AoI's stationary distribution while computing the explicit expression of this distribution is also a tough problem due to the difficulty of computing the inverse of the LST. On the other hand, it is unknown if the method and the obtained formula can be generalized to more general status updating systems, not just for the system with single-server.

In the following part, we will introduce the idea and methods to analyze the AoI of discrete time status updating systems, and talk about their merits compared with those ways dealing with continuous time age of information. By an explicit example, we will show how the results of continuous AoI can be obtained by considering the corresponding discrete time systems.

1.3. Analysis of discrete time AoI: idea and methods

We propose the idea and methods to characterize the steady state AoI of a discrete time status updating system, in which the packet arrivals, the packet service and AoI declines are considered in discrete time slots. Although there are not many, but still have some works analyzing the AoI of discrete system with different queue models. To our best knowledge, the analysis of discrete AoI was proposed for the first time in paper [52]. Using the proof techniques and tools developed to analyze continuous AoI, the authors obtained the average (peak) AoI of $Ber/G/1$ and $G/G/\infty$ queue modeled discrete time status updating system. Notation "Ber" represents that the packet arrival or the service of the packet forms a Bernoulli stochastic process, equivalently, in each time slot a packet arrives (or the packet service is completed) is independent and with an identical probability. Later, using the similar description of age process's sample path as in work [5], in papers [49,50] the expression of discrete AoI's distribution was obtained for the system with First-Come First-Served (FCFS) queue, the preemptive Last-Come First-Served (LCFS) queue, and the bufferless status updating system. Discrete time system with multiple sources are considered in [53]. Under the assumption of Bernoulli packet arrivals and a common general discrete phase-type service time distribution across all the sources, the authors obtained the exact per-source distributions of AoI and peak AoI in matrix-geometric form for three different queueing disciplines, i.e., nonpreemptive bufferless, preemptive bufferless, and nonpreemptive single buffer with replacement.

In our work [54], we obtain the explicit formula of average discrete AoI, $\overline{\Delta}_{Ber/Geo/1/1}$ for bufferless status updating system (actually the service time distribution in [54] is arbitrary), by defining a two-dimensional age process which characterizes the AoI at the destination and the age of packet in service as a whole. The idea we proposed in [54] can be regarded as the discretization of the SHS method, which will be shown to be equally powerful and more flexible when applied to more general systems. We describe all the possible state transfers for every initial state vector, and then establish the stationary equations of the defined two-dimensional discrete age process. These equations are solved completely in paper [54], thus the distribution of AoI can be determined explicitly as one of the marginal distributions of the two-dimensional age process's stationary distribution. Given AoI's distribution, the mean, the variance, and the tail probabilities of the AoI can be easily calculated. The idea that constituting multiple-dimensional age process is then generalized to analyze the discrete AoI for other status updating systems, such as in current paper the AoI of system with $Ber/Geo/1/2$ and $Ber/Geo/1/2^*$ queues are considered simultaneously, which are connected together by the probabilistic packet preemption in system's buffer. In addition, in order to avoid the tedious calculation required to solve the stationary equations and calculate the marginal distribution, we define the Probability Generation Function (PGF) of multiple-dimensional stationary distribution, from which both the AoI's mean and its stationary distribution can be obtained effectively. In fact, except for the system with simple queue models, in most cases it is not easy to find the exact solutions of the probability balance equations. For system's average AoI, in Table 1, we list the results we have obtained about the discrete AoI and the corresponding expressions of continuous system's average AoI. The average AoI $\overline{\Delta}_{Ber/Geo/1/1}$ was obtained in [54] and the other two expressions will be derived in current paper. Apart from the AoI's mean, we also determine the distribution of discrete AoI $\Delta_{Ber/Geo/1/2}$ and $\Delta_{Ber/Geo/1/2^*}$ by writing the PGF as the power series.

As mentioned above, one can see the similarity between our idea and the SHS method and may mistakenly think that what we do is just changing the continuous time into discrete time slots. The power of combining multiple-dimensional state vector description with the PGF method may be underestimated due to the simple assumptions that will be used in current paper, that is the packet arrivals form a Bernoulli process and the packet service time is geometrically distributed. It was known that in order to obtain the complete statistical information, not just its mean of stationary AoI by the method of SHS, one has to solve a group of differential equations, which may be possible for some systems with

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Table 1. Some formulas of average continuous and average discrete age of information

Average continuous and average discrete AoIs
$\overline{\Delta}_{M/M/1/1} = \frac{1}{\mu}\left(1 + \frac{1}{\rho} + \frac{\rho}{1+\rho}\right)$
$\overline{\Delta}_{Ber/Geo/1/1} = \frac{1}{\gamma}\left((1-\gamma) + \frac{1}{\rho_d} + \frac{\rho_d}{1/(1-\gamma)+\rho_d}\right)$
$\overline{\Delta}_{M/M/1/2} = \frac{1}{\mu}\left(1 + \frac{1}{\rho} + \frac{2\rho^2}{1+\rho+\rho^2}\right)$
$\overline{\Delta}_{Ber/Geo/1/2} = \frac{1}{\gamma}\left((1-\gamma) + \frac{1}{\rho_d} + \frac{2\rho_d^2(1-\gamma)(1-\gamma/2)}{1+\rho_d(1-2\gamma)+\rho_d^2(1-\gamma)^2}\right)$
$\overline{\Delta}_{M/M/1/2^*} = \frac{1}{\mu}\left(1 + \frac{1}{\rho} + \frac{\rho^2(1+3\rho+\rho^2)}{(1+\rho+\rho^2)(1+\rho)^2}\right)$
$\overline{\Delta}_{Ber/Geo/1/2^*} = \frac{1}{\gamma}\left((1-\gamma) + \frac{1}{\rho_d} + \frac{\rho_d^2(1-\gamma)[1+3\rho_d(1-\gamma)+\rho_d^2(1-\gamma)(1-2\gamma)]}{[1+\rho_d(1-2\gamma)+\rho_d^2(1-\gamma)^2][1+\rho_d(1-\gamma)]^2}\right)$

simple queues but generally is impossible. In addition, the usage of SHS analysis is heavily restricted because it requires that both the packet arrival process and the packet service process are memoryless, i.e., the interarrival time and the packet service time have to be i.i.d. exponential random variables. In the following, we will explain the merits of considering discrete time system in two aspects.

(1) Calculation: reducing the complexity

Observing that when all the state transitions are described in discrete time slots, the stationary equations characterizing the steady state of the defined age process become a set of linear equations, which is more likely to be solved compared with those differential equations. We will show in this paper that these linear equations can be dealt with using the PGF method even more easily and more effectively. In our another work, we have determined the explicit expression of average AoI and the corresponding AoI's distribution assuming the $Ber/Geo/1/c$ queue is used in the status updating system, where the system's size c can be arbitrary. For the cases $c = 3$ and 4, we obtain that

$$\overline{\Delta}_{Ber/Geo/1/3} = \frac{1}{\gamma}\left((1-\gamma) + \frac{1}{\rho_d} + \frac{\rho_d^2(1-\gamma^2) + 3\rho_d^3(1-5\gamma/3+\gamma^2/3)}{1+\rho_d(1-3\gamma)+\rho_d^2(1-3\gamma+3\gamma^2)+\rho_d^3(1-\gamma)^3}\right) \tag{2}$$

and

$$\overline{\Delta}_{Ber/Geo/1/4} = \frac{1}{\gamma}\Bigg((1-\gamma) + \frac{1}{\rho_d} + \frac{\rho_d^2(1-\gamma) + 2\rho_d^3(1-\gamma)(1-2\gamma) + 4\rho_d^4(1-\gamma)(1-11\gamma/4+9\gamma^2/4-\gamma^3/4)}{1+\rho_d(1-4\gamma)+\rho_d^2(1-4\gamma+6\gamma^2)+\rho_d^3(1-4\gamma+6\gamma^2-4\gamma^3)+\rho_d^4(1-\gamma)^4}\Bigg) \tag{3}$$

Although we have not mentioned yet, the readers should find that those expressions of average continuous and average discrete AoI given in Table 1 are quite similar. We propose following possible relationship:

$$\mu \cdot \overline{\Delta}_{M/M/1/c} = \gamma \cdot \overline{\Delta}_{Ber/Geo/1/c}\big|_{\gamma=0}, \text{ then replacing } \rho_d \text{ with } \rho \tag{4}$$

The relation (4) holds at least for $c = 1$, $c = 2$, and $c = 2^*$. (Actually, we have proved that (4) is also correct when system size c is infinite.) If fortunately equation (4) is appliable in general which we hope so, then from expressions (2) and (3), immediately we have

$$\overline{\Delta}_{M/M/1/3} = \frac{1}{\mu}\left(1 + \frac{1}{\rho} + \frac{\rho^2+3\rho^3}{1+\rho+\rho^2+\rho^3}\right) \tag{5}$$

and

$$\overline{\Delta}_{M/M/1/4} = \frac{1}{\mu}\left(1 + \frac{1}{\rho} + \frac{\rho^2 + 2\rho^3 + 4\rho^4}{1 + \rho + \rho^2 + \rho^3 + \rho^4}\right) \tag{6}$$

Notice that the average continuous AoI (5) and (6) are not derived using any of the three methods we discussed earlier, that is the method based on the sample path of AoI process, the SHS, and the method proposed in [5] and [50]. On the contrary, we first characterize the stationary AoI of corresponding discrete time system, and then obtain the expression of continuous AoI's mean through relationship (4). There is no doubt that the formulas of $\overline{\Delta}_{M/M/1/3}$ and $\overline{\Delta}_{M/M/1/4}$ can be obtained using AoI's SHS analysis, however, the general formula of AoI's mean, i.e., $\overline{\Delta}_{M/M/1/c}$ for arbitrary size c is temporarily unknown. Furthermore, the stationary distribution of discrete AoI can also be determined explicitly from the PGF defined for considered system, while the distribution properties of continuous AoI cannot be revealed easily through either the graphical method or the AoI's SHS analysis. Although it is not possible to accurately reprint the continuous AoI's distribution in every position using the discrete approximation, the difference between them can be reasonably small when the length of time slot is short enough. In current paper, we will determine the distribution expressions of discrete AoI for the system with $Ber/Geo/1/2$ and $Ber/Geo/1/2^*$ queues. Unlike in work [5] and [50], these expressions are straight-forward and not expressed in the form of other transformations.

According to above discussions, from the perspective of deriving average AoI or obtaining AoI's distribution, considering the status updating system in discrete time model is of great significance. To a certain extent, we can even conclude that our method is *stronger* since *more specific* results about AoI have been obtained.

(2) Generalization: in terms of system structure and service time distribution

Recently, using the SHS method, the age analysis has been generalized to the status updating networks with simple structure, especially the system with multiple sources. In this part, we will briefly explain how the discrete age of information is characterized in the multiple-source bufferless system and the two-source system equipped with a size 1 buffer. The system models are depicted in following Figure 1.

Figure 1. (**a**) Status updating system with multiple sources and bufferless server. (**b**) Status updating system with two sources and a size 1 buffer.

Specifically, we assume the packets arrive at the beginning of one time slot and whether the packet service is completed is determined at the end of the time slot. Since the system's random dynamics are considered in time slots, it is possible that in a time slot more than one packet arrives to the server (buffer) which come from different sources. The server has to choose one from them and discards the other packets if the system does not have a buffer. This packet collision problem can be solved by assigning priorities to the packets from different sources, then the packet having the highest priority is selected and put into the server.

In the bufferless system given in the first picture of Figure 1, let r_i be the priority of source s_i, $1 \le i \le N$, and assuming $r_1 > r_2 > \cdots > r_N$, that is the priority of source s_i is over that of s_j if $i < j$. In each time slot, source s_i generates a new packet with probability p_i and the packet generation process is independent of all other sources. Actually, this situation is exactly the generalization of our work in [54] when the status updating system has multiple independent sources. For the given $i \in [1, N]$, it shows that the AoI process corresponding to source s_i can be analyzed separately, thus similar to work [54], introducing a two-dimensional state vector (n_i, m_i) is sufficient to track the real time changes of AoI_i and the age of the packet in server which is from source s_i. In this system, observing that it does not matter whether the service of the packet from s_i can be preempted by other packets who have higher priorities. The state vector transfers from every (n_i, m_i) can be described as in [54], but the transition probabilities need to be modified. For example, for $n_i > m_i \geq 1$, we have

$$\textit{State vector at next time slot} = \begin{cases} (n_i+1, m_i+1) & \text{the packet service is not completed,} \\ (m_i+1, 0) & \text{the service of the packet is over.} \end{cases} \quad (7)$$

if the service process cannot be preempted. Or, it can be decided that

$$\begin{aligned} &\textit{State vector at next time slot} \\ &= \begin{cases} (n_i+1, m_i+1) & \text{no packets of higher priorities arrive, the service is not over,} \\ (n_i+1, 0) & \text{one packet with higher priority comes,} \\ (m_i+1, 0) & \text{no packets with higher priorities arrive, the service is over.} \end{cases} \end{aligned} \quad (8)$$

when packet service preemption is allowable. After all the state transfers are described and their transition probabilities are determined, we can obtain the stationary equations, which can be solved completely as in paper [54], or using the PGF method as in this paper. Like [54], the service time distribution in this case can be arbitrary.

Although there are multiple sources, it sees that the age analysis of each source is easy when the status updating system has no buffer. Notice that in this case no queue is formed before the server, thus there is no chance the packets from different sources are combined. As a result, the packets from every source are totally divided and the AoI of each source can be analyzed separately. The situation is much more difficult if the system has a buffer. As an example, we consider the AoI of each source of a two-source system which is depicted in the second picture of Figure 1.

In order to describe the AoI process of each source, we have to define a six-dimensional state vector $(n_1, n_2, m_1, m_2, l_1, l_2)$ representing the values of two AoI at destination, the age of packet in server, and the age of packet in system's buffer. In every position of the system, apart from the "age", it is necessary to indicate which source the packet comes from. Therefore, a three-dimensional state vector (n, m, l) which does not include this information is not sufficient. Notice that at any time, at most one of m_1 and m_2 are non-zero. This is the same for the parameters l_1 and l_2. When there is a buffer in front of the server, apparently a queue is formed if a packet arrives and finds that the server is currently busy. Each one of two packets in system (one is in server and the other is in buffer) may come from source s_1 or s_2. Of course, these two packets may belong to different sources. That is, packets from two sources are combined, which makes it impossible to describe the age of each source separately. Although the problem becomes complex, theoretically all the state transfers of every initial six-dimensional state vector can be determined explicitly, since the randomness that causes the state vector transfers are limited to random packet arrival, the service of the packet, and the additional packet preemption. Then, according to the balance of probabilities in steady state, the stationary equations are established, this solves the first half of the AoI analysis. The latter half, that is deriving the average AoI from the group of stationary equations can refer to the procedures in this article.

We find that in paper [55] the authors obtained the average continuous AoI for the same two-source status updating system in Figure (1b) using the SHS method. They added another assumption that the packet in server and the packet in buffer must belong to different sources in their second and the third considered situation, and named the policies "source-aware packet management". As we have mentioned above, although the packets from two sources are still combined, after adding this restriction, the complexity of the problem has been greatly reduced. In our discrete two-source model, in this case we can use a five-dimensional state vector, either (n_1, n_2, m_1, m_2, l) or (n_1, n_2, m, l_1, l_2) to record the two AoIs and the age of all the packets. Since two packets cannot belong to the same source, if we use the vector (n_1, n_2, m_1, m_2, l), then when $m_1 > 0$, the packet in buffer with age l must come from source s_2. If instead (n_1, n_2, m, l_1, l_2) is defined, then when $l_1 > 0$, the packet in server is from s_2. It is this one-dimension decrement that the possible formations

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of state vectors are largely reduced. We will discuss the age analysis for this two-source discrete system and derive the expression of average AoI in our later work.

Except the simple status updating networks given in Figure 1, we have also obtained the average discrete AoI for a status updating system with two-stage service, where for simplicity in front of each server, no buffer is equipped. For the system with two parallel servers, the age analysis is more difficult, since some packets may become "ineffective" if one packet which is generated later but arrives to the destination earlier. Some policies need to be identified to deal with these packets, for instance, deleting the packet directly once it becomes ineffective. If nothing is done, when an ineffective packet is obtained at the receiver, the value of AoI will not be reduced.

Another direction of generalization we shall talk about is the distribution of packet service time (while the packet arrival process is still Bernoulli). Now, taking the size 2 status updating system as an example, we explain how the service time distribution is relaxed to be an arbitrary distribution. Using a three-dimensional state vector (n, m, l), we can fully describe the random dynamics including the AoI at the receiver and the age of two packets in system, if both the packet interarrival time and the service time have memoryless properties. In each time slot, the changes of AoI's value and the packet ages depend on random packet arrival which is memoryless and independent, and whether the packet service is over. When the service time distribution is arbitrary, the probability that the service is completed in one time slot is related to the time this packet has experienced in the server. Let S be the random variable of service time, we represent the general distribution as

$$\Pr\{S = i\} = q_i \qquad (i \geq 1) \tag{9}$$

Assuming before current time slot, the packet has stayed in server for j time slots, then the probabilities that determine the state vector transfers should be following two conditional probabilities

$$\Pr\{S = j+1 | S > j\} \text{ and } \Pr\{S > j+1 | S > j\} \tag{10}$$

Therefore, if we have the knowledge about this passed service time j, as before all the state transfers of state vector (n, m, l) can be completely described and the age analysis becomes feasible. Since no one of three parameters n, m, and l can provide this information, it is natural to introduce an extra component, say k, to denote the service time that the packet has consumed, and constitute the four-dimensional state vector (n, m, l, k). In this way, the possible state transfers of this four-dimensional state vector can be described and the transition probabilities can also be determined. For example, let the initial state vector be (n, m, l, k), we have the state transfers and transition probabilities as

$$\begin{aligned} &\textit{Next state vector} \\ &= \begin{cases} (n+1, m+1, l+1, k+1) & \text{the service is not over with prob. } 1 - q_{k+1} / \sum_{i=k+1}^{\infty} q_i \\ (m+1, l+1, 0, 0) & \text{the service completes with prob. } q_{k+1} / \sum_{i=k+1}^{\infty} q_i \end{cases} \end{aligned} \tag{11}$$

where we assume the queue discipline is FCFS and there is no packet preemption. We show that the four parameters n, m, l, and k satisfy the relationships $n > m > l \geq 0$ and $n > m \geq k \geq 0$. The first one holds because n, m and l are three ages of packets generated in chronological order, and $n > m \geq k$ is satisfied since the packet system time m must be larger than or equal to the service time of the packet which is denoted by k. These relations determine which vectors are qualified state vectors. Although we show that the state transfers can be analyzed and the group of stationary equations can be determined by considering the balance of those stationary probabilities, however, it can be expected that solving these equations are not easy. Since the service time probabilities q_i's are arbitrary, the expression of average AoI, if we can determine in later work, will not be closed-formed.

Summarize above discussions, actually we have proved that on the basis of original memoryless status updating system, by introducing an extra component to denote the time the packet has consumed in the server, the age analysis becomes feasible for the situation where the packet service time is arbitrarily distributed. Although it may be difficult to obtain the expressions of system's average AoI, the idea is still appliable when we generalize the size 2 system to a status updating system with arbitrary size c. In one of our work, we have shown that for a size c discrete time status updating system with Bernoulli packet arrivals and geometrically distributed service time, in order to fully characterize the real time transfers of system's AoI and all the packet ages, a $(c+1)$ dimensional state vector $(n, m_1, \cdots, m_c)$ should be defined. By adding an extra state component k which records the service time the packet has experienced in the server, according to previous discussions, the age analysis can be generalized to size c status updating system whose service time distribution is arbitrary (at least we can establish all the stationary equations).

We have to attribute above idea to the paper [56], in which the authors considered timely transmitting the updates over an erasure channel. They assume each update consists of k symbols and the symbol erasure in each time slot is an i.i.d. Bernoulli process. The aim of work [56] is designing an optimal online transmission scheme to minimize the time average AoI and the problem is formulated as an Markov Decision Process (MDP). Although the optimization of AoI is not our interest, the state tuple (δ_t, d_t, l_t) they defined in Section II.A is very enlightening, based on which the transmission policy at next time slot is determined. At t-th time slot, notation δ_t denotes the value of AoI, d_t is the age of next update, i.e., the packet at the head of the queue, and l_t records the number of symbols that has been obtained successfully up to this time slot, these symbols belong to the update which is transmitted currently. Similar timely source coding problem was also discussed in paper [57], the authors also pointed out that the length of the encoded update is equivalent to the serivce time of the update, and their considered system behaves as a discrete time $Geo/G/1$ queue (we use the notation $Ber/G/1$). Therefore, the role of l_t in [56] can be regarded (or redefined) as the service time the current update has consumed. By adding this knowledge, the distribution of the source in these papers, and the service time distribution in the discrete time status updating system which we study in this part can be arbitrary.

In previous paragraphs, we explain the idea and methods that used to study the AoI of discrete time status updating systems. We have shown how the discrete AoI is characterized for the basic system, the system with multiple sources, and the system whose service time distribution is arbitrary. As the part of AoI theory, we believe that discrete AoI deserves more attention and it is meaningful to establish analytical results including the AoI's mean and its distribution for more general systems. In particular, the proposed possible relationship in (4) shows that discussing discrete AoI not only has independent theoretical significance, but also helps to determine certain results about continuous AoI. If one problem is difficult in continuous time model, it is a choice to consider it in discrete time settings.

1.4. The work of current paper

Although we have talked so much about discrete AoI, it is inappropriate to consider all the issues in one article. In current paper, we focus on the stationary AoI of discrete time system with $Ber/Geo/1/2$ and $Ber/Geo/1/2^*$ queue, and discuss the both in single model. We assume the packet in buffer can be probabilistically preempted by the fresher packets from the source, and define the queue model in this scenario as $Ber/Geo/1/2^*/\eta$ where η is the preemption probability. Then, for the case of $\eta = 0$, the queue model of the system reduces to $Ber/Geo/1/2$, while when η is equal to 1, the status updating system with $Ber/Geo/1/2^*$ queue is obtained. For the general case, we will derive the explicit expression of system's average AoI. By writing the defined PGF as power series, for two extreme cases of $\eta = 0$ and $\eta = 1$, the distribution expressions of two discrete AoIs are determined as well.

The rest part of the paper is organized as follows. In Section 2, we describe the model of discrete time status updating system with probabilistic packet preemption. The stationary

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distribution and the mean of system's AoI are also defined. By analyzing the steady state of a three-dimensional stochastic age process, in Section 3 we obtain the explicit formula of the average AoI under general preemption probability using the probability generation function (PGF) method. In Section 4, let $\eta = 0$ and $\eta = 1$, we determine the average AoIs $\overline{\Delta}_{Ber/Geo/1/2}$ and $\overline{\Delta}_{Ber/Geo/1/2^*}$ from the general expression derived previously in Section 3. Furthermore, in order to obtain the stationary distribution of two discrete AoIs, we write the PGF as power series. Then, the coefficient before x^n gives the probability the AoI takes value n for each $n \geq 1$. Numerical results are placed in Section 5. For general case, we illustrate the relationships between average AoI and η and the traffic intensity ρ_d respectively. In addition, the mean and the cumulative probabilities of three discrete AoIs including $\Delta_{Ber/Geo/1/1}$, $\Delta_{Ber/Geo/1/2}$, and $\Delta_{Ber/Geo/1/2^*}$ are depicted. These average discrete AoIs and their corresponding average continuous AoIs are also numerically compared in Section 5. At last, we conclude the paper in Section 6.

2. System Model and Problem Formulation

We depict the model of the status updating system which uses $Ber/Geo/1/2^*/\eta$ queue in Figure 2, in which the packet in system's buffer can be preempted by fresher packet from the source s with probability η. The packet arrivals to the transmitter is assumed to form a Bernoulli stochastic process, that is in each time slot a new packet comes with an identical probability which we denote by p. Packet service time follows the geometric distribution with intensity γ. Updating packet generated at s is transmitted to the destination d through the transmitter, in which a random period of time is consumed. The age of information (AoI) at d is defined as the time elapsed since the generation time of the last obtained packet up to now. Within the time when no packet is received, the value of AoI increases 1 after each time slot ends. Every time when a packet passes the transmitter and arrives to d, the AoI will be reduced to the system time of the obtained packet, which is actually equal to the instantaneous age of this packet.

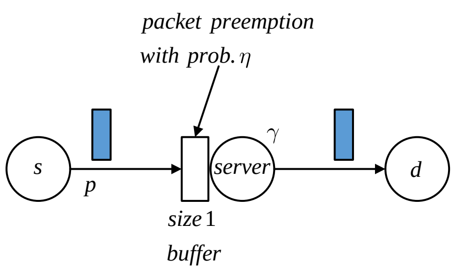

Figure 2. The model of discrete time status updating system with probabilistic packet preemption in system's buffer.

Let $a(k)$ be the value of AoI in kth time slot. The AoI at next time slot, $a(k+1)$ is determined by

$$a(k+1) = \begin{cases} a(k)+1 & \text{if no packet is obtained,} \\ a(k)+1-Y_j & \text{when } j\text{th generated packet arrives to } d. \end{cases} \tag{12}$$

where Y_j is the interarrival time between $(j-1)$th and jth arriving packet.

Notice that these $(j-1)$th and jth packets may be generated discontinuously, since between them some updating packets may be discarded when they arrive and find the system full. Actually, this is exactly the difference between the finite and infinite status updating systems. Based on this observations, in paper [58] we have determined the average AoI and its stationary distribution for infinite size status updating system with Bernoulli packet arrivals and geometric service time.

Denote the stationary AoI for the system with probabilistic packet preemption as $\Delta_{Ber/Geo/1/2^*/\eta}$. We define the time average AoI as follows, which is equal to the mean of the AoI because the age process is assumed to be ergodic. We have

$$\overline{\Delta}_{Ber/Geo/1/2^*/\eta} = \lim_{T\to\infty} \frac{1}{T} \sum_{k=1}^{T} a(k) \tag{13}$$

$$= \lim_{T\to\infty} \frac{1}{T} \sum_{i=1}^{M_T} i \cdot |\{1 \le k \le T : a(k) = i\}| \tag{14}$$

$$= \sum_{i=1}^{\infty} i \cdot \pi_i \tag{15}$$

in which we define $M_T = \max_{1\le k\le T} a(k)$, and for each $i \ge 1$,

$$\pi_i = \lim_{T\to\infty} \frac{|\{1 \le k \le T : a(k) = i\}|}{T} \tag{16}$$

is the probability that stationary AoI takes value i. In fact, probability distribution $\{\pi_i, i \ge 1\}$ forms the stationary distribution of the AoI $\Delta_{Ber/Geo/1/2^*/\eta}$.

The randomness of both packet arrivals and the service time in server, along with the probabilistic packet preemption in system's buffer together makes the AoI at destination change randomly. After one time slot, the value of AoI may increase by 1 if no packet is obtained, or drops to the age of the obtained packet at that time if one such packet is successfully received. In order to fully describe these random dynamics of AoI, we propose to use a three-dimensional state vector to simultaneously record the changes of the AoI, the age of packet in server, and the age of the packet waiting in buffer, and then constitute the three-dimensional stochastic process. Next, the steady-state of this multiple-dimensional age process is analyzed. To obtain the mean and the distribution of AoI, we define the PGF corresponding to the stationary distribution of three-dimensional age process, from which both the AoI's mean and its distribution can be obtained. The detailed analysis of system's AoI is given in following Section 3.

3. AoI Analysis for Status Updating System with Probabilistic Packet Preemption

Define the three-dimensional state vector (n, m, l) where we use n to denote the AoI at destination d, and the other two parameters m, l are the age of packets in system's server and the buffer. In kth time slot, if the server is busy while the buffer is empty, then n_k and m_k are greater than 0 but $l_k = 0$. When both the server and the buffer are empty, we have $m_k = l_k = 0$. In this case, the entire system is empty.

Constituting the following three-dimensional age process

$$AoI_{PP} = \{(n_k, m_k, l_k) : n_k > m_k > l_k \ge 0, k \in \mathbb{N}\} \tag{17}$$

where the subscript "PP" in expression (17) is the abbreviation of *probabilistic preemption*. Notice that when the system is empty, the last two parameters m_k and l_k are both equal to 0, however, we still write $n_k > m_k > l_k$ in equation (17) to indicate the relationship of the AoI and two packet ages when there is one packet in the server and one packet in the buffer. It will be clear later that this relationship facilitates the derivation of probability generation function $H_{PP}(x)$, which is defined in equation (20).

Define three random variables A, B, and F to represent in a time slot whether a packet is generated, if the service of the packet is completed, and if the arriving packet replaces the original one in the buffer. For each possible initial state vector, according to different realizations of r.v.s (A, B, F), the state transfers of three-dimensional state vector (n, m, l) can be described specifically. We list all of them using the Table 2. For example, the third row of the table shows a packet of age l is in buffer and a new packet arrives, since the r.v. A takes value 1. However, $F = 0$ means this new packet will not substitute the original one, meanwhile, $B = 0$ implies that the packet service is not over at this time slot. Summarizing all these events, the beginning state vector (n, m, l) will transfer to $(n+1, m+1, l+1)$ at

Table 2. The state vector transfers of age process AoI_{PP}

Initial state vector	Considered r.v.s	Realizations and next state vector
(n,m,l), $n>m>l\geq 1$	$(A=0,B)$	$(0,0):(n+1,m+1,l+1)$
		$(0,1):(m+1,l+1,0)$
	$(A=1,B,F)$	$(1,0,0):(n+1,m+1,l+1)$
		$(1,0,1):(n+1,m+1,1)$
		$(1,1,0):(m+1,l+1,0)$
		$(1,1,1):(m+1,1,0)$
$(n,m,0)$, $n>m\geq 1$	(A,B)	$(0,0):(n+1,m+1,0)$
		$(0,1):(m+1,0,0)$
		$(1,0):(n+1,m+1,1)$
		$(1,1):(m+1,1,0)$
$(n,0,0)$, $n\geq 1$	$A=0$	$(n+1,0,0)$
	$(A=1,B)$	$(1,0):(n+1,1,0)$
		$(1,1):(1,0,0)$

next time slot, and the transition probability is determined as $p(1-\gamma)(1-\eta)$. The other cases in the third column of Table 2 are obtained through similar discussions.

From the state transfers given in Table 2 and the corresponding transition probabilities, we can establish all the stationary equations that characterize the steady-state of age process AoI_{PP}. Let $\pi_{(n,m,l)}$, $n>m\geq l\geq 0$ be the probability the process stays at state vector (n,m,l) when it reaches the steady-state, we show that these stationary probabilities $\pi_{(n,m,l)}$ satisfy the following equations.

$$\begin{cases} \pi_{(n,m,l)} = \pi_{(n-1,m-1,l-1)}[(1-p)(1-\gamma)+p(1-\gamma)(1-\eta)] & (n>m>l\geq 2) \\ \pi_{(n,m,1)} = \pi_{(n-1,m-1,0)}p(1-\gamma)+\sum_{j=1}^{m-2}\pi_{(n-1,m-1,j)}p(1-\gamma)\eta & (n>m\geq 3) \\ \pi_{(n,2,1)} = \pi_{(n-1,1,0)}p(1-\gamma) & (n\geq 3) \\ \pi_{(n,m,0)} = \pi_{(n-1,m-1,0)}(1-p)(1-\gamma) & \\ \qquad +\sum_{k=n}^{\infty}\pi_{(k,n-1,m-1)}[(1-p)\gamma+p\gamma(1-\eta)] & (n>m\geq 2) \\ \pi_{(n,1,0)} = \pi_{(n-1,0,0)}p(1-\gamma) & \\ \qquad +\sum_{k=n}^{\infty}\pi_{(k,n-1,0)}p\gamma+\sum_{k=n}^{\infty}\sum_{j=1}^{n-2}\pi_{(k,n-1,j)}p\gamma\eta & (n\geq 3) \\ \pi_{(2,1,0)} = \pi_{(1,0,0)}p(1-\gamma)+\sum_{k=2}^{\infty}\pi_{(k,1,0)}p\gamma & \\ \pi_{(n,0,0)} = \pi_{(n-1,0,0)}(1-p)+\sum_{k=n}^{\infty}\pi_{(k,n-1,0)}(1-p)\gamma & (n\geq 2) \\ \pi_{(1,0,0)} = \sum_{n=1}^{\infty}\pi_{(n,0,0)}p\gamma & \end{cases} \tag{18}$$

We explain the stationary equations only for a part of state vectors and show that the other equations in (18) can be determined in the similar manner. Firstly, for the fifth row of (18), the state vector $(n,1,0)$ can be obtained from $(n-1,0,0)$ assuming a new packet arrives and enters the server directly, but the service does not end in single time slot. Next, from the current state vector $(k,n-1,0)$, $k\geq n$, if the service of the age $(n-1)$ packet is completed and at the same time slot there comes a new packet, it is observed that the packet of age $(n-1)$ will be sent to the receiver, which makes the AoI change to n at next time slot. The new packet enters the server, thus the middle parameter of state vector changes to 1. This gives the expected state $(n,1,0)$. Since in this case the buffer is empty, when a new packet comes, it occupies the buffer directly and no packet preemption occurs. At last, we consider the situation where the age process begins with an arbitrary state $(k,n-1,j)$ where $k>n-1>j\geq 1$. As long as the packet service is completed and at the same time there arrives a new packet preempting the original one in buffer, again we will obtain the state vector $(n,1,0)$ after one time slot. Combining all of above cases,

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the stationary equation corresponding to $(n, 1, 0)$ is finally determined. In addition to the fifth row, we also explain the last equation in (18). Observing that in order to obtain the state vector $(1, 0, 0)$, the receiver needs a packet of age 1, meanwhile, the system has to be emptied. This state can be transferred to only from $(n, 0, 0)$ and the service time of the newly arrived packet is restricted to be only one time slot.

To derive the expression of average AoI $\overline{\Delta}_{Ber/Geo/1/2^*/\eta}$, we will not solve equations (18) although this approach is feasible for the AoI analysis of current system. In our work [59], we analyzed the AoI of status updating system with $Ber/Geo/1/1$, $Ber/Geo/1/2$, and $Ber/Geo/1/2^*$ queues, the expression of AoI's stationary distribution was determined for each case. There, we completely solved the stationary equations for each system and obtain the explicit expression for every stationary probability. Notice that this work can be regarded as discrete correspondence of the packet management of continuous AoI in papers [8,9]. Assuming all the probabilities $\pi_{(n,m,l)}$ have been determined by solving equations (18), then we have

$$\Pr\left\{\Delta_{Ber/Geo/1/2^*/\eta} = n\right\} = \begin{cases} \pi_{(1,0,0)} & (n = 1) \\ \pi_{(n,0,0)} + \sum_{l=0}^{n-2} \sum_{m=l+1}^{n-1} \pi_{(n,m,l)} & (n \geq 2) \end{cases} \tag{19}$$

since the probability the AoI takes each n is equal to the sum of all the stationary probabilities with the identical first component. Equation (19) gives the stationary distribution of the AoI, from which we can calculate the average value of AoI as

$$\overline{\Delta}_{Ber/Geo/1/2^*/\eta} = \sum_{n=1}^{\infty} n \cdot \Pr\left\{\Delta_{Ber/Geo/1/2^*/\eta} = n\right\}$$

However, the amount of calculations to solve the equations (18) may be large and apart from this, extra computations are required to determine the AoI's distribution according to formula (19). In fact, with the increase of state vector's dimension, the amount of calculation increases rapidly, both of solving the stationary equations and calculating the probability distribution. Thus, we will determine the mean of AoI and its distribution in another way, i.e., the probability generation function (PGF) method.

For $0 < x \leq 1$, define the probability generation function

$$H_{PP}(x) = \sum_{n=1}^{\infty} x^n \Pr\left\{\Delta_{Ber/Geo/1/2^*/\eta} = n\right\} \tag{20}$$

and we write $H_{PP}(x)$ further as

$$\begin{aligned} H_{PP}(x) &= x \Pr\left\{\Delta_{Ber/Geo/1/2^*/\eta} = 1\right\} + \sum_{n=2}^{\infty} x^n \Pr\left\{\Delta_{Ber/Geo/1/2^*/\eta} = n\right\} \\ &= x\pi_{(1,0,0)} + \sum_{n=2}^{\infty} x^n \left\{\pi_{(n,0,0)} + \sum_{l=0}^{n-2} \sum_{m=l+1}^{n-1} \pi_{(n,m,l)}\right\} && (21) \\ &= \sum_{n=1}^{\infty} x^n \pi_{(n,0,0)} + \sum_{n=2}^{\infty} x^n \sum_{l=0}^{n-2} \sum_{m=l+1}^{n-1} \pi_{(n,m,l)} && (22) \\ &= \sum_{n=1}^{\infty} x^n \pi_{(n,0,0)} + \sum_{l=0}^{\infty} \sum_{m=l+1}^{\infty} \sum_{n=m+1}^{\infty} x^n \pi_{(n,m,l)} && (23) \\ &= \sum_{n=1}^{\infty} x^n \pi_{(n,0,0)} + \sum_{m=1}^{\infty} \sum_{n=m+1}^{\infty} x^n \pi_{(n,m,0)} + \sum_{l=1}^{\infty} \sum_{m=l+1}^{\infty} \sum_{n=m+1}^{\infty} x^n \pi_{(n,m,l)} && (24) \end{aligned}$$

where in (21), we have used the probability expressions (19). Equation (23) is obtained by exchanging the summation order in (22). In last equation (24), we divide the PGF $H_{PP}(x)$ into three parts. It will be seen in following paragraphs the entire function (20) is obtained by determining these parts separately.

According to expression (20), immediately we have

$$H_{PP}(1) = 1, \quad \left.\frac{\mathrm{d}H_{PP}(x)}{\mathrm{d}x}\right|_{x=1} = \overline{\Delta}_{Ber/Geo/1/2^*/\eta} \tag{25}$$

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That is, the average AoI can be obtained from the PGF's derivative at point $x = 1$, and the probability the steady state AoI equals n is determined by the coefficient before the term x^n for every $n \geq 1$.

Now, we determine the PGF $H_{PP}(x)$. For $0 < x \leq 1$, define the functions

$$\begin{aligned} h_1(x) &= \textstyle\sum_{n=1}^{\infty} x^n \pi_{(n,0,0)} \\ h_2(x) &= \textstyle\sum_{m=1}^{\infty} \sum_{n=m+1}^{\infty} x^n \pi_{(n,m,0)} \\ h_3(x) &= \textstyle\sum_{l=1}^{\infty} \sum_{m=l+1}^{\infty} \sum_{n=m+1}^{\infty} x^n \pi_{(n,m,l)} \end{aligned}$$

and

$$h_2^{(m)}(x) = \textstyle\sum_{m=1}^{\infty} \sum_{n=m+1}^{\infty} x^m \pi_{(n,m,0)}$$

We first give following lemma, from which the PGF $H_{PP}(x)$ can be determined completely.

Lemma 1. *For the functions $h_i(x)$, $1 \leq i \leq 3$ and $h_2^{(m)}(x)$, we have*

$$h_1(x) = \frac{p\gamma M_1 x}{1-(1-p)x} + \frac{(1-p)\gamma x}{1-(1-p)x} h_2^{(m)}(x) \tag{26}$$

$$h_2(x) = \frac{p(1-\gamma)x}{1-(1-p)(1-\gamma)x} h_1(x) + \frac{p\gamma x}{[1-(1-p)(1-\gamma)x][1-(1-\gamma)x]} h_2^{(m)}(x) \tag{27}$$

$$h_3(x) = \frac{p(1-\gamma)x}{1-(1-\gamma)x} h_2(x) \tag{28}$$

and it is determined that

$$h_2^{(m)}(x) = \frac{[\gamma + p^2(1-\gamma)\eta - (1-p)(1-\gamma)(1-p\eta)\gamma x] M_2 x}{[1-(1-p)(1-\gamma)x][1-(1-\gamma)(1-p\eta)x]} \tag{29}$$

in which the numbers M_1 and M_2 are given as

$$M_1 = \frac{(1-p)\gamma^2}{(p+\gamma-2p\gamma)\gamma + p^2(1-\gamma)^2} \tag{30}$$

$$M_2 = \frac{p\gamma(1-\gamma)}{(p+\gamma-2p\gamma)\gamma + p^2(1-\gamma)^2} \tag{31}$$

Proof. The Lemma 1 will be proved in Appendix A. □

Using Lemma 1, we calculate the PGF $H_{PP}(x)$ as follows. From (24), it shows

$$\begin{aligned} H_{PP}(x) &= h_1(x) + h_2(x) + h_3(x) \\ &= h_1(x) + h_2(x) + \frac{p(1-\gamma)x}{1-(1-\gamma)x} h_2(x) \\ &= h_1(x) + \frac{1-(1-p)(1-\gamma)x}{1-(1-\gamma)x}\left(\frac{p(1-\gamma)x}{1-(1-p)(1-\gamma)x} h_1(x)\right. \\ &\qquad \left. + \frac{p\gamma x}{[1-(1-p)(1-\gamma)x][1-(1-\gamma)x]} h_2^{(m)}(x)\right) \\ &= \frac{1-(1-p)(1-\gamma)x}{1-(1-\gamma)x} h_1(x) + \frac{p\gamma x}{[1-(1-\gamma)x]^2} h_2^{(m)}(x) \end{aligned} \tag{32}$$

where in (32) we have substituted equation (27).

Using equation (26) and merging the same terms, eventually we obtain

$$H_{PP}(x) = \frac{p\gamma M_1 x[1-(1-p)(1-\gamma)x]}{[1-(1-p)x][1-(1-\gamma)x]} + \frac{\gamma x\{1-(1-p)[2(1-\gamma)+p\gamma]x+(1-p)^2(1-\gamma)^2x^2\}}{[1-(1-p)x][1-(1-\gamma)x]^2} h_2^{(m)}(x) \quad (33)$$

in which the function $h_2^{(m)}(x)$ is given in equation (29).

According to the formula (25), the average AoI of system with probabilistic packet preemption is calculated in following Theorem 1.

Theorem 1. *For the discrete time state updating system having $Ber/Geo/1/2^*/\eta$ queue, assuming the packet waiting in buffer can be preempted by following fresher packets with probability η, then the average age of information of this system is determined as*

$$\overline{\Delta}_{Ber/Geo/1/2^*/\eta} = \frac{(p+\gamma-p\gamma)(p+\gamma)-p\gamma}{p\gamma}M_1 + \frac{(p+\gamma-p\gamma)^2-p\gamma(1-p)}{p\gamma}\cdot\left.\frac{\mathrm{d}h_2^{(m)}(x)}{\mathrm{d}x}\right|_{x=1} + \frac{\left\{(p+\gamma-p\gamma)[1-3(1-p)(1-\gamma)]-2p\gamma(1-p)\right\}p\gamma+\mathrm{Poly}_1}{p^2\gamma^2}M_2 \quad (34)$$

in which we define

$$\mathrm{Poly}_1 = [(p+\gamma-p\gamma)^2-p\gamma(1-p)][2p(1-\gamma)+(1-p)\gamma] \quad (35)$$

and the derivative of $h_2^{(m)}(x)$ at point 1 is calculated as

$$\left.\frac{\mathrm{d}h_2^{(m)}(x)}{\mathrm{d}x}\right|_{x=1} = \left(\frac{1}{p+\gamma-p\gamma}+\frac{p(1-\gamma)(1-p\eta)}{(p+\gamma-p\gamma)[\gamma+p(1-\gamma)\eta]}\right)M_2 \quad (36)$$

Let $p=\rho_d\cdot\gamma$ and substitute numbers M_1, M_2, the average AoI is also written as

$$\begin{aligned}&\overline{\Delta}_{Ber/Geo/1/2^*/\eta}\\ &= \frac{1+\rho_d(1-2\gamma)+\rho_d^2(1-\gamma)^2+2\rho_d^3(1-\gamma)^2}{\rho_d\gamma[1+\rho_d(1-2\gamma)+\rho_d^2(1-\gamma)^2]}\\ &\quad+\frac{(1-\gamma)+\rho_d(1-\gamma)(1-2\gamma)+\rho_d^2(1-\gamma)(1-\gamma+\gamma^2)}{1+\rho_d(1-2\gamma)+\rho_d^2(1-\gamma)^2}\left\{\frac{(2-\gamma)-\rho_d\gamma(1-\gamma)}{\gamma[1+\rho_d(1-\gamma)]}\right.\\ &\quad\left.-\frac{(1-\gamma)-\rho_d(1-\gamma)[1+\gamma-(1-\gamma)\eta]+\rho_d^2\gamma^2(1-\gamma)\eta}{\gamma[1+\rho_d(1-\gamma)(1+\eta)+\rho_d^2(1-\gamma)^2\eta]}\right\}\end{aligned} \quad (37)$$

where $\rho_d = p/\gamma$ is defined as the discrete traffic load.

Proof. The average AoI is determined by first computing the derivative of $H_{PP}(x)$ in (33) and then letting $x=1$. Replacing parameter p with $\rho_d\cdot\gamma$, expression (37) is obtained eventually. Although a certain amount of calculation is required, all the computations are straight-forward.

Here, we only provide the details of obtaining equation (36). From (29), we have

$$
\begin{aligned}
&\left.\frac{\mathrm{d}h_2^{(m)}(x)}{\mathrm{d}x}\right|_{x=1} \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left.\frac{[\gamma + p^2(1-\gamma)\eta - (1-p)(1-\gamma)(1-p\eta)\gamma x]M_2 x}{[1-(1-p)(1-\gamma)x][1-(1-\gamma)(1-p\eta)x]}\right|_{x=1} \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left.\left(M_2\frac{[\gamma + p^2(1-\gamma)\eta]x - (1-p)(1-\gamma)(1-p\eta)\gamma x^2}{1-[(1-p)(1-\gamma)+(1-\gamma)(1-p\eta)]x + (1-p)(1-\gamma)^2(1-p\eta)x^2}\right)\right|_{x=1} \\
&= M_2\frac{\mathrm{Poly}_2\cdot(p+\gamma-p\gamma)[\gamma+p(1-\gamma)\eta] + [\gamma+p^2(1-\gamma)\eta-(1-p)(1-\gamma)(1-p\eta)\gamma]\cdot\mathrm{Poly}_3}{(p+\gamma-p\gamma)^2[\gamma+p(1-\gamma)\eta]^2}
\end{aligned}
\tag{38}
$$

where

$$
\mathrm{Poly}_2 = \gamma + p^2(1-\gamma)\eta - 2(1-p)(1-\gamma)(1-p\eta)\gamma
$$

and

$$
\begin{aligned}
\mathrm{Poly}_3 &= (1-p)(1-\gamma) + (1-\gamma)(1-p\eta) - 2(1-p)(1-\gamma)^2(1-p\eta) \\
&= (1-p)(1-\gamma)[\gamma+p(1-\gamma)\eta] + (p+\gamma-p\gamma)(1-\gamma)(1-p\eta)
\end{aligned}
$$

Notice that

$$
\begin{aligned}
&\gamma + p^2(1-\gamma)\eta - (1-p)(1-\gamma)(1-p\eta)\gamma \\
&= \gamma + p(1-\gamma)\eta - p(1-p)(1-\gamma)\eta - (1-p)(1-\gamma)(1-p\eta)\gamma \\
&= \gamma + p(1-\gamma)\eta - (1-p)(1-\gamma)[\gamma+p(1-\gamma)\eta] \\
&= (p+\gamma-p\gamma)[\gamma+p(1-\gamma)\eta]
\end{aligned}
\tag{39}
$$

Substituting (39) into (38), it shows

$$
\begin{aligned}
\left.\frac{\mathrm{d}h_2^{(m)}(x)}{\mathrm{d}x}\right|_{x=1} &= M_2\left(1 - \frac{(1-p)(1-\gamma)(1-p\eta)\gamma}{(p+\gamma-p\gamma)[\gamma+p(1-\gamma)\eta]} + \frac{(1-p)(1-\gamma)}{p+\gamma-p\gamma} + \frac{(1-\gamma)(1-p\eta)}{\gamma+p(1-\gamma)\eta}\right) \\
&= M_2\left(\frac{1}{p+\gamma-p\gamma} + \frac{p(1-\gamma)(1-p\eta)}{(p+\gamma-p\gamma)[\gamma+p(1-\gamma)\eta]}\right)
\end{aligned}
\tag{40}
$$

which is exactly the equation (36). □

Notice that in definition (20), for each $n \geq 1$, the coefficient of x^n is the probability the AoI equals n. In order to obtain these coefficients, we decompose the PGF $H_{PP}(x)$ into power series. It shows that

$$
\begin{aligned}
H_{PP}(x) = &\frac{p\gamma M_1 x}{\gamma - p}\left(\frac{(1-p)\gamma}{1-(1-p)x} - \frac{p(1-\gamma)}{1-(1-\gamma)x}\right) \\
&- \left(\frac{(1-p)^2\gamma^2 M_2 x^2}{(\gamma-p)[1-(1-p)x]} - \frac{p\gamma(1-p)(1-\gamma)M_2x^2}{(\gamma-p)[1-(1-\gamma)x]} + \frac{p\gamma M_2 x^2}{[1-(1-\gamma)x]^2}\right) \\
&\times\left(\frac{\eta(1-p)(p+\gamma-p\gamma)}{(1-\eta)[1-(1-p)(1-\gamma)x]} - \frac{(1-p\eta)[\gamma+p(1-\gamma)\eta]}{(1-\eta)[1-(1-\gamma)(1-p\eta)x]}\right)
\end{aligned}
\tag{41}
$$

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when preemption probability $\eta \neq 1$, while for the case $\eta = 1$, we have

$$H_{PP}(x) = \frac{p\gamma M_1 x}{\gamma - p}\left(\frac{(1-p)\gamma}{1-(1-p)x} - \frac{p(1-\gamma)}{1-(1-\gamma)x}\right) + \left(\frac{(1-p)^2\gamma^2 M_2 x^2}{(\gamma-p)[1-(1-p)x]} - \frac{p\gamma(1-p)(1-\gamma)M_2x^2}{(\gamma-p)[1-(1-\gamma)x]} + \frac{p\gamma M_2 x^2}{[1-(1-\gamma)x]^2}\right) \times \frac{\gamma + p^2(1-\gamma) - (1-p)^2(1-\gamma)\gamma x}{[1-(1-p)(1-\gamma)x]^2} \tag{42}$$

The details to obtain equations (41) and (42) are given in Appendix B. In the following Section 4, along with the average value of the AoI, we will determine AoI's stationary distribution for two extreme cases $\eta = 0$ and $\eta = 1$.

4. Stationary age of information under two extreme cases

In this Section, we will determine the average AoI of status updating system without packet preemption by setting $\eta = 0$, and when the preemption probability η is equal to 1, the mean of AoI for the $Ber/Geo/1/2^*$ queue modeled system is also derived. In addition, using equations (41) and (42), the stationary distribution of discrete AoI for two cases are also obtained.

Theorem 2. *Assuming the packet arrivals form a Bernoulli process and the service time is geometrically distributed, the average AoI of discrete time status updating system with $Ber/Geo/1/2$ and $Ber/Geo/1/2^*$ queues are calculated as*

$$\overline{\Delta}_{Ber/Geo/1/2} = \frac{1}{\gamma}\left((1-\gamma) + \frac{1}{\rho_d} + \frac{2\rho_d^2(1-\gamma)(1-\gamma/2)}{1+\rho_d(1-2\gamma)+\rho_d^2(1-\gamma)^2}\right) \tag{43}$$

and

$$\overline{\Delta}_{Ber/Geo/1/2^*} = \frac{1}{\gamma}\left((1-\gamma) + \frac{1}{\rho_d} + \frac{\rho_d^2(1-\gamma)\left[1+3\rho_d(1-\gamma)+\rho_d^2(1-\gamma)(1-2\gamma)\right]}{\left[1+\rho_d(1-2\gamma)+\rho_d^2(1-\gamma)^2\right]\left[1+\rho_d(1-\gamma)\right]^2}\right) \tag{44}$$

For each $n \geq 1$, the distribution of the AoI $\Delta_{Ber/Geo/1/2}$ is given by

$$\Pr\{\Delta_{Ber/Geo/1/2} = n\} = \frac{p\gamma^2(1-p)M_1}{(\gamma-p)^2}(1-p)^n - \frac{(\gamma-p^2)(1-p)\gamma^2 M_2}{(\gamma-p)^2}(1-\gamma)^{n-1} - \frac{p\gamma^2(1-p)M_2}{\gamma-p}(n-1)(1-\gamma)^{n-1} + \frac{p\gamma^2 M_2}{2}n(n-1)(1-\gamma)^{n-2} \tag{45}$$

while when the system has full packet preemption, we show that

$$\begin{aligned}
&\Pr\{\Delta_{Ber/Geo/1/2^*} = n\} \\
&= \frac{p\gamma M_1}{\gamma-p}\left(\gamma(1-p)^n - p(1-\gamma)^n\right) \\
&\quad + \frac{(1-p)[\gamma^2 + p(1-\gamma)(p+\gamma)]M_2}{\gamma-p}\left((1-p)^{n-1} - [(1-p)(1-\gamma)]^{n-1}\right) \\
&\quad - \frac{(1-p)\gamma[p+2(1-p)\gamma]M_2}{\gamma-p}\left((1-\gamma)^{n-1} - [(1-p)(1-\gamma)]^{n-1}\right) + p\gamma M_2\Big(A(1-\gamma)^{n-2} \\
&\quad + B(n-1)(1-\gamma)^{n-2} + C[(1-p)(1-\gamma)]^{n-2} + D(n-1)[(1-p)(1-\gamma)]^{n-2}\Big)
\end{aligned} \tag{46}$$

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in which the coefficients A, B, C, and D are determined by

$$A = \frac{2-p}{p^3}\left((1-p)^2\gamma - \frac{2(1-p)[\gamma + p^2(1-\gamma)]}{2-p}\right) \tag{47}$$

$$C = -\frac{(1-p)(2-p)}{p^3}\left((1-p)^2\gamma - \frac{2(1-p)[\gamma + p^2(1-\gamma)]}{2-p}\right) \tag{48}$$

$$D = -\frac{p^2(1-p)\cdot A + (1-p)^2[\gamma + p^2(1-\gamma)]}{p(2-p)} \tag{49}$$

and

$$B = [\gamma + p^2(1-\gamma)] - A - C - D \tag{50}$$

Proof. We first derive two average AoIs in equations (43) and (44) from the general expression (37). Let η be 0, then no packet preemption will occur in system's buffer. The system's queue model reduces to $Ber/Geo/1/2$ and from (37), we can obtain the average AoI $\overline{\Delta}_{Ber/Geo/1/2}$.

In this case, it is easy to show the last two terms within the brace of (37) is calculated to be $1/\gamma$. Thus, we have

$$\begin{aligned}
&\overline{\Delta}_{Ber/Geo/1/2} = \overline{\Delta}_{Ber/Geo/1/2^*/\eta}\Big|_{\eta=0} \\
&= \frac{1+\rho_d(1-2\gamma)+\rho_d^2(1-\gamma)^2+2\rho_d^3(1-\gamma)^2}{\rho_d\gamma[1+\rho_d(1-2\gamma)+\rho_d^2(1-\gamma)^2]} \\
&\quad + \frac{(1-\gamma)+\rho_d(1-\gamma)(1-2\gamma)+\rho_d^2(1-\gamma)(1-\gamma+\gamma^2)}{\gamma[1+\rho_d(1-2\gamma)+\rho_d^2(1-\gamma)^2]} \\
&= \frac{1+\rho_d(2-3\gamma)+\rho_d^2(1-\gamma)(2-3\gamma)+\rho_d^3(1-\gamma)(3-3\gamma+\gamma^2)}{\rho_d\gamma[1+\rho_d(1-2\gamma)+\rho_d^2(1-\gamma)^2]} \\
&= \frac{1}{\rho_d\gamma}\left(1+\rho_d(1-\gamma)+\frac{2\rho_d^3(1-\gamma)(1-\gamma/2)}{1+\rho_d(1-2\gamma)+\rho_d^2(1-\gamma)^2}\right) && (51) \\
&= \frac{1}{\gamma}\left((1-\gamma)+\frac{1}{\rho_d}+\frac{2\rho_d^2(1-\gamma)(1-\gamma/2)}{1+\rho_d(1-2\gamma)+\rho_d^2(1-\gamma)^2}\right) && (52)
\end{aligned}$$

where in equation (51), we use the method of long division.

For the other extreme case of $\eta = 1$, obviously the general expression (37) gives the average AoI $\overline{\Delta}_{Ber/Geo/1/2^*}$. Similarly, we first determine the value of last two terms within the brace. We shows that the difference of last two terms equals

$$\frac{1}{\gamma} - \frac{\rho_d^2(1-\gamma)}{\gamma[1+\rho_d(1-\gamma)^2]} \tag{53}$$

thus the average AoI $\overline{\Delta}_{Ber/Geo/1/2^*}$ is calculated as

$$\overline{\Delta}_{Ber/Geo/1/2^*} = \overline{\Delta}_{Ber/Geo/1/2^*/\eta}\Big|_{\eta=1}$$
$$= \overline{\Delta}_{Ber/Geo/1/2} - \frac{(1-\gamma)+\rho_d(1-\gamma)(1-2\gamma)+\rho_d^2(1-\gamma)(1-\gamma+\gamma^2)}{1+\rho_d(1-2\gamma)+\rho_d^2(1-\gamma)^2} \cdot \frac{\rho_d^2(1-\gamma)}{\gamma[1+\rho_d(1-\gamma)]^2} \tag{54}$$

$$= \frac{1}{\gamma}\Bigg((1-\gamma)+\frac{1}{\rho_d}+\frac{\rho_d^2(1-\gamma)(2-\gamma)}{1+\rho_d(1-2\gamma)+\rho_d^2(1-\gamma)^2} - \frac{\rho_d^2(1-\gamma)^2+\rho_d^3(1-\gamma)^2(1-2\gamma)+\rho_d^4(1-\gamma)^2(1-\gamma+\gamma^2)}{[1+\rho_d(1-2\gamma)+\rho_d^2(1-\gamma)^2][1+\rho_d(1-\gamma)]^2}\Bigg) \tag{55}$$

$$= \frac{1}{\gamma}\left((1-\gamma)+\frac{1}{\rho_d}+\frac{\rho_d^2(1-\gamma)[1+3\rho_d(1-\gamma)+\rho_d^2(1-\gamma)(1-2\gamma)]}{1+\rho_d(1-2\gamma)+\rho_d^2(1-\gamma)^2}\right) \tag{56}$$

In equation (54), since in the case of $\eta = 0$, the difference of last two terms is $1/\gamma$, thus the average AoI $\overline{\Delta}_{Ber/Geo/1/2}$ is obtained. This equation also gives the exact gap between two average AoIs of system with and without packet preemption. Notice that the latter term in (54) is always positive, then the average AoI must get lower when packet preemption strategy is applied. Equation (52) is substituted in (55) and in equation (56), the expression of average AoI $\overline{\Delta}_{Ber/Geo/1/2^*}$ is finally determined.

Next, the distribution of discrete AoI is calculated. Before the expressions (45) and (46) are derived, we first verify that both (45) and (46) are proper probability distributions by providing a specific numerical example.

Numerical Results of two AoI's distributions. Let $p = 1/4$ and $\gamma = 1/2$.

Firstly, from equations (30) and (31), two numbers M_1 and M_2 are determined to be

$$M_1 = \frac{12}{17}, \quad M_2 = \frac{4}{17}.$$

after some simple calculations, for each $n \geq 1$, the expression (45) gives

$$\Pr\{\Delta_{Ber/Geo/1/2} = n\} = \frac{9}{17}\left(\frac{3}{4}\right)^n - \frac{21}{68}\left(\frac{1}{2}\right)^n - \frac{3}{68}(n-1)\left(\frac{1}{2}\right)^{n-1} + \frac{1}{136}n(n-1)\left(\frac{1}{2}\right)^{n-2} \tag{57}$$

To obtain the numerical result of equation (46), it is necessary to determine the four coefficients A, B, C, and D according to expressions (47)-(50). We directly give that

$$A = -\frac{39}{2}, \quad C = \frac{117}{8}, \quad D = \frac{45}{32}, \quad B = 4.$$

After some extra computations, it shows that

$$\Pr\{\Delta_{Ber/Geo/1/2^*} = n\} = \frac{1}{2}\left(\frac{3}{4}\right)^n - \frac{105}{34}\left(\frac{1}{2}\right)^n + \frac{57}{17}\left(\frac{3}{8}\right)^n + \frac{2}{17}(n-1)\left(\frac{1}{2}\right)^{n-2} + \frac{45}{1088}(n-1)\left(\frac{3}{8}\right)^{n-2} \tag{58}$$

It can be checked directly that the sum of both (57) and (58) from $n = 1$ to ∞ are equal to 1. Therefore, expressions (45) and (46) indeed form the proper probability distributions.

In the following, by decomposing (41) and (42) further into several simplest rational fractions, we derive the explicit expressions of AoI's distribution for the system with and without packet preemption.

First of all, for $\eta = 0$, it is easy to prove that the last part of (41) is equal to

$$\frac{-\gamma}{1-(1-\gamma)x}$$

thus we have following equations. It shows that

$$\begin{aligned}
&H_{PP}(x)|_{\eta=0} \\
&= \frac{p\gamma M_1 x}{\gamma - p}\left(\frac{(1-p)\gamma}{1-(1-p)x} - \frac{p(1-\gamma)}{1-(1-\gamma)x}\right) \\
&\quad - \left(\frac{(1-p)^2\gamma^2 M_2 x^2}{(\gamma-p)[1-(1-p)x]} - \frac{p\gamma(1-p)(1-\gamma)M_2x^2}{(\gamma-p)[1-(1-\gamma)x]} + \frac{p\gamma M_2 x^2}{[1-(1-\gamma)x]^2}\right)\cdot\frac{-\gamma}{1-(1-\gamma)x} \\
&= \frac{p\gamma^2(1-p)M_1x}{(\gamma-p)[1-(1-p)x]} - \frac{p^2\gamma(1-\gamma)M_1x}{(\gamma-p)[1-(1-\gamma)x]} \\
&\quad + \frac{(1-p)^2\gamma^3M_2x^2}{(\gamma-p)[1-(1-p)x][1-(1-\gamma)x]} - \frac{p\gamma^2(1-p)(1-\gamma)M_2x^2}{(\gamma-p)[1-(1-\gamma)x]^2} + \frac{p\gamma^2M_2x^2}{[1-(1-\gamma)x]^3} \\
&= \frac{p\gamma^2(1-p)M_1x}{(\gamma-p)[1-(1-p)x]} - \frac{p^2\gamma(1-\gamma)M_1x}{(\gamma-p)[1-(1-\gamma)x]} \\
&\quad + \frac{(1-p)^2\gamma^3M_2x^2}{(\gamma-p)}\left(\frac{1-p}{(\gamma-p)[1-(1-p)x]} - \frac{1-\gamma}{(\gamma-p)[1-(1-\gamma)x]}\right) \\
&\quad - \frac{p\gamma^2(1-p)(1-\gamma)M_2x^2}{(\gamma-p)[1-(1-\gamma)x]^2} + \frac{p\gamma^2M_2x^2}{[1-(1-\gamma)x]^3} \\
&= \left(\frac{p\gamma^2(1-p)M_1x}{\gamma-p} + \frac{(1-p)^3\gamma^3M_2x^2}{(\gamma-p)^2}\right)\sum_{n=0}^{\infty}[(1-p)x]^n \\
&\quad - \left(\frac{p^2\gamma(1-\gamma)M_1x}{\gamma-p} + \frac{(1-p)^2(1-\gamma)\gamma^3M_2x^2}{(\gamma-p)^2}\right)\sum_{n=0}^{\infty}[(1-\gamma)x]^n \\
&\quad - \frac{p\gamma^2(1-p)(1-\gamma)M_2x^2}{\gamma-p}\sum_{n=1}^{\infty}n[(1-\gamma)x]^{n-1} + \frac{p\gamma^2M_2x^2}{2}\sum_{n=2}^{\infty}n(n-1)[(1-\gamma)x]^{n-2}
\end{aligned} \tag{59}$$

Taking the coefficient before x^n, we obtain that

$$\begin{aligned}
&\Pr\{\Delta_{Ber/Geo/1/2} = n\} \\
&= \left(\frac{p\gamma^2(1-p)M_1}{\gamma-p}(1-p)^{n-1} + \frac{(1-p)^3\gamma^3M_2}{(\gamma-p)^2}(1-p)^{n-2}\right) \\
&\quad - \left(\frac{p^2\gamma(1-\gamma)M_1}{\gamma-p}(1-\gamma)^{n-1} + \frac{(1-p)^2(1-\gamma)\gamma^3M_2}{(\gamma-p)^2}(1-\gamma)^{n-2}\right) \\
&\quad - \frac{p\gamma^2(1-p)(1-\gamma)M_2}{\gamma-p}(n-1)(1-\gamma)^{n-2} + \frac{p\gamma^2M_2}{2}n(n-1)(1-\gamma)^{n-2} \\
&= \frac{p\gamma^2(1-p)M_1}{(\gamma-p)^2}(1-p)^n - \frac{(\gamma-p^2)(1-p)\gamma^2M_2}{(\gamma-p)^2}(1-\gamma)^{n-1} \\
&\quad - \frac{p\gamma^2(1-p)M_2}{\gamma-p}(n-1)(1-\gamma)^{n-1} + \frac{p\gamma^2M_2}{2}n(n-1)(1-\gamma)^{n-2}
\end{aligned} \tag{60}$$

this gives the stationary distribution (45) for the system without packet preemption.

On the other hand, when η is equal to 1, the system has full packet preemption. Factoring the PGF in equation (42), we can also determine the stationary distribution of the AoI $\Delta_{Ber/Geo/1/2^*}$ by taking the coefficients of terms x^n for each $n \geq 1$. We give the explicit decomposition below, and from which the distribution of AoI for the system with packet preemption is obtained explicitly.

From equation (42), we show that

$$
\begin{aligned}
&H_{PP}(x)|_{\eta=1} \\
&= \frac{p\gamma M_1 x}{\gamma-p}\left(\frac{(1-p)\gamma}{1-(1-p)x}-\frac{p(1-\gamma)}{1-(1-\gamma)x}\right)+\frac{(1-p)^2\gamma^2 M_2 x^2}{\gamma-p}\left(\frac{\gamma^2+p(1-\gamma)(p+\gamma)}{\gamma^2[1-(1-p)x]}\right. \\
&\quad\left.-\frac{(1-\gamma)[\gamma^2+p(1-\gamma)(p+\gamma)]}{\gamma^2[1-(1-p)(1-\gamma)x]}-\frac{p(1-\gamma)(p+\gamma-p\gamma)}{\gamma[1-(1-p)(1-\gamma)x]^2}\right) \\
&\quad-\frac{p\gamma(1-p)(1-\gamma)M_2x^2}{\gamma-p}\left(\frac{p+2(1-p)\gamma}{p[1-(1-\gamma)x]}-\frac{(1-p)[p+2(1-p)\gamma]}{p[1-(1-p)(1-\gamma)x]}\right. \\
&\quad\left.-\frac{(1-p)(p+\gamma-p\gamma)}{[1-(1-p)(1-\gamma)x]^2}\right)+p\gamma M_2x^2\left(\frac{A}{1-(1-\gamma)x}+\frac{B}{[1-(1-\gamma)x]^2}\right. \\
&\quad\left.+\frac{C}{1-(1-p)(1-\gamma)x}+\frac{D}{[1-(1-p)(1-\gamma)x]^2}\right)
\end{aligned} \tag{61}
$$

in which we determine

$$A=\frac{2-p}{p^3}\left((1-p)^2\gamma-\frac{2(1-p)[\gamma+p^2(1-\gamma)]}{2-p}\right) \tag{62}$$

$$C=-\frac{(1-p)(2-p)}{p^3}\left((1-p)^2\gamma-\frac{2(1-p)[\gamma+p^2(1-\gamma)]}{2-p}\right) \tag{63}$$

$$D=-\frac{p^2(1-p)\cdot A+(1-p)^2[\gamma+p^2(1-\gamma)]}{p(2-p)} \tag{64}$$

and

$$B=[\gamma+p^2(1-\gamma)]-A-C-D \tag{65}$$

Obtaining the second and the third row of (61) is not hard, while for the last row, we give some derivation details in appendix C. Following the same procedures of obtaining (60), according to equation (61), the probability that stationary AoI equals each n is determined by the coefficient of the term x^n.

$$
\begin{aligned}
&H_{PP}(x)|_{\eta=1} \\
&= \frac{p\gamma M_1}{\gamma-p}\left((1-p)\gamma(1-p)^{n-1}-p(1-\gamma)(1-\gamma)^{n-1}\right)+\frac{(1-p)^2\gamma^2M_2}{\gamma-p} \\
&\times\left(\frac{\gamma^2+p(1-\gamma)(p+\gamma)}{\gamma^2}(1-p)^{n-2}-\frac{(1-\gamma)[\gamma^2+p(1-\gamma)(p+\gamma)]}{\gamma^2}[(1-p)(1-\gamma)]^{n-2}\right. \\
&\left.-\frac{p(1-\gamma)(p+\gamma-p\gamma)}{\gamma}(n-1)[(1-p)(1-\gamma)]^{n-2}\right)-\frac{p\gamma(1-p)(1-\gamma)M_2}{\gamma-p} \\
&\times\left(\frac{p+2(1-p)\gamma}{p}(1-\gamma)^{n-2}-\frac{(1-p)[p+2(1-p)\gamma]}{p}[(1-p)(1-\gamma)]^{n-2}\right. \\
&\left.-(1-p)(p+\gamma-p\gamma)(n-1)[(1-p)(1-\gamma)]^{n-2}\right)+p\gamma M_2\Big(A(1-\gamma)^{n-2} \\
&+B(n-1)(1-\gamma)^{n-2}+C[(1-p)(1-\gamma)]^{n-2}+D(n-1)[(1-p)(1-\gamma)]^{n-2}\Big) \\
&= \frac{p\gamma M_1}{\gamma-p}\left(\gamma(1-p)^n-p(1-\gamma)^n\right) \\
&+\frac{(1-p)[\gamma^2+p(1-\gamma)(p+\gamma)]M_2}{\gamma-p}\left((1-p)^{n-1}-[(1-p)(1-\gamma)]^{n-1}\right) \\
&-\frac{(1-p)\gamma[p+2(1-p)\gamma]M_2}{\gamma-p}\left((1-\gamma)^{n-1}-[(1-p)(1-\gamma)]^{n-1}\right)+p\gamma M_2\Big(A(1-\gamma)^{n-2} \\
&+B(n-1)(1-\gamma)^{n-2}+C[(1-p)(1-\gamma)]^{n-2}+D(n-1)[(1-p)(1-\gamma)]^{n-2}\Big)
\end{aligned} \tag{66}
$$

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which determines the distribution of AoI $\Delta_{Ber/Geo/1/2^*}$.

So far, in equations (52), (56), (60), and (66), we have obtained all the results in Theorem 2, thus the proof is completed. □

Actually, we have obtained the explicit expression of AoI's distribution for the system with packet preemption in our early work [59]. Earlier in this paper, we explain that solving the stationary equations is feasible for the easy situations, but cannot be generalized when system structure or queue models become complex. In paper [59], we focused on the discrete time system with three queues, i.e., the $Ber/Geo/1/1$, $Ber/Geo/1/2$, and $Ber/Geo/1/2^*$, and named them "*discrete packet management strategies*". There, we determined the AoI's stationary distribution for each system, and all the cases are dealt with by solving the stationary equations directly. Although the calculations are long, even tedious, these methods still have great significance, especially when the general status updating system is considered where the packet arrival process or the packet service process is arbitrary. It is these methods that the analysis of discrete AoI can break through the limitation of memoryless property which is imposed on the packet arrival and packet service processes in the SHS approach.

In paper [9], based on graphical arguments of the age process, the authors determined the average continuous AoI for the system with $M/M/1/2$ and $M/M/1/2^*$ queue as

$$\Delta_{M/M/1/2} = \frac{1}{\mu}\left(1 + \frac{1}{\rho} + \frac{2\rho^2}{1+\rho+\rho^2}\right) \tag{67}$$

and

$$\Delta_{M/M/1/2^*} = \frac{1}{\mu}\left(1 + \frac{1}{\rho} + \frac{\rho^2(1+3\rho+\rho^2)}{(1+\rho+\rho^2)(1+\rho)^2}\right) \tag{68}$$

In addition, in previous work [54] we have proved that the mean of AoI for bufferless discrete time status updating system is equal to

$$\overline{\Delta}_{Ber/Geo/1/1} = \frac{1}{\gamma}\left((1-\gamma) + \frac{1}{\rho_d} + \frac{\rho_d}{1/(1-\gamma)+\rho_d}\right) \tag{69}$$

while the corresponding continuous system with $M/M/1/1$ queue has average AoI

$$\overline{\Delta}_{M/M/1/1} = \frac{1}{\mu}\left(1 + \frac{1}{\rho} + \frac{\rho}{1+\rho}\right) \tag{70}$$

which was also given in [9].

We list the equations (43), (44), and (67)-(70) in Table 3, notice that this table has been given previously in Table 1 except for the last row, which gives the average continuous and discrete AoI for infinite size status updating system. The mean of discrete AoI $\Delta_{Ber/Geo/1/\infty}$ was obtained recently in our work [58]. It is observed that apart from some additional product factors, the expressions of discrete AoI's mean for the system with Bernoulli packet arrivals and geometric service time are identical to that of continuous system's average AoI, who uses the Poisson-Exponential assumptions. So far, we have obtained enough evidence to propose following relationship between the mean of discrete and continuous AoIs:

$$\mu \cdot \overline{\Delta}_{M/M/1/c} = \gamma \cdot \overline{\Delta}_{Ber/Geo/1/c}\big|_{\gamma=0} \text{ then replacing } \rho_d \text{ by } \rho \tag{71}$$

It is interesting and meaningful to verify the correspondence (71) by calculating the average AoI for more continuous time and discrete time systems. For example, determining the mean of AoI assuming $M/M/1/c$ queue is used in continuous time system and for the discrete time systems with general $Ber/Geo/1/c$ queues, where system's size c is larger than 2.

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Table 3. Some formulas of average continuous and average discrete age of information

Average continuous and average discrete AoIs
$\overline{\Delta}_{M/M/1/1} = \frac{1}{\mu}\left(1 + \frac{1}{\rho} + \frac{\rho}{1+\rho}\right)$
$\overline{\Delta}_{Ber/Geo/1/1} = \frac{1}{\gamma}\left((1-\gamma) + \frac{1}{\rho_d} + \frac{\rho_d}{1/(1-\gamma)+\rho_d}\right)$
$\overline{\Delta}_{M/M/1/2} = \frac{1}{\mu}\left(1 + \frac{1}{\rho} + \frac{2\rho^2}{1+\rho+\rho^2}\right)$
$\overline{\Delta}_{Ber/Geo/1/2} = \frac{1}{\gamma}\left((1-\gamma) + \frac{1}{\rho_d} + \frac{2\rho_d^2(1-\gamma)(1-\gamma/2)}{1+\rho_d(1-2\gamma)+\rho_d^2(1-\gamma)^2}\right)$
$\overline{\Delta}_{M/M/1/2^*} = \frac{1}{\mu}\left(1 + \frac{1}{\rho} + \frac{\rho^2(1+3\rho+\rho^2)}{(1+\rho+\rho^2)(1+\rho)^2}\right)$
$\overline{\Delta}_{Ber/Geo/1/2^*} = \frac{1}{\gamma}\left((1-\gamma) + \frac{1}{\rho_d} + \frac{\rho_d^2(1-\gamma)[1+3\rho_d(1-\gamma)+\rho_d^2(1-\gamma)(1-2\gamma)]}{[1+\rho_d(1-2\gamma)+\rho_d^2(1-\gamma)^2][1+\rho_d(1-\gamma)]^2}\right)$
$\overline{\Delta}_{M/M/1/\infty} = \frac{1}{\mu}\left(1 + \frac{1}{\rho} + \frac{\rho^2}{1-\rho}\right)$
$\overline{\Delta}_{Ber/Geo/1/\infty} = \frac{1}{\gamma}\left((1-\gamma) + \frac{1}{\rho_d} + \frac{\rho_d^2(1-\gamma)}{1-\rho_d}\right)$

5. Numerical Simulation

We provide the numerical results in this Section. For general preemption probability, in the first two pictures of Figure 3, we illustrate the relationships between the average AoI $\overline{\Delta}_{Ber/Geo/1/2^*/\eta}$ and the packet preemption probability η, and the traffic load ρ_d, respectively. The mean of three discrete AoIs including $\overline{\Delta}_{Ber/Geo/1/1}$, $\overline{\Delta}_{Ber/Geo/1/2}$, and $\overline{\Delta}_{Ber/Geo/1/2^*}$ are plotted in Figure 3(c). For comparison, we also provide the numerical simulations of corresponding average continuous AoIs. At last, for three discrete AoIs, we depict their distribution curves and the cumulative probabilities in Figure 4. Notice that in our work [54], the distribution of AoI for the bufferless system was obtained as

$$\Pr\{\Delta_{Ber/Geo/1/1} = n\} = \frac{p(1-p)\gamma^3[(1-p)^n - (1-\gamma)^n]}{(p+\gamma-p\gamma)(\gamma-p)^2} - \frac{(p\gamma)^2 n(1-\gamma)^n}{(p+\gamma-p\gamma)(\gamma-p)} \quad (72)$$

For three different traffic load ρ_d, we first draw the graphs between average AoI $\overline{\Delta}_{Ber/Geo/1/2^*/\eta}$ and the preemption probability η. It is understandable that replacing the packet in buffer with a fresher one can decrease the average AoI at the destination, and the numerical results in Figure 3(a) show that this trend is consistent as the preemption probability becomes large, that is, the mean of AoI is decreasing monotonically when η increases. We mark the values at two extreme points where $\eta = 0$ and $\eta = 1$, which gives the average AoI $\overline{\Delta}_{Ber/Geo/1/2}$ and $\overline{\Delta}_{Ber/Geo/1/2^*}$. Notice that the closer to $\eta = 0$, the behavior of the system is more similar to that of a system using $Ber/Geo/1/2$ queues, and when η gradually gets to 1, a status updating system with $Ber/Geo/1/2^*$ queue is finally obtained. The three curves in Figure 3(a) also show that as traffic load ρ_d increases from 0.4 to 0.45, average AoI of the system with probabilistic packet preemption is reduced, thus the timeliness performance is improved.

In picture 3(b), for three settings of preemption probabilities, i.e., $\eta = 0$, $\eta = 0.5$ and $\eta = 1$, the relationships of average AoI versus traffic intensity ρ_d are illustrated. The topmost curve gives the average AoI of the system without packet preemption because for the case of $\eta = 0$, the average AoI reduces to $\overline{\Delta}_{Ber/Geo/1/2}$. On the other hand, the curve at the bottom corresponds to AoI's mean of the system who has full packet preemption. In order to make the difference among these graphs more significant, we draw the results in the range $\rho_d \geq 0.45$. Three curves in Figure (3b) clearly show that the timeliness of system with complete packet preemption is the best, since when η is set to 1, system's average AoI

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is the lowest. Since the results in Figure 3(a) shows that the average AoI is monotonically decreasing when η increases, thus the graphs of AoI's mean for a system with probabilistic packet preemption is located between the blue and the black lines in Figure 3(b), such as the red line which denotes the average AoI $\overline{\Delta}_{Ber/Geo/1/2^*/0.5}$. In addition, the gap between these curves are not significant for small ρ_d's but becomes large as ρ_d increases.

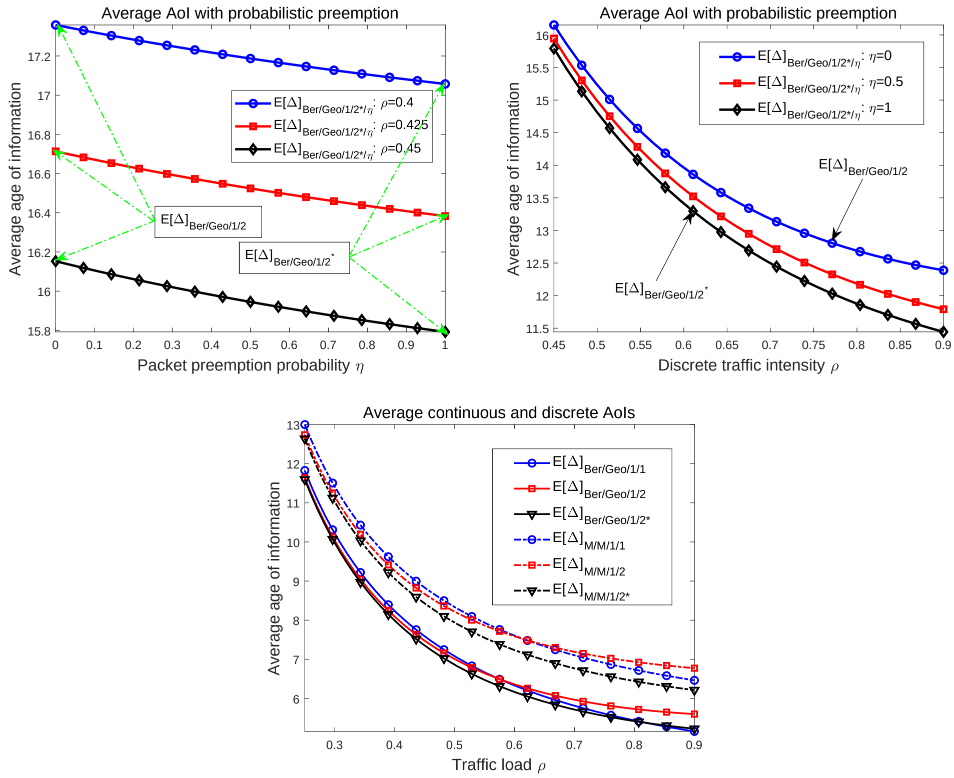

Figure 3. (**a**) Average AoI versus preemption probability η (different traffic load). (**b**) Average AoI versus traffic load ρ_d (different preemption intensity). (**c**) Comparisons of average discrete AoI and average continuous AoI.

From $\rho_d = 0.15$ to 0.9, we depict both the average discrete AoIs and the corresponding continuous average AoIs in Figure 3(c) for bufferless system and size 2 status updating system with and without preemption. Continuous AoIs are denoted by dashed lines and we use solid lines to represent the discrete AoIs. First of all, all the curves are decreasing as ρ_d becomes large and the gaps among them are gradually apparent. For three continuous AoIs, it is observed that the average AoI $\overline{\Delta}_{M/M/1/2^*}$ is the lowest in all the range of traffic load ρ. For other two status updating systems, it is found that the system with $M/M/1/2$ queue has lower average AoI when ρ takes small values, while for high ρ the average AoI of the bufferless system is smaller, thus the timeliness is better. These results are the same for the graphs of discrete AoIs. Notice that, when discrete traffic intensity ρ_d is extremely large (near 0.9), the numerical results show that the average AoI $\overline{\Delta}_{Ber/Geo/1/1}$ can even smaller than $\overline{\Delta}_{Ber/Geo/1/2^*}$.

In Figure 3(c), both of continuous and discrete average AoIs are monotonically decreasing in all the range of ρ_d, however, the monotonicity of the curve between average AoI and ρ_d can only be maintained for small-size status updating systems. It has known that the average AoI of infinite size system, i.e., $\overline{\Delta}_{M/M/1/\infty}$ is not monotonic when the traffic load varies from 0 to 1. Thus, for size c status updating system with Bernoulli packet arrivals and geometrically distributed service time, there must be a *critical size* c^* such that when $c < c^*$, the mean of system's AoI $\overline{\Delta}_{Ber/Geo/1/c}$ is monotonically decreasing as ρ_d tends to 1. While for those cases where system size $c \geq c^*$, the curve has a valley and an optimal ρ_d exists at which the average AoI is minimized. Similarly, for continuous

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average AoI $\overline{\Delta}_{M/M/1/c}$ of the system with general size c, also a c^* exists so that $\overline{\Delta}_{M/M/1/c}$ is always decreasing when $c < c^*$ and the graph of $\overline{\Delta}_{M/M/1/c}$ first falls and then raises for those c's where $c \geq c^*$. In addition, from the alternation of $\overline{\Delta}_{M/M/1/1}$ and $\overline{\Delta}_{M/M/1/2}$, also of $\overline{\Delta}_{Ber/Geo/1/1}$ and $\overline{\Delta}_{Ber/Geo/1/2}$, we can infer that as the function of c, the graphs of $\overline{\Delta}_{M/M/1/c}$ and $\overline{\Delta}_{Ber/Geo/1/c}$ are not monotonic.

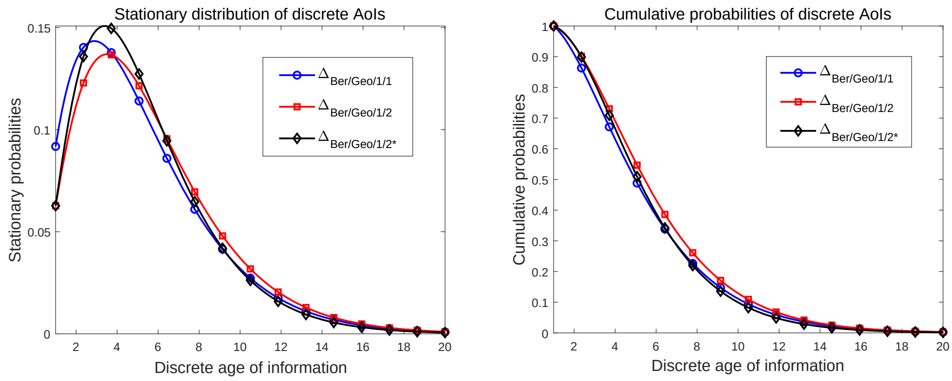

Figure 4. (**a**) Stationary distributions of discrete AoI for bufferless system and the system with and without packet preemption. (**b**) The cumulative probabilities of three discrete AoIs.

At last, the distribution curves and cumulative probabilities of three discrete AoIs are depicted in Figure 4, in which we set a relatively large ρ_d to make the difference between them clear. On the whole, these curves are similar. In Figure (4a), from the distributions of AoI $\Delta_{Ber/Geo/1/1}$ to that of $\Delta_{Ber/Geo/1/2}$, the peak of the curve decreases and the point at which the peak stationary probability is achieved moves slightly to the right. As the AoI becomes large, the distribution curve of system with $Ber/Geo/1/1$ queue drops more sharply. For the distribution corresponding to $\Delta_{Ber/Geo/1/2^*}$, it has the largest peak value among all of three discrete AoIs and the descent speed is the fastest when the value of AoI is large. In addition, it seems that this maximal probability is taken at the same discrete AoI as that of the distribution of $\Delta_{Ber/Geo/1/2}$. We also provide the cumulative probabilities of three discrete AoIs in Figure (4b).

6. Conclusions

In this paper, we consider the stationary AoI of a size 2 status updating system where the packet waiting in buffer can be preempted by fresher packets with the given probability η. We show that this phenomenon may occur in the energy-harvest (EH) nodes of wireless sensor networks where the charging process is stochastic. We constitute a three-dimensional age process and derive the general expression of system's average AoI using the PGF method. Let $\eta = 0$ and $\eta = 1$, the mean of two discrete AoIs $\overline{\Delta}_{Ber/Geo/1/2}$ and $\overline{\Delta}_{Ber/Geo/1/2}$ are determined, and the exact distribution expressions of both AoIs are also obtained by writing the PGF as the power series.

We propose the idea and methods for the analysis of discrete AoIs, that is constituting multiple-dimensional age process and applying the PGF method. Much more introductions and explanations are given to exhibit the usage of the idea and methods to more discrete time status updating systems. With this paper, we have shown how the AoI of basic discrete system is characterized, while for further work, we will focus on the age analysis of system with more general structure, such as system with multi-sources and system having multi-hop packet transmission. As one part of the AoI theory, we believe that the research of discrete AoI deserves more attention.

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Appendix A. Proof of Theorem 1

In this appendix, we derive all the results in Lemma 1 from the stationary equations (18).

Define that

$$M_1 = \sum_{n=1}^{\infty} \pi_{(n,0,0)}, \; M_2 = \sum_{m=1}^{\infty} \sum_{n=m+1}^{\infty} \pi_{(n,m,0)}, \text{ and } M_3 = \sum_{l=1}^{\infty} \sum_{m=l+1}^{\infty} \sum_{n=m+1}^{\infty} \pi_{(n,m,l)},$$

these three numbers are determined at the first place.

According to the last two rows of (18), we have

$$\begin{aligned} M_1 &= \pi_{(1,0,0)} + \sum_{n=2}^{\infty} \pi_{(n,0,0)} \\ &= \left(\sum_{n=1}^{\infty} \pi_{(n,0,0)} \right) p\gamma + \sum_{n=2}^{\infty} \left\{ \pi_{(n-1,0,0)}(1-p) + \sum_{k=n}^{\infty} \pi_{(k,n-1,0)}(1-p)\gamma \right\} \\ &= p\gamma M_1 + (1-p)M_1 + \sum_{\tilde{m}=1}^{\infty} \sum_{\tilde{n}=\tilde{m}+1}^{\infty} \pi_{(\tilde{n},\tilde{m},0)}(1-p)\gamma \qquad \text{(A1)} \\ &= p\gamma M_1 + (1-p)M_1 + (1-p)\gamma M_2 \qquad \text{(A2)} \end{aligned}$$

where in (A1), we have used the substitutions $k = \tilde{n}$ and $n - 1 = \tilde{m}$. From equation (A2), we obtain the first relation

$$p(1-\gamma)M_1 = (1-p)\gamma M_2 \qquad \text{(A3)}$$

Next, we deal with the number M_3 as follows.

$$\begin{aligned} M_3 &= \sum_{l=1}^{\infty} \sum_{m=l+1}^{\infty} \sum_{n=m+1}^{\infty} \pi_{(n,m,l)} \\ &= \sum_{m=2}^{\infty} \sum_{n=m+1}^{\infty} \pi_{(n,m,1)} + \sum_{l=2}^{\infty} \sum_{m=l+1}^{\infty} \sum_{n=m+1}^{\infty} \pi_{(n,m,l)} \qquad \text{(A4)} \end{aligned}$$

Using the first row of (18), the latter sum in equation (A4) equals

$$\begin{aligned} &\sum_{l=2}^{\infty} \sum_{m=l+1}^{\infty} \sum_{n=m+1}^{\infty} \pi_{(n-1,m-1,l-1)}(1-\gamma)(1-p\eta) \\ &= \sum_{l=1}^{\infty} \sum_{m=l+1}^{\infty} \sum_{n=m+1}^{\infty} \pi_{(n,m,l)}(1-\gamma)(1-p\eta) \\ &= (1-\gamma)(1-p\eta)M_3 \qquad \text{(A5)} \end{aligned}$$

and from the second and the third row of (18), the first part of (A4) is calculated as

$$\begin{aligned} &\sum_{n=3}^{\infty} \pi_{(n,2,1)} + \sum_{m=3}^{\infty} \sum_{n=m+1}^{\infty} \pi_{(n,m,1)} \\ &= \sum_{n=3}^{\infty} \pi_{(n-1,1,0)} p(1-\gamma) \\ &\quad + \sum_{m=3}^{\infty} \sum_{n=m+1}^{\infty} \left\{ \pi_{(n-1,m-1,0)} p(1-\gamma) + \sum_{j=1}^{m-2} \pi_{(n-1,m-1,j)} p(1-\gamma)\eta \right\} \\ &= \sum_{n=2}^{\infty} \pi_{(n,1,0)} p(1-\gamma) \\ &\quad + \sum_{m=2}^{\infty} \sum_{n=m+1}^{\infty} \pi_{(n,m,0)} p(1-\gamma) + \sum_{m=3}^{\infty} \sum_{n=m+1}^{\infty} \sum_{j=1}^{m-2} \pi_{(n-1,m-1,j)} p(1-\gamma)\eta \\ &= \sum_{m=1}^{\infty} \sum_{n=m+1}^{\infty} \pi_{(n,m,0)} p(1-\gamma) + \sum_{\tilde{m}=2}^{\infty} \sum_{\tilde{n}=\tilde{m}+1}^{\infty} \sum_{\tilde{l}=1}^{\tilde{m}-1} \pi_{(\tilde{n},\tilde{m},\tilde{l})} p(1-\gamma)\eta \qquad \text{(A6)} \\ &= p(1-\gamma)M_2 + p(1-\gamma)\eta M_3 \qquad \text{(A7)} \end{aligned}$$

where in (A6), we let $n - 1 = \tilde{n}$, $m - 1 = \tilde{m}$, and $j = \tilde{l}$. Equations (A4), (A5), and (A7) together give

$$p(1-\gamma)M_2 = \gamma M_3 \qquad \text{(A8)}$$

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Since the sum of all the stationary probabilities equals 1, thus we have

$$M_1 + M_2 + M_3 = 1 \tag{A9}$$

Combining equations (A3), (A8), and (A9), we can solve that

$$M_1 = \frac{(1-p)\gamma^2}{(p+\gamma-2p\gamma)\gamma + p^2(1-\gamma)^2} \tag{A10}$$

$$M_2 = \frac{p\gamma(1-\gamma)}{(p+\gamma-2p\gamma)\gamma + p^2(1-\gamma)^2} \tag{A11}$$

$$M_3 = \frac{p^2(1-\gamma)^2}{(p+\gamma-2p\gamma)\gamma + p^2(1-\gamma)^2} \tag{A12}$$

We mention that from the fourth, the fifth, and the sixth equation in (18), another relation can also be obtained for the second number M_2, which is given directly as

$$M_2 = p(1-\gamma)M_1 + p\gamma M_2 + p\gamma\eta M_3 + (1-p)(1-\gamma)M_2 + (1-p\eta)\gamma M_3 \tag{A13}$$

which is reduced to (A8) when eliminating M_1 using the relation (A3).

Then, the relationships between functions $h_i(x)$, $1 \le i \le 3$ and $h_2^{(m)}(x)$ are determined through the similar procedures. First of all, we see that

$$\begin{aligned}
h_1(x) &= \sum_{n=1}^{\infty} x^n \pi_{(n,0,0)} \\
&= x\pi_{(1,0,0)} + \sum_{n=2}^{\infty} x^n \left\{ \pi_{(n-1,0,0)}(1-p) + \sum_{k=n}^{\infty} \pi_{(k,n-1,0)}(1-p)\gamma \right\} \\
&= p\gamma M_1 x + \sum_{n=1}^{\infty} x^{n+1}\pi_{(n,0,0)}(1-p) + \sum_{n=2}^{\infty} x^n \sum_{k=n}^{\infty} \pi_{(k,n-1,0)}(1-p)\gamma \\
&= p\gamma M_1 x + (1-p)xh_1(x) + \sum_{\tilde{m}=1}^{\infty} x^{\tilde{m}+1} \sum_{\tilde{n}=\tilde{m}+1}^{\infty} \pi_{(\tilde{n},\tilde{m},0)}(1-p)\gamma \qquad \text{(A14)} \\
&= p\gamma M_1 x + (1-p)xh_1(x) + (1-p)\gamma x h_2^{(m)}(x) \qquad \text{(A15)}
\end{aligned}$$

in which we denote $k = \tilde{n}$ and $n - 1 = \tilde{m}$.

From (A15), we obtain

$$h_1(x) = \frac{p\gamma M_1 x}{1-(1-p)x} + \frac{(1-p)\gamma x}{1-(1-p)x} h_2^{(m)}(x) \tag{A16}$$

Using stationary equations (18), we determine function $h_2(x)$ in the following.

$$\begin{aligned}
h_2(x) &= \sum_{m=1}^{\infty} \sum_{n=m+1}^{\infty} x^n \pi_{(n,m,0)} \\
&= \sum_{n=2}^{\infty} x^n \pi_{(n,1,0)} + \sum_{m=2}^{\infty} \sum_{n=m+1}^{\infty} x^n \\
&\quad \times \left\{ \pi_{(n-1,m-1,0)}(1-p)(1-\gamma) + \sum_{k=n}^{\infty} \pi_{(k,n-1,m-1)}(1-p\eta)\gamma \right\} \\
&= x^2\pi_{(2,1,0)} + \sum_{n=3}^{\infty} x^n \left\{ \pi_{(n-1,0,0)} p(1-\gamma) + \sum_{k=n}^{\infty} \pi_{(k,n-1,0)} p\gamma \right. \\
&\quad \left. + \sum_{k=n}^{\infty} \sum_{j=1}^{n-2} \pi_{(k,n-1,j)} p\gamma\eta \right\} + \sum_{m=1}^{\infty} \sum_{n=m+1}^{\infty} x^{n+1} \pi_{(n,m,0)}(1-p)(1-\gamma) \\
&\quad + \sum_{m=2}^{\infty} \sum_{n=m+1}^{\infty} x^n \sum_{k=n}^{\infty} \pi_{(k,n-1,m-1)}(1-p\eta)\gamma \\
&= x^2 \left\{ \pi_{(1,0,0)} p(1-\gamma) + \sum_{k=2}^{\infty} \pi_{(k,1,0)} p\gamma \right\} + \sum_{n=2}^{\infty} x^{n+1} \pi_{(n,0,0)} p(1-\gamma) \\
&\quad + \sum_{n=3}^{\infty} x^n \sum_{k=n}^{\infty} \pi_{(k,n-1,0)} p\gamma + \sum_{n=3}^{\infty} x^n \sum_{k=n}^{\infty} \sum_{j=1}^{n-2} \pi_{(k,n-1,j)} p\gamma\eta \\
&\quad + (1-p)(1-\gamma)xh_2(x) + (1-p\eta)\gamma x h_3^{(m)}(x) \qquad \text{(A17)}
\end{aligned}$$

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Let $k = \tilde{n}$, $n-1 = \tilde{m}$, and $m-1 = \tilde{l}$, we have

$$\begin{aligned}
&\sum_{m=2}^{\infty}\sum_{n=m+1}^{\infty} x^n \sum_{k=n}^{\infty} \pi_{(k,n-1,m-1)}(1-p\eta)\gamma \\
&= \sum_{\tilde{l}=1}^{\infty}\sum_{\tilde{m}=\tilde{l}+1}^{\infty} x^{\tilde{m}+1} \sum_{\tilde{n}=\tilde{m}+1}^{\infty} \pi_{(\tilde{n},\tilde{m},\tilde{l})}(1-p\eta)\gamma \\
&= (1-p\eta)\gamma x h_3^{(m)}(x) \qquad \text{(A18)}
\end{aligned}$$

where we define

$$h_3^{(m)}(x) = \sum_{l=1}^{\infty}\sum_{m=l+1}^{\infty}\sum_{n=m+1}^{\infty} x^m \pi_{(n,m,l)} \qquad \text{(A19)}$$

and the last term in equation (A17) is obtained.

Continue the calculation of (A17), we obtain that

$$\begin{aligned}
h_2(x) = {} & p(1-\gamma)xh_1(x) + p\gamma x h_2^{(m)}(x) + \sum_{\tilde{m}=2}^{\infty} x^{\tilde{m}+1} \sum_{\tilde{n}=\tilde{m}+1}^{\infty} \sum_{\tilde{l}=1}^{\tilde{m}-1} \pi_{(\tilde{n},\tilde{m},\tilde{l})} p\gamma\eta \\
& + (1-p)(1-\gamma)xh_2(x) + (1-p\eta)\gamma x h_3^{(m)}(x) \qquad \text{(A20)}
\end{aligned}$$

in (A20) we have used the substitutions $k = \tilde{n}$, $n-1 = \tilde{m}$, and $j = \tilde{l}$.

Except the factor $p\gamma\eta x$, the sum in (A20) is equal to

$$\begin{aligned}
& x^2 \sum_{n=3}^{\infty} \pi_{(n,2,1)} + x^3 \sum_{n=4}^{\infty} \left[\pi_{(n,3,1)} + \pi_{(n,3,2)}\right] + x^4 \sum_{n=5}^{\infty} \left[\pi_{(n,4,1)} + \pi_{(n,4,2)} + \pi_{(n,4,3)}\right] + \cdots \\
&= \sum_{m=2}^{\infty} x^m \sum_{n=m+1}^{\infty} \pi_{(n,m,1)} + \sum_{m=3}^{\infty} x^m \sum_{n=m+1}^{\infty} \pi_{(n,m,2)} + \sum_{m=4}^{\infty} x^m \sum_{n=m+1}^{\infty} \pi_{(n,m,3)} + \cdots \\
&= \sum_{l=1}^{\infty}\sum_{m=l+1}^{\infty} x^m \sum_{n=m+1}^{\infty} \pi_{(n,m,l)} \\
&= h_3^{(m)}(x) \qquad \text{(A21)}
\end{aligned}$$

Substituting the result (A21) into equation (A20) and merging the same terms gives

$$h_2(x)[1-(1-p)(1-\gamma)x] = p(1-\gamma)xh_1(x) + p\gamma x h_2^{(m)}(x) + \gamma x h_3^{(m)}(x) \qquad \text{(A22)}$$

We compute $h_3^{(m)}(x)$ right now while determine the other function $h_2^{(m)}(x)$ in the end. As before, from the equations in (18), we have that

$$\begin{aligned}
h_3^{(m)}(x) &= \sum_{l=1}^{\infty}\sum_{m=l+1}^{\infty}\sum_{n=m+1}^{\infty} x^m \pi_{(n,m,l)} \\
&= \sum_{m=2}^{\infty}\sum_{n=m+1}^{\infty} x^m \pi_{(n,m,1)} + \sum_{l=2}^{\infty}\sum_{m=l+1}^{\infty}\sum_{n=m+1}^{\infty} x^m \pi_{(n-1,m-1,l-1)}(1-\gamma)(1-p\eta) \\
&= \sum_{n=3}^{\infty} x^2 \pi_{(n,2,1)} + \sum_{m=3}^{\infty}\sum_{n=m+1}^{\infty} x^m \Big\{\pi_{(n-1,m-1,0)} p(1-\gamma) \\
&\qquad + \sum_{j=1}^{m-2} \pi_{(n-1,m-1,j)} p(1-\gamma)\eta\Big\} + (1-\gamma)(1-p\eta)xh_3^{(m)}(x) \\
&= \sum_{n=3}^{\infty} x^2 \pi_{(n-1,1,0)} p(1-\gamma) + \sum_{m=2}^{\infty}\sum_{n=m+1}^{\infty} x^{m+1} \pi_{(n,m,0)} p(1-\gamma) \\
&\qquad + \sum_{m=3}^{\infty}\sum_{n=m+1}^{\infty} x^m \sum_{j=1}^{m-2} \pi_{(n-1,m-1,j)} p(1-\gamma)\eta + (1-\gamma)(1-p\eta)xh_3^{(m)}(x) \\
&= p(1-\gamma)xh_2^{(m)}(x) + p(1-\gamma)\eta x h_3^{(m)}(x) + (1-\gamma)(1-p\eta)xh_3^{(m)}(x) \\
&= p(1-\gamma)xh_2^{(m)}(x) + (1-\gamma)xh_3^{(m)}(x) \qquad \text{(A23)}
\end{aligned}$$

which shows that

$$h_3^{(m)}(x) = \frac{p(1-\gamma)x}{1-(1-\gamma)x} h_2^{(m)}(x) \qquad \text{(A24)}$$

Combining equations (A22) and (A24) yields the relation

$$h_2(x) = \frac{p(1-\gamma)x}{1-(1-p)(1-\gamma)x} h_1(x) + \frac{p\gamma x}{[1-(1-p)(1-\gamma)x][1-(1-\gamma)x]} h_2^{(m)}(x) \qquad \text{(A25)}$$

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Now, we deal with function $h_3(x)$. Similarly to the process of obtaining (A23), we have

$$\begin{aligned} h_3(x) &= \sum_{m=2}^{\infty}\sum_{n=m+1}^{\infty} x^n \pi_{(n,m,1)} + \sum_{l=2}^{\infty}\sum_{m=l+1}^{\infty}\sum_{n=m+1}^{\infty} x^n \pi_{(n-1,m-1,l-1)}(1-\gamma)(1-p\eta) \\ &= \sum_{n=3}^{\infty} x^n \pi_{(n,2,1)} + \sum_{m=3}^{\infty}\sum_{n=m+1}^{\infty} x^n \Big\{ \pi_{(n-1,m-1,0)} p(1-\gamma) \\ &\quad + \sum_{j=1}^{m-2} \pi_{(n-1,m-1,j)} p(1-\gamma)\eta \Big\} + (1-\gamma)(1-p\eta)xh_3(x) \\ &= \sum_{n=3}^{\infty} x^n \pi_{(n-1,1,0)} p(1-\gamma) + \sum_{m=2}^{\infty}\sum_{n=m+1}^{\infty} x^{n+1} \pi_{(n,m,0)} p(1-\gamma) \\ &\quad + \sum_{m=3}^{\infty}\sum_{n=m+1}^{\infty} x^n \sum_{j=1}^{m-2} \pi_{(n-1,m-1,j)} p(1-\gamma)\eta + (1-\gamma)(1-p\eta)xh_3(x) \\ &= p(1-\gamma)xh_2(x) + p(1-\gamma)\eta x h_3(x) + (1-\gamma)(1-p\eta)xh_3(x) \\ &= p(1-\gamma)xh_2(x) + (1-\gamma)xh_3(x) \end{aligned} \tag{A26}$$

from which we derive

$$h_3(x) = \frac{p(1-\gamma)x}{1-(1-\gamma)x} h_2(x) \tag{A27}$$

So far, we have obtained the relations (26)-(28) in equations (A16), (A25), and (A27). To complete the proof of Lemma 1, the remaining part is determining the last function $h_2^{(m)}(x)$. It shows that

$$\begin{aligned} h_2^{(m)}(x) &= \sum_{m=1}^{\infty}\sum_{n=m+1}^{\infty} x^m \pi_{(n,m,0)} \\ &= \sum_{n=2}^{\infty} x\pi_{(n,1,0)} + \sum_{m=2}^{\infty}\sum_{n=m+1}^{\infty} x^m \\ &\quad \times \Big\{ \pi_{(n-1,m-1,0)}(1-p)(1-\gamma) + \sum_{k=n}^{\infty} \pi_{(k,n-1,m-1)}(1-p\eta)\gamma \Big\} \\ &= x\pi_{(2,1,0)} + \sum_{n=3}^{\infty} x \Big\{ \pi_{(n-1,0,0)} p(1-\gamma) + \sum_{k=n}^{\infty} \pi_{(k,n-1,0)} p\gamma \\ &\quad + \sum_{k=n}^{\infty}\sum_{j=1}^{n-2} \pi_{(k,n-1,j)} p\gamma\eta \Big\} + \sum_{m=1}^{\infty}\sum_{n=m+1}^{\infty} x^{m+1} \pi_{(n,m,0)}(1-p)(1-\gamma) \\ &\quad + \sum_{m=2}^{\infty}\sum_{n=m+1}^{\infty} x^m \sum_{k=n}^{\infty} \pi_{(k,n-1,m-1)}(1-p\eta)\gamma \\ &= x\Big\{ \pi_{(1,0,0)} p(1-\gamma) + \sum_{k=2}^{\infty} \pi_{(k,1,0)} p\gamma \Big\} + \sum_{n=2}^{\infty} x\pi_{(n,0,0)} p(1-\gamma) \\ &\quad + \sum_{n=3}^{\infty} x \sum_{k=n}^{\infty} \pi_{(k,n-1,0)} p\gamma + \sum_{n=3}^{\infty} x \sum_{k=n}^{\infty}\sum_{j=1}^{n-2} \pi_{(k,n-1,j)} p\gamma\eta \\ &\quad + (1-p)(1-\gamma)xh_2^{(m)}(x) + (1-p\eta)\gamma x h_3^{(l)}(x) \end{aligned} \tag{A28}$$

Similarly, use the substitutions $k = \tilde{n}$, $n-1 = \tilde{m}$, $m-1 = \tilde{l}$, we write

$$\begin{aligned} &\sum_{m=2}^{\infty}\sum_{n=m+1}^{\infty} x^m \sum_{k=n}^{\infty} \pi_{(k,n-1,m-1)}(1-p\eta)\gamma \\ &= \sum_{\tilde{l}=1}^{\infty}\sum_{\tilde{m}=\tilde{l}+1}^{\infty} x^{\tilde{l}+1} \sum_{\tilde{n}=\tilde{m}+1}^{\infty} \pi_{(\tilde{n},\tilde{m},\tilde{l})}(1-p\eta)\gamma \\ &= (1-p\eta)\gamma x h_3^{(l)}(x) \end{aligned} \tag{A29}$$

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thus the last term in equation (A28) is obtained, where $h_3^{(l)}(x)$ is defined and determined as

$$\begin{aligned}
h_3^{(l)}(x) &= \sum_{l=1}^{\infty}\sum_{m=l+1}^{\infty} x^l \sum_{n=m+1}^{\infty} \pi_{(n,m,l)} \\
&= \sum_{m=2}^{\infty} x \sum_{n=m+1}^{\infty} \pi_{(n,m,1)} + \sum_{l=2}^{\infty}\sum_{m=l+1}^{\infty} x^l \sum_{n=m+1}^{\infty} \pi_{(n-1,m-1,l-1)}(1-\gamma)(1-p\eta) \\
&= x\sum_{n=3}^{\infty} \pi_{(n,2,1)} + \sum_{m=3}^{\infty} x \sum_{n=m+1}^{\infty} \Big\{ \pi_{(n-1,m-1,0)} p(1-\gamma) \\
&\quad + \sum_{j=1}^{m-2} \pi_{(n-1,m-1,j)} p(1-\gamma)\eta \Big\} + (1-\gamma)(1-p\eta)xh_3^{(l)}(x) \\
&= x\sum_{n=3}^{\infty} \pi_{(n-1,1,0)} p(1-\gamma) + \sum_{m=2}^{\infty} x \sum_{n=m+1}^{\infty} \pi_{(n,m,0)} p(1-\gamma) \\
&\quad + \sum_{m=3}^{\infty} x \sum_{n=m+1}^{\infty} \sum_{j=1}^{m-2} \pi_{(n-1,m-1,j)} p(1-\gamma)\eta + (1-\gamma)(1-p\eta)xh_3^{(l)}(x) \\
&= p(1-\gamma)M_2 x + p(1-\gamma)\eta M_3 x + (1-\gamma)(1-p\eta)xh_3^{(l)}(x) \\
&= [\gamma + p(1-\gamma)\eta]M_3 x + (1-\gamma)(1-p\eta)xh_3^{(l)}(x) \qquad \text{(A30)}
\end{aligned}$$

in last equation (A30) we have used the relation (A8), which says $p(1-\gamma)M_2 = \gamma M_3$. From (A30), we obtain the result

$$h_3^{(l)}(x) = \frac{[\gamma + p(1-\gamma)\eta]M_3 x}{1-(1-\gamma)(1-p\eta)x} \qquad \text{(A31)}$$

Come back to the calculation of function $h_2^{(m)}(x)$, equation (A28) shows

$$\begin{aligned}
h_2^{(m)}(x)[1-(1-p)(1-\gamma)x] &= p(1-\gamma)M_1 x + p\gamma M_2 x + p\gamma\eta M_3 x + (1-p\eta)\gamma x h_3^{(l)}(x) \\
&= \left[\gamma + p^2(1-\gamma)\eta\right] M_2 x + (1-p\eta)\gamma x h_3^{(l)}(x) \qquad \text{(A32)}
\end{aligned}$$

in which we have used equations (A3), (A8) to replace numbers M_1 and M_3. Substituting the expression (A31), after some extra operations, we determine that

$$h_2^{(m)}(x) = \frac{[\gamma + p^2(1-\gamma)\eta - (1-p)(1-\gamma)(1-p\eta)\gamma x]M_2 x}{[1-(1-p)(1-\gamma)x][1-(1-\gamma)(1-p\eta)x]} \qquad \text{(A33)}$$

this eventually completes the proof of Lemma (1).

Appendix B. Proof of Equations (41) and (42)

In equation (33), we show that

$$\begin{aligned}
H_{PP}(x) &= \frac{p\gamma M_1 x[1-(1-p)(1-\gamma)x]}{[1-(1-p)x][1-(1-\gamma)x]} \\
&\quad + \frac{\gamma x\{1-(1-p)[2(1-\gamma)+p\gamma]x + (1-p)^2(1-\gamma)^2x^2\}}{[1-(1-p)x][1-(1-\gamma)x]^2} h_2^{(m)}(x) \qquad \text{(A34)}
\end{aligned}$$

Equation (41) and (42) are obtained by decomposing each part of (A34). For the first part, we assume

$$\begin{aligned}
\frac{1-(1-p)(1-\gamma)x}{[1-(1-p)x][1-(1-\gamma)x]} &= \frac{A}{1-(1-p)x} + \frac{B}{1-(1-\gamma)x} \\
&= \frac{(A+B) - [A(1-\gamma) + B(1-p)]x}{[1-(1-p)x][1-(1-\gamma)x]} \qquad \text{(A35)}
\end{aligned}$$

and according to the coefficients of correspoding terms, we obtain

$$A + B = 1, \quad A(1-\gamma) + B(1-p) = (1-p)(1-\gamma) \qquad \text{(A36)}$$

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which determine A and B as

$$A = \frac{(1-p)\gamma}{\gamma - p}, \text{ and } B = \frac{p(1-\gamma)}{\gamma - p} \tag{A37}$$

Therefore,

$$\frac{p\gamma M_1 x[1-(1-p)(1-\gamma)x]}{[1-(1-p)x][1-(1-\gamma)x]} = \frac{p\gamma M_1 x}{\gamma - p}\left(\frac{(1-p)\gamma}{1-(1-p)x} - \frac{p(1-\gamma)}{1-(1-\gamma)x}\right) \tag{A38}$$

For the second part of expression (A34), let

$$\begin{aligned} &\frac{1-(1-p)[2(1-\gamma)+p\gamma]x+(1-p)^2(1-\gamma)^2x^2}{[1-(1-p)x][1-(1-\gamma)x]^2} \\ = &\frac{A}{1-(1-p)x} + \frac{B}{1-(1-\gamma)x} + \frac{C}{[1-(1-\gamma)x]^2} \\ = &\frac{(A+B+C)-[(A+B)(1-\gamma)+A(1-\gamma)+B(1-p)+C(1-p)]x+c_2x^2}{[1-(1-p)x][1-(1-\gamma)x]^2} \end{aligned}$$

in which

$$c_2 = [A(1-\gamma)+B(1-p)](1-\gamma)$$

Thus, we have

$$\begin{cases} 1 = A+B+C & (1) \\ (1-p)[2(1-\gamma)+p\gamma] = (A+B)(1-\gamma)+A(1-\gamma)+B(1-p)+C(1-p) & (2) \\ (1-p)^2(1-\gamma) = A(1-\gamma)+B(1-p) & (3) \end{cases} \tag{A39}$$

Substituting relation (3) and $A+B = 1-C$ into the second relation of (A39), we obtain

$$(1-p)[2(1-\gamma)+p\gamma] = (1-C)(1-\gamma)+(1-p)^2(1-\gamma)+C(1-p) \tag{A40}$$

from which we can solve that $C = p$.

Then, according to the equations

$$A+B = 1-p \;\text{ and }\; A(1-\gamma)+B(1-p) = (1-p)^2(1-\gamma)$$

the other two numbers are obtained to be

$$A = \frac{(1-p)^2\gamma}{\gamma-p}, \quad B = -\frac{p(1-p)(1-\gamma)}{\gamma-p} \tag{A41}$$

thus the factorization of the second part is obtained.

The last part, that is the function $h_2^{(m)}(x)$ is dealt with similarly. Omitting the straightforward calculations, we directly give that

$$h_2^{(m)}(x) = -\frac{\eta(1-p)(p+\gamma-p\gamma)}{(1-\eta)[1-(1-p)(1-\gamma)x]} + \frac{(1-p\eta)[\gamma+p(1-\gamma)\eta]}{(1-\eta)[1-(1-\gamma)(1-p\eta)x]} \tag{A42}$$

Notice that $(1-\eta)$ is contained in the denominator of fractions in equation (A42), thus $\eta \neq 1$. When $\eta = 1$, equation (29) shows that

$$h_2^{(m)}(x)\Big|_{\eta=1} = \frac{\gamma+p^2(1-\gamma)-(1-p)^2(1-\gamma)\gamma x}{[1-(1-p)(1-\gamma)x]^2} \tag{A43}$$

Summarize above results, both the equations (41) and (42) are determined.

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Appendix C. Factorization of last part of equation (42)

We write

$$\begin{aligned}
&\frac{p\gamma M_2 x^2}{[1-(1-\gamma)x]^2}\cdot\frac{\gamma+p^2(1-\gamma)-(1-p)^2(1-\gamma)\gamma x}{[1-(1-p)(1-\gamma)x]^2}\\
&=p\gamma M_2 x^2\left(\frac{A}{1-(1-\gamma)x}+\frac{B}{[1-(1-\gamma)x]^2}+\frac{C}{1-(1-p)(1-\gamma)x}+\frac{D}{[1-(1-p)(1-\gamma)x]^2}\right)\\
&=p\gamma M_2 x^2\left(\frac{(A+B)-A(1-\gamma)x}{[1-(1-\gamma)x]^2}+\frac{(C+D)-C(1-p)(1-\gamma)x}{[1-(1-p)(1-\gamma)x]^2}\right)
\end{aligned}\tag{A44}$$

Merging the terms in bracket of (A44) and according to corresponding coefficients, it shows that

$$\begin{cases}
A+B+C+D=\gamma+p^2(1-\gamma) & (1)\\
2(A+B)(1-p)+2(C+D)+A+C(1-p)=(1-p)^2\gamma & (2)\\
(A+B)(1-p)^2+2A(1-p)+(C+D)+2C(1-p)=0 & (3)\\
A(1-p)+C=0 & (4)
\end{cases}\tag{A45}$$

Relation (4) shows that $C=-A(1-p)$ and using the first relationship, the second and the third row of equations (A45) are equivalent to

$$\begin{cases}
2(1-p)[\gamma+p^2(1-\gamma)-(C+D)]+2(C+D)+A-A(1-p)^2=(1-p)^2\gamma\\
(1-p)^2[\gamma+p^2(1-\gamma)-(C+D)]+2A(1-p)+(C+D)-2A(1-p)^2=0
\end{cases}\tag{A46}$$

which gives

$$p(2-p)(C+D)=-2p(1-p)A-(1-p)^2[\gamma+p^2(1-\gamma)]\tag{A47}$$

and

$$2p(C+D)=-p(2-p)A+(1-p)^2\gamma-2(1-p)[\gamma+p^2(1-\gamma)]\tag{A48}$$

Combining equations (A47) and (A48), the coefficient A is solved as

$$A=\frac{2-p}{p^3}\left((1-p)^2\gamma-\frac{2(1-p)[\gamma+p^2(1-\gamma)]}{2-p}\right)\tag{A49}$$

and immediately

$$C=-A(1-p)=-\frac{(1-p)(2-p)}{p^3}\left((1-p)^2\gamma-\frac{2(1-p)[\gamma+p^2(1-\gamma)]}{2-p}\right)\tag{A50}$$

Using equation (A47), we have

$$\begin{aligned}
D&=\frac{-2p(1-p)A-(1-p)^2[\gamma+p^2(1-\gamma)]}{p(2-p)}-C\\
&=\frac{-2p(1-p)A-(1-p)^2[\gamma+p^2(1-\gamma)]}{p(2-p)}+A(1-p)\\
&=\frac{p^2(1-p)A-(1-p)^2[\gamma+p^2(1-\gamma)]}{p(2-p)}
\end{aligned}\tag{A51}$$

and in the end, the last number B is determined by

$$B=[\gamma+p^2(1-\gamma)]-A-C-D\tag{A52}$$

So far, all the coefficients are obtained and the decomposition is totally determined.

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