The Nonlinear Analysis related to Schauder Fixed Points, Best Approximation, Birkhoff-Kellogg Problems and Principle of Leray-Schauder Alternatives in p-Vector Spaces

Dedicated to Professor Shih-sen Chang on his 90th Birthday

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Abstract

It is known that the class of p-normed spaces $(0 < p \leq 1)$ is an important generalization of usual normed spaces with rich topological and geometrical structure, but the most of tools and general principles in nature with nonlinearity have not been developed yet, thus the main goal of this paper is to develop tools for nonlinear analysis under the framework of p-vector spaces. In particular, we first develop the general fixed point theorems which provide solutions to answer Schauder conjecture since 1930's in the affirmative for p-vector spaces when $p = 1$ (which is just general topological vector spaces); then the one best approximation result for upper semi-continuous mappings is given, which is used as a powerful tool to establish fixed points for non-self set-valued mappings with either inward or outward set conditions; and finally we establish comprehensive existence results of solutions for Birkhoff-Kellogg Problems, and the general principle of nonlinear alternative by including Leray-Schauder alternative and related results.

The results given in this paper not only include the corresponding results in the existing literature as special cases, but also expected to be useful tools for the study of nonlinear problems arising from theory to practice.

Keywords: Nonlinear analysis, Best approximation, Birkhoff-Kellogg theorem, Schauder Conjecture, fixed point theorem, Nonlinear alternative, Leray-Schauder alternative, p-convex, p-Inward and p-Outward set, p-vector space.

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1. Introduction

It is known that the class of p-normed spaces ($0 < p \leq 1$) is an important generalization of usual normed spaces, and it has a rich topological and geometrical structure, and related study has received a lot of attention (e.g., see Balachandran [5], Bayoumi [6], Bayoumi et al.[7], Bernuées and Pena [9], Ding [23], Gal and Goldstein [32], Gholizadeh et al.[33], Jarchow [37], Kalton [39]-[40], Kalton et al.[41], Part[61], Qiu and Rolewicz [66], Rolewicz [69], Simons [75], Tabor et al.[79], Tan [80], Wang [82], Xiao and Lu [85], Xiao and Zhu[86]-[87], Yuan [91], and many others).

However, to the best of our knowledge, the corresponding basic tools and associated results in the category of nonlinear functional analysis have not been well developed, thus the goal of this paper is first to develop tools for nonlinear analysis under the framework of p-vector spaces. In particular, we first develop the general fixed point theorems which provide solutions to answer Schauder conjecture since 1930's in the affirmative for p-vector spaces when $p = 1$ (which is just general topological vector spaces); then the one best approximation result for upper semi-continuous mappings is given, which is used as a powerful tool to establish fixed points for non-self set-valued mappings with either inward or outward set conditions; and finally we establish comprehensive existence results of solutions for Birkhoff-Kellogg Problems, and the general principle of nonlinear alternative by including Leray-Schauder alternative and related results. The results given in this paper do not only include the corresponding results in the existing literature as special cases, but also would become useful new tools for the study of nonlinear problems from social science, engineering, applied mathematics and related topics and areas.

Before discussing the study of the best approximation and related nonlinear analysis tools under the framework of p-vector spaces, we like first to share with readers that that though most of results in nonlinear analysis are normally highly associated with the convexity hypotheses under the local convex topological vector spaces (of course, including normed spaces, and Banach spaces, nice metric spaces), it seems surprise that p-vector spaces which in general do not have the local convex structure comparing with locally convex spaces, but they are provide some nice properties in the nature way with some kinds of nice approximation and better (i.e., the bigger) structures for the so-called the convexities of p-convex subset play very important roles for us to describe Birkhoff and Kellogg problems, and related nonlinear problems (such as fixed point problem and so on) in topological vector spaces (TVS) based on p-vector spaces's behaviors for p in $(0, 1)$ (as one p-vector space E reduces to TVS when $p = 1$), and also see the corresponding results and properties as pointed by the Remark 2.1 (1), Lemma 2.1(ii) and Lemma 2.3 below).

Here, in particularly, we recall that since the first Birkhoff-Kellogg theorem was introduced and proved by Birkhoff and Kellogg [10] in 1922 in discussing the existence of solutions for the equation $x = \lambda F(x)$, where λ is a real parameter, and F is a general nonlinear non-self mapping defined on an open convex subset U of a topological vector space E, now the general form of the Birkhoff-Kellogg problem is to find the so-called an invariant direction for the nonlinear set-valued mappings F, i.e., to find $x_0 \in \partial\overline{U}$ and $\lambda > 0$ if $\lambda x_0 \in F(x_0)$.

Since Birkhoff and Kellogg theorem given by Birkhoff and Kellogg in 1920's, the study on Birkhoff-Kellogg problem has been received a lot of attention by scholars since then. For example, in 1934, one of the fundamental

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results in nonlinear functional analysis, famously called the Leray-Schauder alternative by Leray and Schauder [46] was established via topological degree. Thereafter, certain other types of Leray-Schauder alternatives were proved using different techniques other than topological degree, see works given by Granas and Dugundji [34], Furi and Pera [31] in the Banach space setting and applications to the boundary value problems for ordinary differential equations in noncompact problems, a general class of mappings for nonlinear alternative of Leray-Schauder type in normal topological spaces, and some Birkhoff-Kellogg type theorems for general class mappings in topological vector spaces by Agarwal et al.[1], Agarwal and O'Regan [2]-[3], Park [59]. In particular, recently by using the Leray-Schauder type coincidence theory to establish some Birkhoff-Kellogg problem, Furi-Pera type results for a general class of mappings by O'Regan [54] and references wherein. In this paper, based on the application of our best approximation as a tool, the general principle for the existence of solutions for Birkhoff-Kellogg problems and related nonlinear alternatives will be established, which then also allows us to give general existence of Leray-Schaduer type and related fixed point theorems for non-self mappings in general p-vector spaces for $(0 < p \leq 1)$. These new results given in this paper not only include the corresponding results in the existing literature as special cases, but also expected to be useful tools for the study of nonlinear problems arising from theory to practice.

Now we give a brief introduction on the best approximation theorem related to the development of the nonlinear analysis as a powerful tool with some background.

We all know that the beat approximation in nature related to fixed points for non-self mappings, which tightly link with the classical Leray-Schauder alternative based on the Leray-Schauder continuation theorem by Leray and Schauder [46], which is a remarkable result in nonlinear analysis; and in addition, there exist several continuation theorems, which have many applications to the study of nonlinear functional equations (see O'Regan and Precup [55]). Historically, it seems that the continuation theorem is based on the idea of obtaining a solution of a given equation, starting from one of the solutions of a simpler equation. The essential part of this theorem is the "Leray-Schauder boundary condition". It seems the "continuation method" was initiated by Poincare [64], Bernstein [8]. Certainly, Leray and Schauder[46] in 1934 gave the first abstract formulation of "continuation principle" using the topological degree (see also Granas and Dugundji [34], Isac [38], Rothe [70]-[71], Zeidler [92]). But in this paper, we will see how the best approximation method could be used for the study of fixed point theorems in p-vector space for $0 < p \leq 1$ which as a basic tool, will help us to develop the principle of nonlinear alterative, Leray-Schauder alternative, fixed point theorems of Rothe, Petryshyn, Atlman type for non-self mappings, and related fixed point, nonlinear alternative with different boundary conditions. Moreover, the new results given in this paper are highly expected to become useful tools for the study on optimization, nonlinear programming, variational inequality, complementarity, game theory, mathematical economics, and related other social science area.

It is well-known that the best approximation is one of very important aspects for the study of nonlinear problems related to the problems on their solvability for partial differential equations, dynamic systems, optimization, mathematical program, operation research; and in particularly, the one approach well accepted for study of nonlinear problems in optimization, complementarity problems and of variational inequalities problems and so on, strongly based on today called Fan's best approximation theorem given by Fan [29] in 1969 which acts as a very powerful

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tool in nonlinear analysis, and see the book of Singh et al.[76] for the related discussion and study on the fixed point theory and best approximation with the KKM-map principle), among them, the related tools are Rothe type and principle of Leray-Schauder alterative in topological vector spaces (TVS), and local topological vector spaces (LCS) which are comprehensively studied by Chang et al.[18], Chang et al.[19]-[21], Carbone and Conti [14], Ennassik and Taoudi [25], Ennassik et al.[26], Isac [38], Granas and Dugundji [34], Kirk and Shahzad [43], Liu [48], Park [62], Rothe [70]-[71], Shahzad [72]-[74], Xu [88], Yuan [89]-[91], Zeidler [92], and references wherein.

On the other hand, since the celebrated so-called KKM principle established in 1929 in [44], was based on the celebrated Sperner combinatorial lemma and first applied to a simple proof of the Brouwer fixed point theorem. Later it became clear that these three theorems are mutually equivalent and they were regarded as a sort of mathematical trinity (Park [62]). Since Fan extended the classical KKM theorem to infinite-dimensional spaces in 1961 by Fan [28]-[30], there have been a number of generalizations and applications in numerous areas of nonlinear analysis, and fixed points in TVS and LCS as developed by Browder [11]-[13] and related references wherein. Among them, Schauder's fixed point theorem [77] in normed spaces is one of the powerful tools in dealing with nonlinear problems in analysis. Most notably, it has played a major role in the development of fixed point theory and related nonlinear analysis and mathematical theory of partial and differential equations and others. A generalization of Schauder's theorem from normed space to general topological vector spaces is an old conjecture in fixed point theory which is explained by the Problem 54 of the book "The Scottish Book" by Mauldin [50] as stated as Schauder's conjecture: "Every nonempty compact convex set in a topological vector space has the fixed point property, or in its analytic statement, does a continuous function defined on a compact convex subset of a topological vector space to itself have a fixed point?" Recently, this question has been recently answered by the work of Ennassik and Taoudi [25], Ennassik et al.[26] by using the p-seminorm methods under p-vector spaces; and also singificant contribution by Cauty [16], plus corresponding contributions by Askoura and Godet-Thobie [4], Cauty [15], Chang [17], Chang et al.[18], Chen [22], Dobrowolski [24], Gholizadeh et al.[33], Isac [38], Li [47], Nhu [51], Okon [52], Park [61]-[63], Reich [67], Smart [78], Weber [83]-[84], Xiao and Lu [85], Xiao and Zhu [86]-[87], Yuan [89]-[91] and related references wherein under the general framework of p-vector spaces for even non-self set-valued mappings $(0 < p \leq 1)$.

Though the goal of this paper is to establish the general new tools of nonlinear analysis based on the new best approximation theorem for p-vector spaces by applying the KKM principle from abstract convex spaces for $0 < p \leq 1$, actually we will see that as the applications of the best approximations, a number of new results which are Birkhoff-Kellogg type, related nonlinear alternative, and fixed point theorems for non-self set-valued with boundary conditions, Rothe, Petryshyn type, Altman type, Leray-Schedule types and related others are established. These results indeed also provide solutions for Schauder's conjecture in the affirmative way with the extension to non-self mappings under general p-vector spaces $(0 < p \leq 1)$ which may not locally convex (see Kalton [39]-[40], Kalton et al.[41], Jarchow [37], Roloewicz [69], Fan [27]-[30], Singh et al.[76], and related references).

The paper has six sections. Section 1 is the introduction. Section 2 describes general concepts for the p-convex subsets of topological vector spaces $(0 < p \leq 1)$. In Section 3, then some basic results of KKM principle related

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to abstract convex spaces are given. In Section 4, as the application of the KKM principle in abstract convex spaces which including p-convex vector spaces as a special class $(0 < p \leq 1)$ by combining the embedding lemma for compact p-convex subsets from topological vector spaces into locally p-convex spaces, we provide a general fixed point theorems for upper semi-continuous set-valued mappings for compact p-convex subsets in vector spaces for both inward and outward sets. In section 5, this is the main part of the paper as the general best approximation result is given first for p-vector spaces, which then is used as a powerful tool to establish the general principle for Birkhoff-Kellogg alternative, and related nonlinear alterative and by including Leray-Schauder alternative, Rothe type and Altman type associated with different boundary conditions are established under p-convex vector spaces for $(0 < p \leq 1)$. In section 6, we give a number of new results based on general principles of Birkhoff-Kellogg, and Leray-Schauder alternative given by Section 5 with considering the general existence of solutions for Birkhoff-Kellogg problems and fixed points of non-self mappings with different boundary conditions. .

For the convenience of our discussion, throughout this paper, all p-convex topological vector spaces and the compact p-convex sets are always assumed to be Hausdorff, and $0 < p \leq 1$ unless specified.

2. The Basic Results of p-Vector Spaces

We now recall some notion and definitions for p-convex topological vector spaces which will be used below (see Jarchow [37], Kalton [39], Rolewicz [69], Bayoumi [6], Gholizadeh et al.[33], or Ennassik and Taoudi [25], Xiao and Lu [85], Xiao and Zhu [86],).

Definition 2.1. A set A in a vector space X is said to be p-convex if for any $x, y \in A$, $s, t \geq 0$, we have $(1-t)^{1/p}x + t^{1/p}y \in A$, whenever $0 \leq t \leq 1$. If A is 1-convex, it is simply called convex (for $p = 1$) in general vector spaces.

Definition 2.2. If A is a subset of a topological vector space X, the closure of A is denoted by $\overline{A}$, then the p-convex hull of A and its closed p-convex hull denoted by $C_p(A)$, and $\overline{C}_p(A)$, respectively, which is the smallest p-convex set containing A, and the smallest closed p-convex set containing A, respectively.

Definition 2.3. Let A be p-convex and $x_1, \cdots, x_n \in A$, and $t_i \geq 0, \sum_1^n t_i^{\mathrm{p}} = 1$. Then $\sum_1^n t_i x_i$ is called a p-convex combination of $\{x_i\}$ for $i = 1, 2, \cdots, n$. If $\sum_1^n |t_i|^{\mathrm{p}} \leq 1$, then $\sum_1^n t_i x_i$ is called an absolutely p-convex combination. It is easy to see that $\sum_1^n t_i x_i \in A$ for a p-convex set A.

Definition 2.4. A subset A of a vector space X is called circled (or balanced) if $\lambda A \subset A$ holds for all scalars λ satisfying $\|\lambda\| \leq 1$. We say that A is absorbing if for each $x \in X$, there is a real number $\rho_x > 0$ such that $\lambda x \in A$ for all $\lambda > 0$ with $|\lambda| \leq \rho_x$.

By the definition 2.4, it is easy to see that the system of all circled subsets of X is easily seen to be closed under the formation of linear combinations, arbitrary unions, and arbitrary intersections. In particular, every set $A \subset X$ determines a smallest circled subset $\hat{A}$ of X in which it is contained: $\hat{A}$ is called the circled hull of A. It is clear that $\hat{A} = \cup_{|\lambda| \leq 1} \lambda A$ holds, so that A is circled if and only if (in short, iff) $\hat{A} = A$. We use $\overline{\hat{A}}$ to denote for the closed

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circled hull of $A \subset X$.

In addition, if X is a topological vector space, we use the $int(A)$ to denote the interior of set $A \subset X$ and if $0 \in int(A)$, then $int(A)$ is also circled; and use the ∂A to denote the boundary of A in X.

Definition 2.5. A topological vector space is said to be locally p-convex if the origin has a fundamental set of absolutely p-convex 0-neighborhoods. This topology can be determined by p-seminorms which are defined in the obvious way (see P.52 of Bayoumi [6], Jarchow [37] or Rolewicz [69]).

Definition 2.6. Let X is a vector space and R^+ is a non-negative part of a real line R. Then a mapping $P : X \longrightarrow R^+$ is said to be an p-seminorm if it satisfies the requirements for $(0 < p \leq 1)$

(i) $P(x) \geq 0$ for all $x \in X$;

(ii) $P(\lambda x) = |\lambda|^p P(x)$ for all $x \in X$ and $\lambda \in R$;

(iii) $P(x + y) \leq P(x) + P(y)$ for all $x, y \in X$.

An p-seminorm P is called an p-norm if $x = 0$ whenever $P(x) = 0$. A vector space with a specific p-norm is called an p-normed space. Of course if $p = 1$, X is is a usual normed space.

By Lemma 3.2.5 of Balachandra [5], the following proposition gives a necessary and sufficient condition for an p-seminorm to be continuous.

Proposition 2.1. Let X be a topological vector space, P is an p-seminorm on X and $V := \{x \in X : P(x) < 1\}$. Then P is continuous if and only if $0 \in int(V)$, where $int(V)$ is the interior of V.

Now given an p-seminorm P, the p-seminorm topology determined by P (in short, the p-topology) is the class of unions of open balls $B(x, \epsilon) := \{y \in X : P(y - x) < \epsilon\}$ for $x \in X$ and $\epsilon > 0$.

Definition 2.7. A topological vector space X is said to be locally p-convex if it has a 0-basis consisting of p-convex neighborhoods for $(0 < p \leq 1)$. If $p = 1$, X a usual locally convex space.

We also need following notion for the so-called p-gauge (see Balachandra [5]).

Definition 2.8. Let A be an absorbing subset of a vector space X. For $x \in X$ and $0 < p \leq 1$, set $P_A = \inf\{\alpha > 0 : x \in \alpha^{\frac{1}{p}} A\}$, then the non-negative real-valued function P_A is called the p-gauge (gauge if $p = 1$). The p-gauge of A is also known as the Minkowski p-functional.

By Proposition 4.1.10 of Balachandra [5], we have the following proposition.

Proposition 2.2. Let A be an absorbing subset of X. Then p-gauge P_A has the following properties:

(i) $P_A(0) = 0$;

(ii) $P_A(\lambda x) = |\lambda|^p P_A(x)$ if $\lambda \geq 0$;

(iii) $P_A(\lambda x) = |\lambda|^p P_A(x)$ for all $\lambda \in R$ provided A is circled;

(iv) $P_A(x + y) \leq P_A(x) + P_A(y)$ for all $x, y \in A$ provided A is p-convex.

In particular, P_A is a p-seminorm if A is absolutely p-convex (and also absorbing).

Recall that a given p-seminorm, is said to be an p-norm if $x = 0$ whenever $P(x) = 0$. A vector space with a specific p-norm is called an p-normed space. The p-norm of an element $x \in E$ will usually be denoted by $\|x\|_p$. If $p = 1$, X is a usual normed space. If X is an p-normed space, then (X, d_p) is a metric linear space with a translation

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invariant metric d_p such that $d_p = d_p(x, y) = \|x - y\|_p$ for $x, y \in X$. We point out that$p$-normed spaces are very important in the theory of topological vector spaces. Specifically, a Hausdorff topological vector space is locally bounded if and only if it is an p-normed space for some p-norm $\|\cdot\|_p$, where $0 < p \leq 1$ (see p.114 of Jarchow [37]). we note that examples of p-normed spaces include such as $L^p(\mu)$ - spaces and Hardy spaces H_p, $0 < p < 1$, endowed with their usual p-norms.

Remark 2.1. We like to make the following important two points as follows:

(1) First, by the fact that (e.g., see Kalton et al.[41], or Ding [23]), there is no open convex non-void subset in $L^p[0, 1]$ (for $0 < p < 1$) except $L^p[0, 1]$ itself, this means that p-normed paces with $0 < p < 1$ are not necessarily locally convex. Moreover, we know that every p-normed space is locally p-convex; and incorporating Lemma 2.3 below, it seems that p-vector spaces (for $0 < p \leq 1$) is a nicer space as we can use p-vector space to approximate (Hausdorff) topological vector spaces (TVS) in terms of Lemma 2.1 (ii) below for the convex subsets in TVS by using a bigger p-convex subsets in p-vector spaces for $p \in (0, 1)$ by also considering Lemma 2.3 below, in this way, it seems P-vector spaces seems having better properties in terms of p-convexity than the usually $(1-)$ convex subsets used in TVS.

(2) Second, it is worthwhile noting that a 0-neighborhood in a topological vector space is always absorbing by Lemma 2.1.16 of Balachandra [5], or Proposition 2.2.3 of Jarchow [37]).

. We also note that by Proposition 4.1.12 of Balachandra [5], we have the following proposition.

Proposition 2.3. Let A be a subset of a vector space X, which is absolutely p-convex $(0 < p \leq 1)$ and absorbing. Then, we have that

(i) The p-gauge P_A is a p-seminorm such that if $B_1 := \{x \in X : P_A(x) < 1\}$, and $\overline{B_1} = \{x \in X : P_A(x) \leq 1\}$. then $B_1 \subset A \subset \overline{B_1}$; in particular, $kerP_A \subset A$, where $kerP_A := \{x \in X : P_A(x) = 0\}$.

(ii) $A = B_1$ or $\overline{B_1}$ according as A is open or closed in the P_A-topology.

Remark 2.2. Let X be a topological vector space and let U be an open absolutely p-convex neighborhood of the origin, and let ϵ be given. If $y \in \epsilon^{\frac{1}{p}}U$, then $y = \epsilon^{\frac{1}{p}}u$ for some $u \in U$ and $P_U(y) = P_U(\epsilon^{\frac{1}{p}}u) = \epsilon P_U(u) \leq \epsilon$ (as $u \in U$ implies that $P_U(u) \leq 1$). Thus, P_U is continuous at $zero$, and therefore, P_U is continuous everywhere. Moreover, we have $U = \{x \in X : P_U(x) < 1\}$.

Indeed, since U is open and the scalar multiplication is continuous, we have that for any $x \in U$, there exists $0 < t < 1$ such that $x \in t^{\frac{1}{p}}U$ and so $P_U(x) \leq t < 1$.This shows that $U \subset \{x \in X : P_U(x) < 1\}$. The conclusion follows by Proposition 2.3 above.

The following result is a very important and useful result which allows use to make the approximation for convex subsets in topological vector spaces by p-convex subsets in p-convex vector spaces. For the reader's self-contained in reading, we provide a sketch of proof below (see also Lemma 2.1 of Ennassik and Taoudi [26], Remark 2.1 of Qiu and Rolewicz [66]).

Lemma 2.1. Let A be a subset of a vector space X, then we have

(i) If A is r-convex, with $0 < r < 1$, then $\alpha x \in A$ for any $x \in A$ and any $0 < \alpha \leq 1$.

(ii) If A is convex and $0 \in A$, then A is s-convex for any $s \in (0,1]$.

(iii) If A is r-convex for some $r \in (0,1)$, then A is s-convex for any $s \in (0,r]$.

Proof. (i) As $r \leq 1$, by the fact that "for all $x \in A$ and all $\alpha \in [2^{(n+1)(1-\frac{1}{r})}, 2^{n(1-\frac{1}{r})}]$, we have $\alpha x \in A$" is true for all integer $n \geq 0$. Taking into account that the fact that $(0,1] = \cup_{n\geq 0}[2^{(n+1)(1-\frac{1}{r})}, 2^{n(1-\frac{1}{r})}]$, thus the result is obtained.

(ii) Assume that A is a convex subset of X with $0 \in A$ and take a real number $s \in (0,1]$. we show that A is s-convex. Indeed, let $x, y \in A$ and $\alpha, \beta > 0$ with $\alpha^s + \beta^s = 1$. Since A is convex, then $\frac{\alpha}{\alpha+\beta}x + \frac{\beta}{\alpha+\beta}y \in A$. Keeping in mind that $0 < \alpha + \beta < \alpha^s + \beta^s = 1$, it follows that $\alpha x + \beta y = (\alpha+\beta)(\frac{\alpha}{\alpha+\beta}x + \frac{\beta}{\alpha+\beta}y) + (1-\alpha-\beta)0 \in A$.

(iii) Now, assume that A is r-convex for some $r \in (0,1)$ and pick up any real $s \in (0,r]$. We show that A is s-convex. To see this, let $x, y \in A$ and $\alpha, \beta > 0$ such that $\alpha^s + \beta^s = 1$. First notice that $0 < \alpha^{\frac{r-s}{r}} \leq 1$ and $0 < \beta^{\frac{r-s}{r}} \leq 1$, which imply that $\alpha^{\frac{r-s}{r}}x \in A$ and $\beta^{\frac{r-s}{r}} \in A$. By the r-convexity of A and the equality $(\alpha^{\frac{s}{r}})^r + (\beta^{\frac{s}{r}})^r = 1$, it follows that $\alpha x + \beta y = \alpha^{\frac{s}{r}}(\alpha^{\frac{r-s}{r}}x) + \beta^{\frac{s}{r}}(\beta^{\frac{r-s}{r}}y) \in A$. This competes the sketch of the proof. $\square$

Remark 2.3. We like to point out that the results (i) and (iii) of Lemma 2.1 do not hold for $p = 1$. Indeed, any singleton $\{x\} \subset X$ is convex in topological vector spaces; but if $x \neq 0$, then it is not p-convex for any $p \in (0,1)$.

We also need the following Proposition which is proposition 6.7.2 of Jarchow [37].

Proposition 2.4. Let K be compact in a topological vector X and $(1 < p \leq 1)$. Then the closure $\overline{C}_p(K)$ of the p-convex hull, and the closure $\overline{AC}_p(K)$ of absolutely p-convex hull of K are compact if and only if $\overline{C}_p(K)$ and $\overline{AC}_p(K)$ are complete, respectively.

We also need following fact, which is a special case of Lemma 2.4 of Xiao and Zhu [86].

Lemma 2.2. Let C be a bounded closed p-convex subset of p-seminorm X with $0 \in intC$, where $(0 < p \leq 1)$. For every $x \in X$ define an operator by $r(x) := \frac{x}{\max\{1, (P_C(x))^{\frac{1}{p}}\}}$, where P_C is the Minkowski p-functional of C. Then C is a retract of X and $r : X \to C$ is a continuous such that

(1) if $x \in C$, then $r(x) = x$;

(2) if $x \notin C$, then $r(x) \in \partial C$;

(3) if $x \notin C$, then the Minkowski p-functional $P_C(x) > 1$.

Proof. Taking $s = p$ in Lemma 2.4 of Xiao and Zhu [86], Proposition 2.3 and Remark 2.2, thus the proof is compete. $\square$

Remark 2.4. As discussed by Remark 2.2, Lemma 2.2 still holds if "the bounded closed p-convex subset C of the p-normed space $(X, \|\cdot\|_p)$" is replaced by that "X is a p-seminorm vector space and C is a bounded closed absorbing p-convex subset with $0 \in intC$ of X".

Before we close this section, we like to point out that the structure of p-convexity when $p \in (0,1)$ is really different from what we normally have for the concept of "convexity" used in topological vector spaces (TVS), in particular, maybe the following fact is one of reasons for us to use better (p-convex) structures in p-vector spaces to approximate the corresponding structure of the convexity used in TVS (i.e., the p-vector space when $p = 1$). Based on the discussion in P.1740 of Xiao and Zhu [86](see also Bernués and Pena [9] and Sezer et al.[93]), we have the following fact.

Lemma 2.3. Let x be a point of p-vector space E, where assume $0 < p < 1$, then the p-convex hull and the closure of $\{x\}$ is given by

$$C_p(\{x\}) = \begin{cases} \{tx : t \in (0,1]\}, & \text{if } x \neq 0, \\ \{0\}, & \text{if } x = 0; \end{cases} \tag{1}$$

and

$$\overline{C_p(\{x\})} = \begin{cases} \{tx : t \in [0,1]\}, & \text{if } x \neq 0, \\ \{0\}, & \text{if } x = 0. \end{cases} \tag{2}$$

But note that if x is a given one point in p-vector space E, when $p = 1$, we have that $\overline{C_1(\{x\})} = C_1(\{x\}) = \{x\}$, This shows significantly different for the structure of p-convexity between $p = 1$ and $p \neq 1$!

As an application of Lemma 2.3, we have the following fact for (set-valued) mappings with non-empty closed p-convex values in p-vector spaces for $p \in (0,1)$, which are truly different from any (set-valued) mappings defined in topological vector spaces (i.e., for a p-vector space with $p = 1$).

Lemma 2.4. Let U be a non-empty subset of a p-vector space E (where $0 < p < 1$), with zero $0 \in U$, and assume a (set-valued) mapping $T : U \to 2^E$ is with non-empty closed p-convex values. Then T has at least one fixed point in U, actually we have $0 \in \cap_{x \in U} T(x) \neq \emptyset$.

Proof. For each $x \in U$, as $T(x)$ is non-empty closed p-convex, by Lemma 2.3, we have at leat $0 \in T(x)$. It implies that $0 \in \cap_{x \in U} T(x)$ and thus zero of E is a fixed point of T. This completes the proof. □

Finally, for a a given p-convex subset C of a given p-vector space E with that zero $0 \in int(C)$ with the p-seminorm P (for example, thinking of the P-seminorm which is indeed the P_U, which is the Minkowski p-functional of U), we denoted by $d_P(x, C) := \inf\{P_U(x - y) : y \in C\}$ the distance of $\{x\}$ with the set C in E for $0 < p \leq 1$.

For the convenience of our discussion again, throughout this paper, we assume all p-convex vector space X discussed is complete unless specified $(0 < p \leq 1)$.

3. The KKM Principle in Abstract Convex Spaces

As mentioned in the introduction, since Knaster, Kuratowski and Mazurkiewicz (in short, KKM)[44] in 1929 obtained the so-called KKM principle (theorem) to give a new proof for the Brouwer fixed point theorem in finite dimensional spaces; and later in 1961, Fan [28] (see also Fan [30]) extended the KKM principle (theorem) to any topological vector spaces and applied it to various results including the Schauder fixed point theorem. Since then there have appeared a large number of works devoting applications of the KKM principle (theorem). In 1992, such research field was called the KKM theory first time by Park [56], then the KKM theory has been extended to general abstract convex spaces by Park [60](see also Park [61] and [62]) which actually include locally p-convex spaces $(0 < p \leq 1)$ as a special class.

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Here we first give some notion and a brief introduction on the abstract convex spaces which play important role for the development of KKM principle and related applications. Once again, the corresponding comprehensive discussion on KKM theory and its various applications to nonlinear analysis and related topics, we refer to Mauldin [50], Granas and Dugundji [34], Park [62] and [63], Yuan [90]-[91] and related comprehensive reference there.

Let $\langle D\rangle$ denote the set of all nonempty finite subsets of a given non-empty set D, and 2^D denotes the family of all subsets of D. We have the following definition of abstract convex spaces which essentially introduced by Park [60].

Definition 3.1. An abstract convex space $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a set-valued mapping $\Gamma : \langle D\rangle \to 2^E$ with nonempty values $\Gamma_A := \Gamma(A)$ for each $A \in \langle D\rangle$, such that the Γ-convex hull of any $D' \subset D$ is denoted and defined by $\mathrm{co}_\Gamma D' := \cup\{\Gamma_A | A \in \langle D'\rangle\} \subset E$.

A subset X of E is said to be a Γ-convex subset of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D'\rangle$, we have $\Gamma_N \subseteq X$, that is, $\mathrm{co}_\Gamma D' \subset X$. For the convenience of our discussion, in the case $E = D$, the space $(E, E; \Gamma)$ is simply denoted by $(E; \Gamma)$ unless specified.

Definition 3.2. Let $(E, D; \Gamma)$ be an abstract convex space and Z a topological space. For a set-valued mapping (or say, multimap) $F : E \to 2^Z$ with nonempty values, if a set-value mapping $G : D \to 2^Z$ satisfies $F(\Gamma_A) \subset G(A) := \bigcup_{y\in A} G(y)$ for all $A \in \langle D\rangle$, then G is called a KKM mapping with respect to F. A KKM mapping $G : D \to 2^E$ is a KKM mapping with respect to the identity map 1_E.

Definition 3.3. The partial KKM principle for an abstract convex space $(E, D; \Gamma)$ is that, for any closed-valued KKM mapping $G : D \to 2^E$, the family $\{G(y)\}_{y\in D}$ has the finite intersection property. The KKM principle is that, the same property also holds for any open-valued KKM mapping.

An abstract convex space is called a (partial) KKM space if it satisfies the (partial) KKM principle (resp.). We now gave some known examples of (partial) KKM spaces (see Park [60], and also [61]) as follows.

Definition 3.4. A ϕ_A-space $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$ consists of a topological space X, a nonempty set D, and a family of continuous functions $\phi_A : \Delta_n \to 2^X$ (that is, singular n-simplices) for $A \in \{D\}$ with $|A| = n + 1$. By putting $\Gamma_A := \phi_A(\Delta_n)$ for each $A \in \langle D\rangle$, the triple $(X, D; \Gamma)$ becomes an abstract convex space.

Definition 3.5. For a ϕ_A-space $(X, D; \{\phi_A\})$, we recall that any set-valued mapping $G : D \to 2^X$ satisfying $\phi_A(\Delta_J) \subset G(J)$ for each $A \in \langle D\rangle$ and $J \in \langle A\rangle$ is called a KKM mapping.

By the definition, it is clear that every ϕ_A-space is a KKM space, thus we have the following fact (see Lemma 1 of Park [61]).

Lemma 3.1. Let $(X, D; \Gamma)$ be a ϕ_A-space and $G : D \to 2^X$ a set-valued (multimap) with nonempty closed [resp. open] values. Suppose that G is a KKM mapping, then $\{G(a)\}_{a\in D}$ has the finite intersection property.

Definition 3.6. We recall that a topological vector space is said to be locally p-convex if the origin has a fundamental set of absolutely p-convex 0-neighborhoods. This topology can be determined by p-seminorms which are defined in the obvious way (see Jarchow [37], or P.52 of Bayoumi [6]).

Now we have a new KKM space as follows inducted by the concept of p-convexity (see Park [61]).

Lemma 3.2. Suppose that X is a subset of topological vector space E and $p \in (0, 1]$, and D is a nonempty subset of X such that $C_p(D) \subset X$. Let $\Gamma_N := C_p(N)$ for each $N \in \langle D \rangle$. Then $(X, D; \Gamma)$ is a ϕ_A-space.

Proof. Since $C_p(D) \subset X$, Γ_N is well-defined. For each $N = \{x_0, x_1, \cdots, x_n\} \subset D$, we define $\phi_N : \Delta_n \to \Gamma_N$ by $\sum_{i=0}^{n} t_i e_i \mapsto \sum_{i=0}^{n} (t_i)^{\frac{1}{p}} x_i$. Then clearly $(X, D; \Gamma)$ is a ϕ_A-space. This completes the proof. □

4. The Fixed Points of Set-Valued Mappings in p-Vector Spaces

In this section, we will establish fixed point theorems for upper semi-continuous mappings for non-compact p-convex subsets in p-vector spaces, which will be a tool used in section 5 to conduct the best approximation for non-self set-valued mappings in p-vector spaces, and related to Rothe and Leray-Schaude typees, for which we will discuss in section 5 below. Here, we first gather together some definitions and known facts needed in this section.

Definition 4.1. Let X and Y be two topological spaces. A set-valued mapping (also saying, multifunction) $T : X \longrightarrow 2^Y$ is a point to set function such that for each $x \in X$, $T(x)$ is a subset of Y. The mapping T is said to be upper semi-continuous (USC) if the subset $T^{-1}(B) := \{x \in X : T(x) \cap B \neq \emptyset\}$ (resp., the set $\{x \in X : T(x) \subset B\}$) is closed (resp., open) for any closed (resp., open) subset B in Y. The function $T : X \to 2^Y$ is said to be lower semi-continuous (LSC) if the set $T^{-1}(A)$ is open for any open subset A in Y.

As an application of KKM principle for general abstract convex spaces with the help of embedding lemma for Hausdorff compact p-convex subsets from topological vector spaces (TVS) into locally p-convex vector spaces, we have the following a general existence result for the "approximation" of fixed points for upper and lower semi-continuous set-valued mappings in p-convex vector spaces for $0 < p \leq 1$ (see the corresponding related results given by Theorem 2.7 of Gholizadeh et al. [33], Theorem 5 of Park [61] and related discussion wherein).

The following result is originally given by given by Yuan [91], here we provide the sketch of its proof for the purpose of reading's self-containing.

Theorem 4.1. Let A be a p-convex compact subset of a locally p-convex vector space X, where $0 < p \leq 1$. Suppose that $T : A \to 2^A$ is lower (resp. upper) semi-continuous with non-empty p-convex values. Then for any given U which is a p-convex neighborhood of zero, there exists $x_U \in A$ such that $T(x_U) \cap (x_U + U) \neq \emptyset$.

Proof. Suppose U is any given element $\mathbf{U}$, which is a symmetric open neighborhood V of U for which $\overline{V} + \overline{V} \subset U$ in locally p-convex neighborhood of zero, we prove the results by two cases for T is LSC and USC.

Case 1, by assuming T is lower semi-continuous: As X is locally p-convex vector space, suppose that $\mathbf{U}$ is the family of neighborhoods of 0 in X. For any element U of $\mathbf{U}$, there is a symmetric open neighborhood V of U for which $\overline{V} + \overline{V} \subset U$. Since A is compact, so there exist $x_0, x_1, \cdots, x_n$ in A such that $A \subset \cup_{i=0}^{n}(x_i + V)$. By using the fact that A is p-convex, we find $D := \{b_0, b_2, \cdots, b_n\} \subset A$ for which $b_i - x_i \in V$ for all $i \in \{0, 1, \cdots, n\}$ and we define C by $C := C_p(D) \subset A$. By the fact that T is LSC, it follows that the subset $F(b_i) := \{c \in C : T(c) \cap (x_i + V) = \emptyset\}$ is open in C (as the set $x_i + V$ is open) for each $i \in \{0, 1, \cdots, n\}$. For any $c \in C$, we have $\emptyset \neq T(c) \cap A \subset T(c) \cap \cup_{i=0}^{n}(x_i + V)$, it follows that $\cap_{i=0}^{n} F(b_i) = \emptyset$. Now applying Lemma 3.1 and Lemma 3.2, which implies that that there is $N := \{b_{i_0}, b_{i_1}, \cdots, b_{i_k}\} \in \langle D \rangle$ and $x_U \in C_p(N) \subset A$ for which $x_U \notin F(N)$, and so $T(x_u) \cap (x_{i_j} + V) \neq \emptyset$ for all $j \in \{0, 1, \cdots, k\}$. As $b_i - x_i \in V$ and $\overline{V} + \overline{V} \subset U$, which imply that $x_{i_j} + \overline{V} \subset b_{i_j} + U$,

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which means that $T(x_U) \cap ((b_{i_j} + U) \neq \emptyset$, it follows that $N \subset \{c \in C : T(x_U) \cap (c + U) \neq \emptyset\}$. By the fact that the subsets $C, T(x_U)$ and U are p-convex, we have that $x_U \in \{c \in C : T(x_U) \cap (c + U) \neq \emptyset\}$, which means that $T(x_U) \cap (x_U + U) \neq \emptyset$.

Case 2, by assuming T is upper semi-continuous: We define $F(b_i) := \{c \in C : T(c) \cap (x_i + \overline{V}) = \emptyset\}$, which is then open in C (as the subset $x_i + \overline{V}$ is closed) for each $i = 0, 1, \cdots, n$. Then the argument is similar to the proof for the case T is USC, and by applying Lemma 3.1 and Lemma 3.2 again, it follows that there exists $x_U \in A$ such that $T(x_U) \cap (x_U + U) \neq \emptyset$. This completes the proof. $\square$

By Theorem 4.1, we have the following Fan-Glicksberg fixed point theorems (Fan [27]) in locally p-convex vector spaces for $(0 < p \leq 1)$, which also improve or generalize the corresponding results given by Yuan [90], Xiao and Lu [85], Xiao and Zhu [86]-[87] into locally p-convex vector spaces.

Theorem 4.2. Let A be a p-convex compact subset of a locally p-convex vector space X, where $0 < p \leq 1$. Suppose that $T : A \to 2^A$ is upper semi-continuous with non-empty p-convex closed values. Then T has one fixed point.

Proof. Assume $\mathbf{U}$ is the family of neighborhoods of 0 in X, and $U \in \mathbf{U}$, by Theorem 4.1, there exists $x_U \in A$ such that $T(x_U) \cap (x_U + U) \neq \emptyset$. Then there exists $a_U, b_U \in A$ for which $b_U \in T(a_U)$ and $b_U \in a_U + U$. Now, two nets $\{a_U\}$ and $\{b_U\}$ in Graph(T), which is a compact graph of mapping T as A is compact and T is semi-continuous, we may assume that a_U has a subnet converging to a, and $\{b_U\}$ has a subnet converging to b. As $\mathbf{U}$ is the family of neighborhoods for 0, we should have $a = b$ (e.g., by the Hausdorff separation property), and $a = b \in T(b)$ due to the fact that Graph(T) is close (e.g., see Lemma 1.1 of Yuan [89]), thus the proof is compete. $\square$

For a given set A in vector space X, we denote by "$lin(A)$" the "linear hull" of A in X.

Definition 4.2. Let A be a subset of a topological vector space X and let Y be another topological vector space. We shall say that A can be linearly embedded in Y if there is a linear map $L : lin(A) \to Y$ (not necessarily continuous) whose restriction to A is a homeomorphism.

The following embedded Lemma 4.1 is a significant result due to Theorem 1 of Kalton [39], which says though not every compact convex set can be linearly imbedded in a locally convex space (e.g., see Roberts [68] and Kalton et al.[41]), but for p-convex sets when $0 < p < 1$, every compact p-convex set in topological vector spaces be considered as a subset of a locally p-convex vector space, hence every such set has sufficiently many p-extreme points.

Secondly, by the property (ii) of Lemma 2.1 above, each convex subset of topological vector space is always p-convex for $0 < p \leq 1$, thus it is possible for us to transfer the problem involved p-convex subsets from topological vector spaces into the locally p-convex vector spaces, which indeed allows us to establish the existence of fixed points for upper semi-continuous set-valued mappings for compact p-convex subsets in topological vector spaces $(0 < p \leq 1)$ to cover case when the underlying is just a topological vector space, which provides the answer for Schauder's conjecture in the affirmative.

Lemma 4.1. Let K be a compact p-convex subset $(0 < p < 1)$ of a topological vector space X. Then, K can be linearly embedded in a locally p-convex topological vector space.

Proof. It is Theorem 1 of Kalton [39], which completes the proof. □

Remark 4.1. At this point, it is important to note that Lemma 4.1 does not hold for $p = 1$. By Theorem 9.6 of Kalton et al.[41], it was shown that the spaces $L_p = L_p(0, 1)$, where $0 < p < 1$, contain compact convex sets with no extreme points, which thus cannot be linearly embedded in a locally convex space, see also Roberts [68].

Now we have the following fixed point theorems of upper semi-continuous set-valued mappings for compact p-convex subsets in topological vector spaces (which may not be locally convex). The single-valued version was first given by Theorem 3.3 of Ennassik and Taoudi [25] applying the p-seminorm argument in p-vector spaces for $0 < p \leq 1$. Here we apply KKM theory to establish the set-valued versions for upper semi-continuous mappings, and the technique skill used here called "functional method".

Theorem 4.3. If K is a nonempty compact p-convex subset of a topological vector space X, with $(0 < p \leq 1)$, then any upper semi-continuous set-valued mappings $T : K \to 2^K$ with non-empty p-convex closed value, has at least a fixed point.

Proof. We complete the argument by following two steps.

First step, by assuming that K is p-convex, with $0 < p < 1$. By Lemma 4.1, it follows that K can be linearly embedded in a locally p-convex space E, which means that there exists a linear map $L : lin(K) \to E$ whose restriction to K is a homeomorphism. Define the mapping $S : L(K) \to L(K)$ by $(Lx) := L(Tx)$ for $x \in X$. This mapping is easily checked to be well defined. It is also continuous since L is a homeomorphism and T is continuous on K. Furthermore, the set $L(K)$ is compact, being the image of a compact set under a continuous mapping. It is also p-convex since it is the image of an p-convex set under a linear mapping. Then, by Theorem 4.2, there exists $x \in K$ such that $S(Lx) = Lx$. As $L(Tx) = Lx$, which implies (since L is a homeomorphism) that $Tx = x$.

Second step, consider $p = 1$, by the fact K is convex. Choose an arbitrary $x_0 \in K$ and put $K_0 := \{x - x_0 : x \in K\}$. Then we have that, K_0 is a compact convex subset of X which contains the zero element. By Lemma 2.1 (ii), we also conclude that K_0 is p-convex for any $p \in (0, 1)$. Now, define the mapping $R : K_0 \longrightarrow K_0$ by $R(x - x_0) := Tx - x_0$ for each $x \in K$. Clearly, R is continuous. Now applying the result from the first step to R, we conclude that there exists $x \in K$ such that $R(x - x_0) = x - x_0$, which implies that $Tx = x$. Thus the proof is complete. □

Remark 4.2. Theorem 4.3 says that each compact p-convex subsets $(0 < p \leq 1)$ in topological vector spaces, has the fixed point property, which does not only include or improve most available results for fixed point theorems in the existing literature as special cases. In particular, the single-valued version of Theorem 4.3 was first given by Ennassik and Taoudi (Theorem 3.3 of [25]) by using the p-seminorm method, here this paper it was given by combing both KKM theory and p-convex structure. Here we also mention a number of related works and authors, see Mauldin [50], Granas and Dugundji [34], Ennassik and Taoudi [26], Park [62]-[63] and references wherein). Indeed Theorem 4.3 provides an answer to Schauder conjecture for set-valued version under the general p-vector spaces which include topological vector spaces as a special of classes.

We recall that for two given topological spaces X and Y, and a set-valued mapping $T : X \to 2^Y$ is said to

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be compact if there is compact subset set C in Y such that $F(X)(=\{y \in F(X), x \in X\})$ is contained in C, i.e., $F(X) \subset C$. Now we have the following non-compact version of fixed point theorems for compact set-valued mappings defined on a general p-convex subset in p-vector spaces for $0 < p \leq 1$.

Theorem 4.4 (Non-Compact Schauder Fixed Point Theorem). If C is a nonempty closed p-convex subset of a vector space E with $(0 < p \leq 1)$ and $T : C \to 2^C$ being an upper semi-continuous set-valued mapping with non-empty closed p-convex values, if $F(C)$ is contained in a compact subset of C, then T has at least one fixed point.

Proof. As T is compact, there exists a compact subset A in C such that $T(C)(:=\{T(x) : x \in C\}) \subset A$. Let $K := \overline{C}_p(A)$, the closure of the p-convex hull of set A in C. Then K is compact p-convex by Proposition 2.4, and the mapping $T : K \to 2^K$ is upper semi-continuous with non-empty closed p-convex values. Now by Theorem 4.3, it follows that T has a fixed point $x \in K \subset C$ such that $x \in T(x)$. This completes the proof. □

As an immediate consequence of Theorem 4.3, we have following result which gives an affirmative answer to Schauder's conjecture for upper semi-continuous version in topological vector spaces (TVS).

Corollary 4.1. If K is a nonempty compact convex subset of a topological vector space X, then any upper semi-continuous set-valued mappings $T : K \to 2^K$ with non-empty closed convex values has at least a fixed point.

Proof. Apply Theorem 4.3 with $p = 1$, this completes the proof. □

Corollary 4.1 also improve or unifies corresponding results given by Askoura and Godet-Thobie [4], Cauty [15], Cauty [16], Chen [22], Isac [38], Li [47], Nhu [51], Okon [52], Park [63], Reich [67], Smart [78], Yuan [90], Theorem 3.3 of Ennssik and Taoudi [26], Theorem 3.14 of Gholizadeh et al.[33], Xiao and Lu [85], Xiao and Zhu [86]-[87] under the framework of topological vector spaces for set-valued mappings instead of single-valued functions.

In this section, as an application of KKM principle for abstract convex spaces with Kalton's remarkable embedded lemma [39] for compact p-convex sets in topological vector spaces, we establish the general fixed point theorems of upper semi-continuous set-valued mappings, which allow us to answer Schauder's conjecture in the affirmative.

In addition, comparing with topological method or related arguments used by Askoura et al.[4], Cauty [15]-[16], Nhu [51], Reich[67] and others, the way used in this paper provides an accessible way for which we called "functional analysis approach" for the study of nonlinear problems under the category of nonlinear functional analysis for p-convex vector spaces $(0 < p \leq 1)$ with the combination of KKM theory as a powerful tool.

5. The Best Approximation and Applications to Birkhoff-Kellogg Alternative in p-Vector Spaces

The goal of this section is first to establish one general best approximation, which in turn is used as a tool to derive the general principle for the existence of solutions for Birkhoff-Kellogg Problems. Here, we recall that since the first Birkhoff-Kellogg theorem was introduced and proved by Birkhoff and Kellogg [10] in 1922 in discussing the existence of solutions for the equation $x = \lambda F(x)$, where λ is a real parameter, and F is a general nonlinear non-self mapping defined on an open convex subset U of a topological vector space E, now the general form of the

Birkhoff-Kellogg problem is to find the so-called an invariant direction for the nonlinear set-valued mappings F, i.e., to find $x_0 \in \overline{U}$ and $\lambda > 0$ such that $\lambda x_0 \in F(x_0)$.

We know that after Birkhoff and Kellogg theorem given by Birkhoff and Kellogg in 1920's, the study on Birkhoff-Kellogg problem has been received a lot of attention by scholars since then, for example, one of the fundamental results in nonlinear functional analysis, called the Leray-Schauder alternative by Leray and Schauder [46] in 1934, was established via topological degree. Thereafter, certain other types of Leray-Schauder alternatives were proved using different techniques other than topological degree, see work given by Granas and Dugundji [34], Furi and Pera [31] in the Banach space setting and applications to the boundary value problems for ordinary differential equations, and a general class of mappings for nonlinear alternative of Leray-Schauder type in normal topological spaces, and also Birkhoff-Kellogg type theorems for general class mappings in TVS by Agarwal et al.[1], Agarwal and O'Regan [2]-[3], Park [59]; in particular, recently by using the Leray-Schauder type coincidence theory to establish some Birkhoff-Kellogg problem, Furi-Pera type results for a general class of mappings by O'Regan [54] and references wherein.

In this section, based on the application of our best approximation first established below, the general principle for the existence of solutions for Birkhoff-Kellogg problems and related nonlinear alternatives will be established, which then also allows us to give general existence of Leray-Schaduer type and related fixed point theorems for non-self mappings in general p-vector spaces for $(0 < p \leq 1)$. These new results given in this paper not only include the corresponding results in the existing literature as special cases, but also expected to be useful tools for the study of nonlinear problems arising from theory to practice.

We also note that the general nonlinear alternative related to the Leray-Schauder alternative under the framework of p-vector spaces for $(0 < p \leq 1)$ in this section, would be useful tools for the study of nonlinear problems. In addition, the corresponding results in the existing literature for Birkhoff-Kellogg problems and the Leray-Schauder alternative have been studied comprehensively by Granas and Dugundji [34], Isac [38], Park [60]-[62], Carbone and Conti [14], Chang and Yen [19], Chang et al.[20]-[21], Kim et al.[42], Shahzad [74]-[72], Singh [76]; and in particular, many general forms recently given by O'Regan [53] and references wherein.

Let C be a subset of a p-vector space E and $x \in E$ for $0 < p \leq 1$. Then the p-Inward set $I^p_C(x)$ and p-Outward set $O^p_C(x)$ are defined by

$I^p_C(x) := \{x + r(y-x) : y \in C,$ for any $r \geq 0$ (1) if $0 \leq r \leq 1,$ with $(1-r)^p + r^p = 1;$ or (2) if $r \geq 1,$ with $(\frac{1}{r})^p + (1 - \frac{1}{r})^p = 1\}$; and

$O^p_C(x) := \{x + r(y-x) : y \in C,$ for any $r \leq 0$ (1) if $0 \leq |r| \leq 1,$ with $(1-|r|)^p + |r|^p = 1;$ or (2) if $|r| \geq 1,$ with $(\frac{1}{|r|})^p + (1 - \frac{1}{|r|})^p = 1\}$.

Obviously, when $p = 1$, the both Inward and Outward sets $I^p_C(x)$, $O^p_C(x)$ are reduced to the definition for the Inward set $I_C(x)$ and the Outward set $O_C(x)$, respectively in topological vector spaces introduced by Halpern and Bergman [36] and used for the study of non-self mappings related to nonlinear functional analysis in the literature. In this paper, we will mainly focus on the study of the p-Inward set $I^p_U(x)$ for the best approximation and related to the boundary condition for the existence of the fixed points in p-vector spaces. By the special property of p-convex

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concept when $p \in (0,1)$ and $p = 1$, we have the following fact.

Lemma 5.1 Let C be a subset of a p-vector space E and $x \in E$, where for $0 < p \leq 1$. Then for both p-Inward and Outward sets $I_C^p(x)$ and $O_C^p(x)$ defined above, we have

(I) when $p \in (0,1)$, $I_C^p(x) = [\{x\} \cup C]$, and $O_C^p(x) = [\{x\} \cup \{2x\} \cup -C]$,

(II) when $p = 1$, in general $[\{x\} \cup C] \subset I_C^p(x)$, and $[\{x\} \cup \{2x\} \cup -C] \subset O_C^p(x)$.

Proof. First, when $p \in (0,1)$, by the definitions of $I_C^p(x)$, the only real number $r \geq 0$ satisfying the equation $(1-r)^p + r^p = 1$ for $r \in [0,1]$ is $r = 0$ or $r = 1$, and when $r \geq 1$, the equation $(\frac{1}{r})^p + (1 - \frac{1}{r})^p = 1$ implies that $r = 1$. The same reason for $O_C^p(x)$, it follows that $r = 0$ and $r = -1$.

Secondly when $p = 1$, all $r \geq 0$, and all $r \leq 0$ satisfy the requirement of definition for $I_C^p(x)$ and $O_C^p(x)$, respectively, thus the proof is compete. $\square$

Now we have the following general existence best approximation result for non-self upper semi-continuous set-valued mappings in p-vector spaces for $(0 < p \leq 1)$.

Theorem 5.1 (Best approximation for inward sets). Let U be an open p-convex subset of the p-vector spaces E $(0 \leq p \leq 1)$ the zero $0 \in U$, and C a closed p-convex subset of E with also zero $0 \in C$, and assume $F : \overline{U} \cap C \to 2^C$ is a compact and upper semi-continuous mapping with non-empty closed p-convex values. In addition, for each $x \in \partial_C U$ with $y \in F(x)$ (actually only requiring that for $y \in F(x) \cap (C \setminus \overline{U})$), $P_U^{\frac{1}{p}}(y) - 1 \leq P_U^{\frac{1}{p}}(y - x)$ for $0 < p \leq 1$ (this is trivial when $p = 1$). Then there exist $x_0 \in \overline{U} \cap X$ and $y_0 \in F(x_0)$ such that $P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, \overline{I_{\overline{U}}^p(x_0)} \cap C)$, where P_U is the Minkowski p-functional of U. More precisely, we have the following either (I) or (II) holding:

(I) F has a fixed point $x_0 \in U \cap C$ (i.e., $0 = P_U(y_0 - x_0) = P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, \overline{I_{\overline{U}}^p(x_0)} \cap C)$);

(II) there exists $x_0 \in \partial_C(U)$ and $y_0 \in F(x_0)$ with

$$P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, \overline{I_{\overline{U}}^p(x_0)} \cap C) > 0.$$

Proof. As the mapping F is compact, there exists a compact subset $C_1 \subset C$ such that $F(\overline{U} \cap C) \subset C_1$. Now let $C_0 := Cp(\{0\} \cup C_1)$, the closure of the p-convex hull of set $\{0\} \cup C_1$ in E. Then C_0 is compact p-convex in E by Proposition 2.4 (see also Proposition 6.7.2 and 6.7.3 of Jarchow [37]), and also $C_0 \subset C$. By the fact that $0 \in U \cap C_0$ and we know that $U \cap C_0 \neq \emptyset$. Now let $r : E \to U$ defined by

$$r(x) := \frac{x}{\max\{1, (P_U(x))^{\frac{1}{p}}\}}$$

for each $x \in E$, where P_U is the Minkowski p-functional of U. Since $0 \in int(U) = U$, it follows that r is continuous by Lemma 2.2 and Since C_0 is p-convex and $0 \in C_0$, we have that $r(C_0) \subset C_0$, thus $r(U \cap C_0) \subset U \cap C_0$. Let $X_0 := \overline{U} \cap C_0$ and f be the restriction of r to X_0 and let $T(x) := F \circ f(x)$ for each $x \in C$, we have that $T(X_0) \subset X_0$. By the fact that X_0 is compact p-convex, and the mapping $T : X_0 \to 2^{X_0}$ is upper semi-continuous with non-empty compact p-convex values, now by Theorem 4.4, the mapping $T = F \circ f$ has a fixed point $z_0 \in X_0 = \overline{U} \cap C_0 \subset \overline{U} \cap C$ such that $z_0 \in T(z_0) = F \circ f(z_0)$. We define $x_0 := f(z_0) \in X_0$, then $x_0 \in (f \circ F)(x_0)$ and thus there exists $y_0 \in F(x_0)$ such that $x_0 = f(y_0)$.

Now we have the two cases: (i) $y_0 \in \overline{U} \cap C_0 \subset \overline{U} \cap C$; or (ii) $y_0 \in C \setminus \overline{U}$.

First, if $y_0 \in \overline{U} \cap C$, then $x_0 = f(y_0) = y_0$ by the definition of f is the restriction of r on $X = \overline{U} \cap C$, and then we have the following best approximation of Fan's type (as $x_0 \in \overline{U} \cap C \subset \overline{I^p_{\overline{U}}(x_0)} \cap C)$)

$$P_U(y_0 - x_0) = 0 = d_P(y_0, \overline{U} \cap C) = d_p(y_0, \overline{I_{\overline{U}}(x_0)} \cap C),$$

which means that $x_0 = y_0 \in F(x_0)$ which is a fixed point of the mapping F defined on $\overline{U} \cap C$.

Second, if $y_0 \in C \setminus \overline{U}$, then $x_0 = f(y_0) = \frac{y_0}{(P_U(y_0))^{\frac{1}{p}}}$, which means that $P_U(x_0) = P_U(\frac{y_0}{(P_U(y_0))^{\frac{1}{p}}}) = \frac{P_U(y_0)}{P_U(y_0)} = 1$, thus we have that $x_0 \in \partial_C(U)$ by Lemma 2.2. Now for any $x \in \overline{U} \cap C$, it follows that $P_U(x) \leq 1$ and note that

$$P_U(y_0 - x_0) = P_U(y_0 - \frac{y_0}{(P_U(y_0))^{\frac{1}{p}}}) = \frac{((P_U(y_0))^{\frac{1}{p}} - 1)^p}{P_U(y_0)} P_U(y_0) = ((P_U(y_0))^{\frac{1}{p}} - 1)^p.$$

On the other hand, for $0 < p \leq 1$, by the assumption we have $((P_U(y_0))^{\frac{1}{p}} - 1)^{\frac{1}{p}} \leq P_U(y_0 - x)$. By the following inequality

$$P_U(y_0) - 1 \leq P_U(y_0) - P_U(x) = P_U((y_0 - x) + x) - P_U(x) \leq P_U(y_0 - x),$$

we show that $P_U(x_0 - y_0) \leq P_U(y_0 - x)$ for any $x \in \overline{U} \cap C$. Thus when $y_0 \notin \overline{U}$, we have the following best approximation of Fan's type:

$$P_U(y_0 - x_0) = \inf\{P_U(y_0 - z) : z \in \overline{U} \cap C\} = d_P(y_0, \overline{U} \cap C) > 0,$$

and we also have that when $y_0 \notin \overline{U}$, it has $P_U(y_0 - x_0) = ((P_U(y_0))^{\frac{1}{p}} - 1)^p > 0$ for $0 < p \leq 1$.

Now we go to show indeed the following best approximation of Fan's type is also true:

$$P_U(y_0 - x_0) = \inf\{P_U(y_0 - z) : z \in \overline{U} \cap C\} = d_P(y_0, \overline{U} \cap C) = d_P(y_0, I^p_{\overline{U}}(x_0) \cap C) = d_P(y_0, \overline{I^p_{\overline{U}}(x_0)} \cap C).$$

By the fact that $\overline{U} \cap C) \subset I^p_{\overline{U}}(x_0) \cap C$, let $z \in I^p_{\overline{U}}(x_0) \cap C \setminus (\overline{U} \cap C)$, we first claim that $P_U(y_0 - x_0) \leq P_U(y_0 - z)$.

If not, we have $P_U(y_0 - x_0) > P_U(y_0 - z)$. As $z \in I^p_{\overline{U}}(x_0) \cap C \setminus (\overline{U} \cap C)$, there exists $y \in \overline{U}$ and a non-negative number c (actually $c \geq 1$ as shown soon below) with $z = x_0 + c(y - x_0)$. Since $z \in C$, but $z \notin \overline{U} \cap C$, it implies that $z \notin \overline{U}$. By the fact that $x_0 \in \overline{U}$ and $y \in \overline{U}$, we must have the constant $c \geq 1$; otherwise, it implies that $z(= (1-c)x_0 + cy) \in \overline{U}$, this is impossible as by our assumption, $z \notin \overline{U}$. Therefore we know that $c \geq 1$, this implies that $y = \frac{1}{c}z + (1 - \frac{1}{c})x_0 \in C$ (as both $x_0 \in C$ and $z \in C$). On the other hand, as $z \in I^p_{\overline{U}}(x_0) \cap C \setminus (\overline{U} \cap C)$, and $c \geq 1$ with $(\frac{1}{c})^p + (1 - \frac{1}{c})^p = 1$, it then follows that

$$P_U(y_0 - y) = P_U[\frac{1}{c}(y_0 - z) + (1 - \frac{1}{c})(y_0 - x_0)] \leq [(\frac{1}{c})^p P_U(y_0 - z) + (1 - \frac{1}{c})^p P_U(y_0 - x_0)] < P_U(y_0 - x_0),$$

which contradicts that $P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C)$ as shown above since $y \in \overline{U} \cap C$, we must have $P_U(y_0 - x_0) \leq P_U(y_0 - y)$! This helps us to complete the claim: $P_U(y_0 - x_0) \leq P_U(y_0 - z)$ for any $z \in I^p_{\overline{U}}(x_0) \cap C \setminus (\overline{U} \cap C)$, which means that the following best approximation of Fan's type holding:

$$0 < d_P(y_0, \overline{U} \cap C) = P_U(y_0 - x_0) = d_p(y_0, I^p_{\overline{U}}(x_0) \cap C).$$

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Now by the continuity of P_U, it follows that the following best approximation of Fan type is also true:

$$0 < P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, I^p_{\overline{U}}(x_0) \cap C) = d_p(y_0, \overline{I^p_{\overline{U}}(x_0)} \cap C).$$

The proof is complete. □

Remark 5.1. Based on the Proof of Theorem 5.1, we have that: 1) For the condition "$x \in \partial_C U$ with $y \in F(x)$, $P_U^{\frac{1}{p}}(y) - 1 \leq P_U^{\frac{1}{p}}(y - x)$ for $0 < p \leq 1$", indeed we only need that for "$x \in \partial_C U$ with $y \in F(x) \cap (C \setminus \overline{U})$), $P_U^{\frac{1}{p}}(y) - 1 \leq P_U^{\frac{1}{p}}(y - x)$ for $0 < p \leq 1$"; and 2) when p=1, we have the similar best approximation result based for TVS based on outward sets below (see Theorem 3 of Park [58] and related discussion by references wherein).

For the p-vector space with $p = 1$ being a topological vector space E, we have the following best approximation for the outward set $\overline{O_{\overline{U}}(x_0)}$ based on the point $\{x_0\}$ respect the convex subset U in E.

Theorem 5.2 (Best approximation for outward sets). Let U be an open convex subset of the topological vector spaces E (i.e., the p-vector space with $p = 1$) the zero $0 \in U$, and C a closed convex subset of E with also zero $0 \in C$, and assume $F : \overline{U} \cap C \to 2^C$ is a compact and upper semi-continuous mapping with non-empty closed convex values. Then there exist $x_0 \in \overline{U} \cap X$ and $y_0 \in F(x_0)$ such that $P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, \overline{O_{\overline{U}}(x_0)} \cap C)$, where P_U is the Minkowski p-functional of U. More precisely, we have the following either (I) or (II) holding:

(I) F has a fixed point $x_0 \in U \cap C$ (i.e., $0 = P_U(y_0 - x_0) = P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, \overline{O_{\overline{U}}(x_0)} \cap C)$);

(II) there exists $x_0 \in \partial_C(U)$ and $y_0 \in F(x_0)$ with

$$P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, O_{\overline{U}}(x_0) \cap C) = d_p(y_0, \overline{O_{\overline{U}}(x_0)} \cap C) > 0.$$

Proof. We define a new mapping $F_1 : \overline{U} \cap C \to 2^C$ by $F_1(x) := \{2x\} - F(x)$ for each $x \in \overline{U} \cap C$, then F_1 is also compact and upper semi-continuous mapping with non-empty closed convex values, and F_1 satisfies all hypothesis of Theorem 5.1 wit $p = 1$. By Theorem 5.1, there exists $x_0 \in \overline{U} \cap X$ and $y_1 \in F_1(x_0)$ such that $P_U(y_1 - x_0) = d_P(y_1, \overline{U} \cap C) = d_p(y_1, \overline{I_{\overline{U}}(x_0)} \cap C)$. More precisely, we have the following either (I) or (II) holding:

(I) F_1 has a fixed point $x_0 \in U \cap C$ (so $0 = P_U(y_1 - x_0) = P_U(y_1 - x_0) = d_P(y_1, \overline{U} \cap C) = d_p(y_1, \overline{I_{\overline{U}}(x_0)} \cap C)$);

(II) there exists $x_0 \in \partial_C(U)$ and $y_1 \in F_1(x_0)$ with

$$P_U(y_1 - x_0) = d_P(y_1, \overline{U} \cap C) = d_p(y_1, \overline{O_{\overline{U}}(x_0)} \cap C) > 0.$$

Now for any $x \in O_{\overline{U}}(x_0)$, there exist $r < 0, u \in \overline{U}$ such that $x = x_0 + r(u - x_0)$. Let $x_1 = 2x_0 - x$, then $x_1 = 2x_0 - x_0 - r(u - x_0) = x_0 + (-r)(u - x_0) \in I_{\overline{U}}(x_0)$. Let $y_1 = 2x_0 - y_0$, for some $y_0 \in F(x_0)$. As we have $P_U(y_1 - x_0) = d_P(y_1, \overline{U} \cap C) = d_p(y_1, \overline{I_{\overline{U}}(x_0)} \cap C)$, it follows that $P_U(y_1 - x_0) \leq P_U(y_1 - x_1)$, which implies that

$$P_U(x_0 - y_0) = P_U(y_1 - x_0) \leq P_U(y_1 - x_1) = P_U(2x_0 - y_0 - (2x_0 - x)) = P_U(y_0 - x)$$

for all $x \in O_{\overline{U}}(x_0)$. Thus we have $P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, O_{\overline{U}}(x_0) \cap C)$ and by the continuity of P_U, it follows that

$$P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, \overline{O_{\overline{U}}(x_0)} \cap C) > 0.$$

This completes the proof. $\square$

Now by the application of Theorem 5.1 and Theorem 5.2, the Remark 5.1 and the argument used in Theorem 5.1, we have the the following general principle for the existence of solutions for Birkhoff-Kellogg Problems in p-Vector Spaces, where $(0 < p \leq 1)$.

Theorem 5.3 (The Principle of Birkhoff-Kellogg alternative). Let U be an open p-convex subset of the p-vector spaces E $(0 \leq p \leq 1)$ with the zero $0 \in U$, and C a closed p-convex subset of E with also zero $0 \in C$. Assume the $F : \overline{U} \cap C \to 2^C$ is a compact and upper semi-continuous mapping with non-empty closed p-convex values, then it has at least one of the following two properties:

(I) F has a fixed point $x_0 \in U \cap C$ such that $x_0 \in F(x_0)$,

(II) there exists $x_0 \in \partial_C(U)$ and $y_0 \in F(x_0)$ and $\lambda \in (0,1)$ such that $x_0 = \lambda y_0 \in \lambda F(x_0)$. In addition, if each $x \in \partial_C U$ with $y \in F(x)$, $P_U^{\frac{1}{p}}(y) - 1 \leq P_U^{\frac{1}{p}}(y - x)$ (this is trivial when $p = 1$; and actually only requiring for $y \in F(x) \cap (C \diagdown \overline{U})$), $P_U^{\frac{1}{p}}(y) - 1 \leq P_U^{\frac{1}{p}}(y - x)$), the best approximation between x_0 and y_0 is given by $P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, \overline{I_{\overline{U}}^p(x_0)} \cap C) > 0$.

Proof. If (I) is not the case, then (II) is proved by the Remark 5.1 and by following the proof in Theorem 5.1 for the case (ii): $y_0 \in C \diagdown \overline{U}$, with $y_0 = f(x_0) \in f(x_0)$. Indeed, As $y_0 \notin \overline{U}$, it follows that $P_U(y_0) > 1$, and $x_0 = f(y_0) = y_0 \frac{1}{(P_U(y_0))^{\frac{1}{p}}}$. Now let $\lambda = \frac{1}{(P_U(y_0))^{\frac{1}{p}}}$, we have $\lambda < 1$ and $x_0 = \lambda y_0$ with $y_0 \in F(x_0)$. Finally, the additionally assumption in (II) allows us to have the best approximation between x_0 and y_0 obtained by following the proof of Theorem 5.1 as $P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, \overline{I_{\overline{U}}^p(x_0)} \cap C) > 0$. This completes the proof. $\square$

As an application of Theorem 5.2 for the non-self set-valued mappings discussed in Theorem 5.3 with outward set condition, we have the following general principle of Birkhoff-Kellogg alternative in topological vector spaces (TVS).

Theorem 5.4 (The Principle of Birkhoff-Kellogg alternative in TVS). Let U be an open convex subset of the topological vector space E (i.e., $p = 1$) with the zero $0 \in U$, and C a closed convex subset of E with also zero $0 \in C$. Assume the $F : \overline{U} \cap C \to 2^C$ is a compact and upper semi-continuous mapping with non-empty closed convex values, then it has at least one of the following two properties:

(I) F has a fixed point $x_0 \in U \cap C$ such that $x_0 \in F(x_0)$,

(II) there exists $x_0 \in \partial_C(U)$ and $y_0 \in F(x_0)$ and $\lambda \in (0,1)$ such that $x_0 = \lambda y_0$. In addition, the best approximation between x_0 and y_0 is given by $P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, \overline{I_{\overline{U}}(x_0)} \cap C) > 0$.

On the other hand, by the Proof of Theorems 5.1 and 5.2, we note that for case (II) of Theorem 5.2, the assumption "each $x \in \partial_C U$ with $y \in F(x)$, $P_U^{\frac{1}{p}}(y) - 1 \leq P_U^{\frac{1}{p}}(y - x)$" is only used to guarantee the best approximation "$P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, \overline{I_{\overline{U}}^p(x_0)} \cap C) > 0$", thus we have the following Leray-Schauder alternative in p-vector spaces, which, of course, includes the corresponding results in TVS as special cases.

Theorem 5.5 (Leray-Schauder Nonlinear Alternative). Let C a closed p-convex subset of p-seminorm vector space E with $0 \leq p \leq 1$ and the zero $0 \in C$. Assume the $F : C \to 2^C$ is a compact and upper semi-continuous mapping with non-empty closed p-convex values, and let $\varepsilon(F) := \{x \in C : x \in \lambda F(x),\ \text{for some } 0 < \lambda < 1\}$. Then either F has a fixed point in C or the set $\varepsilon(F)$ is unbounded.

Proof. By assuming the case (I) is not true, i.e., F has no fixed point, then we claim that the set $\varepsilon(F)$ is unbounded. Otherwise, assume the set $\varepsilon(F)$ is bounded. and assume P is the continuous p-seminorm for E, then there exists $r > 0$ such that the set $B(0, r) := \{x \in E : P(x) < r\}$, which contains the set $\varepsilon(F)$, i.e., $\varepsilon(F) \subset B(0, r)$, which means for any $x \in \varepsilon(F)$, $P(x) < r$. Then $B(0.r)$ is an open p-convex subset of E and the zero $0 \in B(0, r)$ by Lemma 2.2 and Remark 2.4. Now let $U := B(0, r)$ in Theorem 5.3, it follows that for the mapping $F : B(0, r) \cap C \to 2^C$ satisfies all general conditions of Theorem 5.3, and we have that any $x_0 \in \partial_C B(0, r)$, no any $\lambda \in (0, 1)$ such that $x_0 = \lambda y_0$, where $y_0 \in F(x_0)$. Indeed, for any $x \in \varepsilon(F)$, it follows that $P(x) < r$ as $\varepsilon(F) \subset B(0, r)$, but for any $x_0 \in \partial_C B(0, r)$, we have $P(x_0) = r$, thus the conclusion (II) of Theorem 5.3 does not have hold. By Theorem 5.3 again, F must have a fixed point, but this contradicts with our assumption that F is fixed point free. This completes the proof. □

Now assume a given p-vector space E equipped with the P-seminorm (by assuming it is continuous at zero) for $0 < p \leq 1$, then we know that $P : E \to R^+$, $P^{-1}(0) = 0$, $P(\lambda x) = |\lambda|^p P(x)$ for any $x \in E$ and $\lambda \in R$. Then we have the following useful result for fixed points due to Rothe and Altman types in p-vector spaces, which plays important roles for optimization problem, variational inequality, complementarity problems.

Corollary 5.1. Let U be an open p-convex subset of p-vector space E and zero $0 \in U$, plus C is a closed p-convex subset of E with $U \subset C$, where $0 < p \leq 1$. Assume that $F : \overline{U} \to C$ is a compact single-valued mapping with non-empty closed p-convex values. If one of the following is satisfied,

(1) (Rothe type condition): $P_U(F(x)) \leq P_U(x)$ for all $x \in \partial U$,

(2) (Petryshyn type condition): $P_U(x) \leq P_U(F(x) - x)$ for all $x \in \partial U$,

(3) (Altman type condition): $|P_U(F(x))|^{\frac{2}{p}} \leq [P_U(F(x) - x)]^{\frac{2}{p}} + [P_U(x)]^{\frac{2}{p}}$ for all $x \in \partial U$,

then F as a fixed point.

Proof. By the conditions (1), (2) and (3), it follows that the conclusion of (II) in Theorem 5.3 "there exist $x_0 \in \partial_C(U)$ and $\lambda \in (0, 1)$ such that $x_0 = \lambda F(x_0)$" does not hold, thus by the alternative of Theorem 5.3, F has a fixed point. This completes the proof. □.

By the fact that when $p = 1$, each p-vector space is a topological vector space, and thus we have the following classical Fan's best approximation result as a powerful tool for the study in the optimization, mathematical programming, games theory, and mathematical economics, and others related topics in applied mathematics.

Corollary 5.2 (Fan's best approximation). Let U be an open convex subset of a topological space E with the zero $0 \in U$, and C a closed convex subset of E with also zero $0 \in C$, and assume $F : \overline{U} \cap C \to 2^C$ is a compact and upper semi-continuous mapping with non-empty closed convex values, and assume P_U being the Minkowski p-functional of U in E. Then there exist $x_0 \in \overline{U} \cap X$ and $y_0 \in T(x_0)$ such that $P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, \overline{I_{\overline{U}}(x_0)} \cap C)$. More precisely, we have the following either (I) or (II) holding, where $W_{\overline{U}}(x_0)$ is either inward set $I_{\overline{U}}(x_0)$, or the outward set $O_{\overline{U}}(x_0)$:

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(I) F has a fixed point $x_0 \in U \cap C$ (so that $0 = P_U(y_0 - x_0) = P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, \overline{W_{\overline{U}}(x_0)} \cap C)$);

(II) there exists $x_0 \in \partial_C(U)$ and $y_0 \in F(x_0)$ with

$$P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, \overline{W_{\overline{U}}(x_0)} \cap C) = P_U(y_0) - 1 > 0.$$

Proof. When $p = 1$, then it automatically satisfies that the inequality: $P_U^{\frac{1}{p}}(y) - 1 \leq P_U^{\frac{1}{p}}(y - x)$, and indeed we have that for $x_0 \in \partial_C(U)$, with $y_0 \in F(x_0)$, we have $P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, \overline{W_{\overline{U}}(x_0)} \cap C) = P_U(y_0) - 1$. The conclusions are given by Theorem 5.1 and Theorem 5.2. The proof is complete. $\square$

We like to point out the similar results on Rothe and Leray-Schauder alternative have been developed by Isac [38], Park [57], Potter [65], Shahzad [72]-[74], Xiao and Zhu [86], and related references wherein as tools of nonlinear analysis in topological vector spaces. As mentioned above, when $p = 1$ and take F as a continuous mapping, then we obtain the version of Lerary-Schauder in TVS and this we omit it statement in details.

6. The Solutions of Birkhoff-Kellogg Problems and Principle of Leray-Schauder Alternatives in *p*-Vector Spaces

As applications of results in Section 5 above, we new establish general results for the existence of solutions for Birkhoff-Kellogg problem, and the principle of Leray-Schauder alternatives in p-vector spaces for $0 < p \leq 1$.

Theorem 6.1 (Birkhoff-Kellogg alternative in p-vector spaces). Let U be an open p-convex subset of the p-vector spaces E (where, $0 \leq p \leq 1$) with the zero $0 \in U$, and C a closed p-convex subset of E with also zero $0 \in C$, and assume $F : \overline{U} \cap C \to 2^C$ is a compact and upper semi-continuous mapping with non-empty closed p-convex values. In addition, for each $x \in \partial_C(U)$ with $y \in F(x)$, $P_U^{\frac{1}{p}}(y) - 1 \leq P_U^{\frac{1}{p}}(y - x)$ for $0 < p \leq 1$ (this is trivial when $p = 1$), where P_U is the Minkowski p-functional of U. Then we have that either (I) or (II) holding below:

(I) there exists $x_0 \in \overline{U} \cap C$ such that $x_0 \in F(x_0)$,

(II) there exists $x_0 \in \partial_C(U)$ with $y_0 \in F(x_0)$ and $\lambda > 1$ such that $\lambda x_0 = y_0 \in F(x_0)$ (i.e., $F(x_0) \cap \{\lambda x_0 : \lambda > 1\} \neq \emptyset$).

Proof. By following the argument and symbols used in the proof of Theorem 5.1, we have that either

(1) F has a fixed point $x_0 \in U \cap C$; or

(2) there exists $x_0 \in \partial_C(U)$ and $y_0 \in F(x_0)$ with $x_0 = f(y_0)$ such that

$$P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, \overline{I_{\overline{U}}(x_0)} \cap C) = P_U(y_0) - 1 > 0,$$

where $\partial_C(U)$ denotes the boundary of U relative to C in E, and f is the restriction of the continuous retraction r respect to the set U in E.

If F has no fixed point, then above (2) holds and $x_0 \notin F(x_0)$. As given by the proof of Theorem 5.1, we have that $y_0 \in F(x_0)$ and $y_0 \notin \overline{U}$, thus $P_U(y_0) > 1$ and $x_0 = f(y_0) = \frac{y_0}{(P_U(y_0))^{\frac{1}{p}}}$, which means $y_0 = (P_U(y_0))^{\frac{1}{p}} x_0$. Let $\lambda = (P_U(y_0))^{\frac{1}{p}}$, then $\lambda > 1$ and we have $\lambda x_0 = y_o \in F(x_0)$. This completes the proof. $\square$.

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Theorem 6.2 (Birkhoff-Kellogg alternative in TVS). Let U be an open convex subset of a topological space E with the zero $0 \in U$, and C a closed convex subset of E with also zero $0 \in C$, and assume $F : \overline{U} \cap C \to 2^C$ is a compact and upper semi-continuous mapping with non-empty closed convex values. Then we have the following either (I) or (II) holding, where $W_{\overline{U}}(x_0)$ is either inward set $I_{\overline{U}}(x_0)$, or the outward set $O_{\overline{U}}(x_0)$:

(I) there exists $x_0 \in \overline{U} \cap C$ such that $x_0 \in F(x_0)$,

(II) there exists $x_0 \in \partial_C(U)$ with $y_0 \in F(x_0)$ and $\lambda > 1$ such that $\lambda x_0 = y_0 \in F(x_0)$ (i.e., $F(x_0) \cap \{\lambda x_0 : \lambda > 1\} \neq \emptyset$).

Proof. When $p = 1$, then it automatically satisfies that the inequality: $P_U^{\frac{1}{p}}(y) - 1 \leq P_U^{\frac{1}{p}}(y - x)$, and indeed we have that for $x_0 \in \partial_C(U)$, with $y_0 \in F(x_0)$, we have $P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, \overline{W_{\overline{U}}(x_0)} \cap C) = P_U(y_0) - 1$. The conclusions are given by Corollary 5.2. The proof is complete. $\square$

Indeed we have the following fixed points for non-self mappings in p-vector spaces for $0 < p \leq 1$ under different boundary conditions.

Theorem 6.3 (Fixed Points of non-self mappings). Let U be an open p-convex subset of the p-vector spaces E (where, $0 \leq p \leq 1$) with the zero $0 \in U$, and C a closed p-convex subset of E with also zero $0 \in C$, and assume $F : \overline{U} \cap C \to 2^C$ is a compact and upper semi-continuous mapping with non-empty closed p-convex values. In addition, for each $x \in \partial_C(U)$ with $y \in F(x)$, $P_U^{\frac{1}{p}}(y) - 1 \leq P_U^{\frac{1}{p}}(y - x)$ for $0 < p \leq 1$ (this is trivial when $p = 1$), where P_U is the Minkowski p-functional of U. If F satisfies any one of the following conditions for any $x \in \partial_C(U) \diagdown F(x)$

(i) For each $y \in F(x)$, $P_U(y - z) < P_U(y - x)$ for some $z \in \overline{I_{\overline{U}}(x)} \cap C$;

(ii) For each $y \in F(x)$, there exists λ with $|\lambda| < 1$ such that $\lambda x + (1 - \lambda) y \in \overline{I_{\overline{U}}(x)} \cap C$;

(iii) $F(x) \subset \overline{I_{\overline{U}}(x)} \cap C$;

(iv) $F(x) \cap \{\lambda x : \lambda > 1\} = \emptyset$;

(v) $F(\partial U)) \subset \overline{U} \cap C$;

(vi) For each $y \in F(x)$, $P_U(y - x) \neq ((P_U(y))^{\frac{1}{p}} - 1)^p$;

then F must has a fixed point.

Proof. By following the argument and symbols used in the proof of Theorem 5.1, we have that either

(1) F has a fixed point $x_0 \in U \cap C$; or

(2) there exists $x_0 \in \partial_C(U)$ and $y_0 \in F(x_0)$ with $x_0 = f(y_0)$ such that

$$P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = d_p(y_0, \overline{I_{\overline{U}}(x_0)} \cap C) = P_U(y_0) - 1 > 0,$$

where $\partial_C(U)$ denotes the boundary of U relative to C in E, and f is the restriction of the continuous retraction r respect to the set U in E.

First, suppose that F satisfies the condition (i), if F has no fixed point, then above (2) holds and $x_0 \notin F(x_0)$. Then by the condition (i), it follows that $P_U(y_0 - z) < P_U(y_0 - x_0)$ for some $z \in \overline{I_{\overline{U}}(x)} \cap C$, this contradicts with the best approximation equations given by (2) above, thus F mush have a fixed pint.

Second, suppose that F satisfies the condition (ii), if F has no fixed point, then above (2) holds and $x_0 \notin F(x_0)$.

Then by the condition (ii), there exists $\lambda > 1$ such that $\lambda x_0 + (1-\lambda)y_0 \in \overline{I_{\overline{U}}(x)} \cap C$. It follows that

$$P_U(y_0 - x_0) \leq P_U(y_0 - (\lambda x_0 + (1 - \lambda y_0)) = P_U(\lambda(y_0 - x_0)) = |\lambda|^p P_U(y_0 - x_0) < P_U(y_0 - x_0)$$

this is impossible and thus F must have a fixed point in $\overline{U} \cap C$.

Third, suppose that F satisfies the condition (iii), i.e., $F(x) \subset \overline{I_{\overline{U}}(x)} \cap C$;, then the (2), we have that $P_U(y_0 - x_0)$ and thus $x_0 = y_0 \in F(x_0)$, which means F has a fixed point.

Forth, suppose that F satisfies the condition (iv), if if F has no fixed point, then above (2) holds and $x_0 \notin F(x_0)$. As given by the proof of Theorem 5.1, we have that $y_0 \notin \overline{U}$, thus $P_U(y_0) > 1$ and $x_0 = f(y_0) = \frac{y_0}{(P_U(y_0))^{\frac{1}{p}}}$, which means $y_0 = (P_U(y_0))^{\frac{1}{p}} x_0$, where $(P_U(y_0))^{\frac{1}{p}} > 1$, this contradicts with the assumption (iv), thus F must have a fixed point. in $\overline{U} \cap C$.

Fifth, suppose that F satisfies the condition (v), then $x_0 \notin F(x_0)$. As $x_0 \in \partial_C U$, now by the condition (v) $F(\partial U) \subset \overline{U} \cap C$, it follows that for any $y_0 \in F(x_0)$, we have $y_0 \in \overline{U} \cap C$, thus $y \notin \overline{U} \setminus \cap C$, which implies that $0 < P_U(y_0 - x_0) = d_P(y_0, \overline{U} \cap C) = 0$, this is impossible, thus F must have a fixed point. Here, like pointed out by Remark 5.1, we know that based on the condition (v),F has a fixed point by applying $F(\partial U) \subset \overline{U} \cap C$ is enought, not needing the general hypothesis: "for each $x \in \partial_C(U)$ with $y \in F(x)$, $P_U^{\frac{1}{p}}(y) - 1 \leq P_U^{\frac{1}{p}}(y - x)$ for $0 < p \leq 1$".

Finally, suppose that F satisfies the condition (vi), if F has no fixed point, then above (2) holds and $x_0 \notin F(x_0)$. Then the condition (v) implies that $P_U(y_0 - x_0) \neq ((P_U(y))^{\frac{1}{p}} - 1)^p$, but the our proof in Theorem shows that $P_U(y_0 - x_0) = ((P_U(y))^{\frac{1}{p}} - 1)^p$, this is impossible, thus F must have a fixed point. Then the proof is complete. $\square$

Now by taking the set C in Theorem 6.1 as the whole p-vector space E itself, we have the following general results for non-self upper semi-continuous set-valued mappings under the application of Theorem 6.1 above by including Rothe, Petryshyn, Altman and Leray-Schauder types' fixed points as special cases.

Taking $p = 1$ and $C = E$ in Theorem 6.3, we have the following fixed points for non-self upper semi-continuous set-valued mappings associated with inward or outward sets in topological vector spaces (TVS) as follows.

Theorem 6.4 (Fixed Points of non-self mappings with boundary conditions). Let U be an open convex subset of the topological vector spaces E with the zero $0 \in U$, and assume $F : \overline{U} \to 2^E$ is a compact and upper semi-continuous mapping with non-empty closed convex values. If F satisfies any one of the following conditions for any $x \in \partial(U) \setminus F(x)$

(i) For each $y \in F(x)$, $P_U(y - z) < P_U(y - x)$ for some $z \in \overline{I_{\overline{U}}(x)}$ (or $z \in \overline{O_{\overline{U}}(x)}$)

(ii) For each $y \in F(x)$, there exists λ with $|\lambda| < 1$ such that $\lambda x + (1 - \lambda)y \in \overline{I_{\overline{U}}(x)}$ (or, $\overline{O_{\overline{U}}(x)}$).

(iii) $F(x) \subset \overline{I_{\overline{U}}(x)}$ (or $\overline{O_{\overline{U}}(x)}$)

(iv) $F(x) \cap \{\lambda x : \lambda > 1\} = \emptyset$;

(v) $F(\partial(U)) \subset \overline{U}$;

(vi) For each $y \in F(x)$, $P_U(y - x) \neq P_U(y) - 1$;

then F must has a fixed point.

In what follow, based on the best approximation theorem in p-vector space, we will also give fixed point theorems

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for non-self mappings, which would play important roles in nonlinear analysis in p-vector space as shown below to be used to give the principles of Rothe type and Leray-Schauder alternatives in next section.

First, as discussed by Remark 5.1, the proof of Theorem 5.2, with the strongly boundary condition "$F(\partial(U)) \subset \overline{U} \cap C$" only, we can prove that F has a fixed point, thus we have the following fixed point theorem of Rothe type in p-vector spaces.

Theorem 6.5 (Rothe Type). Let U be an open p-convex subset of the p-vector spaces E (where, $0 \leq p \leq 1$) with the zero $0 \in U$. Assume $F : \overline{U} \to 2^E$ is a compact and upper semi-continuous mapping with non-empty closed p-convex values with that $F(\partial(U)) \subset \overline{U}$, then F must has a fixed point.

Now as applications of Theorem 6.5, we give the following Leray-Schauder Alternative in p-vector spaces for non-self set-valued mappings associated with the boundary condition which often appear in the applications.

Theorem 6.6 (The Leray-Schauder Alternative in p-Vector Spaces). Let E be a p-convex vector space, where $0 < p \leq 1$, $B \subset E$ a closed p-convex such that $0 \in intB$. Let $F : [0,1] \times B \to E$ be upper semi-continuous set-valued with non-empty closed p-convex values, and the set $F([0,1] \times B)$ be relatively compact in E. If the following assumptions are satisfied:

(1) $x \notin F(t,x)$ for all $x \notin \partial B$ and $t \in [0,1]$,

(2) $F(\{0\} \times \partial B) \subset B$,

then there is an element $x^* \in B$ such that $x^* \in F(1, x^*)$.

Proof. For any $n \in N$, we consider the mapping

$$F_n(x) = \begin{cases} F(\dfrac{1 - P_B(x)}{\epsilon_n}, \dfrac{x}{P_B(x)}), & \text{if } 1 - \epsilon \leq P_B(x) \leq 1, \\ F(1, \dfrac{X}{1 - \epsilon_n}), & \text{if } P_B(x) < 1 - \epsilon_n, \end{cases} \tag{3}$$

where P_B is the Minkowski p-functional of B and $\{\epsilon_n\}_{n \in N}$ is a sequence of real numbers such that $\lim_{n \to \infty} \epsilon_n = 0$ and $0 < \epsilon_n < \frac{1}{2}$ for any $n \in N$. We observe that for each $n \in N$, the mapping F_n is upper semi-continuous with non-empty closed p-convex values on B and $F_n(B)$ is relatively compact in E. From assumption (2), we have that $f_n(\partial B) \subset B$, and the assumptions of Theorem 6.5 (see also Theorem 4.4) are satisfied, then for each $n \in N$, there exists an element $u_n \in B$ such that $u_n \in F_n(u_n)$.

We first prove the following statement: "It is impossible to have an infinite number of the elements u_n satisfy the following inequality: $1 - \epsilon_n \leq P_B(u_n) \leq 1$."

If not, we assume to have an infinite number of the elements u_n satisfy the following inequality:

$$1 - \epsilon_n \leq P_B(u_n) \leq 1.$$

As $F_n(B)$ is relatively compact and by the definition of mappings F_n, we have that $\{u_n\}_{n \in N}$ is contained in a compact set in E. Without loss of the generality (indeed, each compact set is also countably compact), we define the sequence $\{t_n\}_{n \in N}$ by $t_n := \frac{1 - P_B(u_n)}{\epsilon}$ for each $n \in N$. Then we have that $\{t_n\}_{n \in N} \subset [0,1]$ and we may assume

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that $\lim_{n\to\infty} t_n = t \in [0,1]$. The corresponding subsequence of $\{u_n\}_{n\in N}$ is denoted again by $\{u_n\}_{n\in N}$ and it also satisfies the inequality: $1-\epsilon_n \leq P_B(u_n) \leq 1$, which implies that $\lim_{n\to\infty} P_B(u_n) = 1$.

Now let u^* be an accumulation point of $\{u_n\}_{n\in N}$, thus have $\lim_{n\to\infty}(t_n, \frac{u_n}{P_B(u_n)}, u_n) = (t, u^*, u^*)$. By the fact that F is compact, we have assume that $u_n \in F(t_n, \frac{u_n}{P_B(u_n)})$ for each $n \in N$, it follows that $u^* \in F(t, u^*)$, this contradicts with the assumption (1) as we have $\lim_{n\to\infty} P_B(u_n) = 1$ (which means that $u^* \in \partial B$, this is impossible).

Thus it is impossible to have that "to have an infinite number of elements u_n satisfy the inequality: $1-\epsilon_n \leq P_B(u_n) \leq 1$" , which means that there is only a finite number of elements of sequence $\{u_n\}_{n\in N}$ satisfying the inequality: $1-\epsilon_n \leq P_B(u_n) \leq 1$. Now, without loss of the generality, for $n \in N$, we have the following inequality:

$$P_B(u_n) < 1-\epsilon_n.$$

By the fact that $\lim_{n\to}(1-\epsilon_n) = 1$, $u_n \in F(1, \frac{u_n}{1-\epsilon})$ for all $n \in N$ and assume that $\lim_{n\to} u_n = u^*$, then the upper semi-continuity of F with non-empty closed values implies that the graph of F is closed, and by the fact $u_n \in F(1, \frac{u_n}{1-\epsilon})$, it implies that $u^8 \in F(1, u^*)$. This completes the proof. □

As a special case of Theorem 6.6, we have the following principle for the implicit form of Leray-Schauder type alternative for set-valued mappings in p-vector spaces for $0 < p \leq 1$.

Corollary 6.1 (The Implicit Leray-Schauder Alternative). Let E be a p-convex vector space, where $0 < p \leq 1$, $B \subset E$ a closed p-convex such that $0 \in intB$. Let $F : [0,1] \times B \to E$ be continuous with non-empty values, and the set $F([0,1] \times B)$ be relatively compact in E. If the following assumptions are satisfied:

(1) $F(\{0\} \times \partial B) \subset B$,

(2) $x \notin F(0, x)$ for all $x \in \partial B$,

then at least one of the following properties is satisfied:

(i) there exists $x^* \in B$ such that $x^* \in F(1, x^*)$; or

(ii) there exists $(\lambda^*, x^*) \in (0,1) \times \partial B$ such that $x^* \in F(\lambda^*, x^*)$.

Proof. The result is an immediate consequence of Theorem 6.6, this completes the proof. □

We like to point out the similar results on Rothe and Leray-Schauder alternative have been developed by Isac [38], Park [57], Potter [65], Shahzad [72]-[74], and related references wherein as tools of nonlinear analysis in topological vector spaces.

When $p = 1$ and take F as a continuous mapping, then we obtain the version of Lerary-Schauder in TVS and this we omit it statement in details.

Before closing the discussion of this paper, we like to point out that the best approximation, fixed point theorems and the general principle of Leray-Schauder alternative for non-self mappings established in this paper would play important roles for the nonlinear analysis under the framework of p-vector spaces, as did before for nonlinear analysis for topological vector spaces.

As the goal of this paper is develop a number of new analysis tools in nature in the category of nonlinear analysis under the general framework of p-vector spaces for $(0 < p \leq 1)$, and they are expected to become useful tools for the study on optimization, nonlinear programming, variational inequality, complementarity, game theory, mathematical economics, and related other social science area. In particular, we first establish one best approximation, acting as a tool to establish the principle of nonlinear alternative, which then allows us to give general principle of nonlinear alternative under the general framework of p-vector spaces for $(0 < p \leq 1)$. The results given in this paper on the principle of nonlinear alterative and Leray-Schauder type not only include the corresponding results in the existing literature as special cases, but also would become very useful new tools for many new problems from both social science, engineering, applied mathematics and related areas.

Finally, like what mentioned in the beginning of this paper, we do expect that nonlinear results and principles of the best approximation theorem established in this paper would play a very important role for the nonlinear analysis under the general framework of p-vector spaces for $(0 < p \leq 1)$, as shown by those results give in Section 5 and Section 6 above for the fixed point theorems of non-self mappings, principle of nonlinear alternative, Rothe type, Leray-Schauder alternative which do not include corresponding results in the existing literature as special cases, but also would be important tools for the study of optimization, nonlinear programming, variational inequality, complementarity, game theory, mathematical economics and related topics and area forthcoming.

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Compliance with Ethical Standards

The author declares that he has no conflict of interest.

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