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Article

Self-Consistent Maxwell-Dirac Theory

Sergey A. Rashkovskiy

Institute for Problems in Mechanics of the Russian Academy of Sciences, Vernadskogo Ave., 101/1, Moscow, 119526, Russia; rash@ipmnet.ru

Abstract: We show that quantum mechanics can be constructed as a classical field theory that correctly describes all basic quantum effects, by combining the Maxwell and Dirac equations. It is shown that for a self-consistent union of the Maxwell and Dirac equations into a unified classical field theory, it is necessary to introduce an additional short-range tensor field, which compensates the intrinsic electrostatic field of the electron wave inside the atom. For the combined Maxwell-Dirac field, the stress-energy tensor is constructed. We show that in the nonrelativistic limit this theory naturally transforms into the self-consistent Maxwell-Pauli theory and allows describing all basic quantum effects in the framework of classical field theory without any quantization.

Keywords: quantum mechanics; classical field theory; unified Maxwell-Dirac theory; stress-energy tensor

1. Introduction

It was shown in [1-6] that the basic quantum effects can be completely and consistently described within the framework of classical field theory without any quantization, if the electron field described by the wave function is also considered in the spirit of classical field theory, and the wave equations of quantum mechanics (Schrödinger, Pauli, Klein-Gordon and Dirac equations) be considered as field equations of the classical electron field by analogy with how Maxwell equations are field equations of the classical electromagnetic field. At the same time, as shown in [2-6], in order to obtain the correct spectra of spontaneous emission and describe the basic quantum effects, the Schrödinger equation should not include the intrinsic electrostatic field of the electron wave (i.e., the electrostatic field created by the electron wave itself), but the non-stationary (radiative) component of the intrinsic electromagnetic field of the electron wave should be included [1-6]. The results of works [1-6] indicate the possibility of constructing a unified classical theory of electromagnetic and electron fields. However, a simple formal combination of the equations of classical electrodynamics and quantum mechanics leads to incorrect spontaneous emission spectra. This is due to the fact that in the nonlinear Schrödinger (Dirac, Pauli, Klein-Gordon) equation, which is obtained as a result of such a combination, the potential of the own electrostatic field of the electron wave appears, which changes the eigenvalues of the equation compared to the usual (linear) wave equation. In works [2-6], the nonlinear Schrödinger equation was considered, in which the own electrostatic field of the electron wave was artificially discarded, which is unnatural. It seems more natural to combine quantum mechanics and classical Maxwell electrodynamics within the framework of classical field theory, in which the intrinsic electrostatic field of the electron wave will automatically be excluded from the equation describing the electron field, while the nonstationary (radiative) component of the intrinsic electromagnetic field will be taken into account.

In [7], a variant of combining the Maxwell and Pauli classical fields is proposed, which has the above properties. The theory [7] is described by the equations

$$i\hbar\frac{\partial\psi}{\partial t}=\left[\frac{1}{2m_e}\left(\frac{\hbar}{i}\nabla+\frac{e}{c}\mathbf{A}_\Sigma\right)^2-e\varphi_\Sigma-e\boldsymbol{\sigma}\mathbf{G}+\frac{e\hbar}{2m_ec}\boldsymbol{\sigma}\mathbf{H}_\Sigma\right]\psi \quad (1)$$

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$$\frac{1}{c^2}\frac{\partial^2 \mathbf{G}}{\partial t^2} - \Delta \mathbf{G} + \varkappa^2 \mathbf{G} = 4\pi e(\psi^* \boldsymbol{\sigma} \psi) \tag{2}$$

$$\mathbf{H}_\Sigma = \mathbf{H} + \mathbf{H}_e \tag{3}$$

$$\varphi_\Sigma = \varphi + \varphi_e, \qquad \mathbf{A}_\Sigma = \mathbf{A} + \mathbf{A}_e \tag{4}$$

$$\mathrm{rot}\mathbf{H}_e = \frac{1}{c}\frac{\partial \mathbf{E}_e}{\partial t} + \frac{4\pi}{c}\mathbf{j} \tag{5}$$

$$\mathrm{div}\mathbf{E}_e = 4\pi\rho \tag{6}$$

where

$$\mathbf{H}_e = \mathrm{rot}\mathbf{A}_e, \qquad \mathbf{E}_e = -\frac{1}{c}\frac{\partial \mathbf{A}_e}{\partial t} - \nabla\varphi_e \tag{7}$$

$$\rho = -e\psi^*\psi \tag{8}$$

$$\mathbf{j} = \frac{e\hbar}{2m_e i}[(\nabla\psi^*)\psi - \psi^*\nabla\psi] - \frac{e^2}{m_e c}\mathbf{A}_\Sigma\psi^*\psi - \frac{e\hbar}{2m_e}\mathrm{rot}(\psi^*\boldsymbol{\sigma}\psi) \tag{9}$$

the index "Σ" refers to the total electromagnetic field; the index "e" refers to the intrinsic electromagnetic field created by the charged electron wave; the electron wave itself is considered as a classical field and is described by the spinor ψ; parameters without index refer to the external electromagnetic field, i.e. to a field created by external (with respect to the field ψ) charges and currents; $\varkappa$ is a constant that has the dimension reciprocal of the length and satisfies the condition [7]

$$\varkappa a_B \ll 1 \tag{10}$$

where a_B is the Bohr radius. The constant $\varkappa$, which satisfies condition (10), makes the field $\mathbf{G}$ short-range, exponentially decaying at distances of the order of a_B from the source [7]. The constant $\varkappa$ has been introduced into equation (2) in order to explain why the field $\mathbf{G}$ has not yet been experimentally discovered, and also why the electron field of an atom does not "feel" its own electrostatic field, but at the same time "feels" the electrostatic fields created by other atoms and ions.

The system of equations (1)-(9) is closed and self-consistent. It differs from a simple formal combination of the Maxwell and Pauli equations in that the Pauli equation (1) includes an additional term $-e\boldsymbol{\sigma}\mathbf{G}$ containing a new real vector field $\mathbf{G}$ that satisfies equation (2). As shown in [7], this fundamentally changes the solutions of the combined system of Maxwell-Pauli equations, and allows correctly describing the experimentally observed effects.

The introduction of the $-e\boldsymbol{\sigma}\mathbf{G}$ term into equation (1) can be considered as a renormalization of the theory, however, unlike the formal (artificial) renormalization introduced in QED, this renormalization has a simple and natural meaning, as the effect of some classical field $\mathbf{G}$ satisfying equation (2) , which was previously not taken into account in the theory.

From equations (5) and (6) it follows that ρ and $\mathbf{j}$, determined by relations (8) and (9), play the role of the electric charge density and electric current density of the Pauli field (electron field). Thus, the Pauli field (electron field) has an electric charge and an electric current continuously distributed in space, which are not reduced to point charged particles and their motion. The electric charge of the Pauli field contained in some region Ω of space is equal to $q = \int \rho dV$, where the integral is taken over the region Ω. In particular, the electric charge of the electron field of an electrically neutral atom whose nucleus has the charge Ze is equal to $\int \rho dV = -Ze$, where $Z = 1,2, \ldots$, and the integral is taken over the entire space. Taking into account (8), we come to the conclusion that the wave function in an electrically neutral atom satisfies the normalization condition

$$\int \psi^* \psi dV = Z \tag{11}$$

which only for $Z = 1$ goes over to the normalization of the wave function usually accepted in quantum mechanics.

As shown in [2, 7], the classical electron field has an angular momentum and a magnetic moment, which have two components:

$$\mathbf{L} = \mathbf{L}_{or} + \mathbf{S} \tag{12}$$

$$\mathbf{M} = \mathbf{M}_{or} + \boldsymbol{\mu} \tag{13}$$

where $\mathbf{L}_{or}$ and $\mathbf{M}_{or}$ are the convective (orbital) components of the angular momentum and the magnetic moment of the electron field associated with currents, while $\mathbf{S}$ and $\boldsymbol{\mu}$ are the intrinsic angular momentum (spin) of the electron field and its intrinsic magnetic moment associated with the spin. The intrinsic angular momentum (spin) of the electron field is continuously distributed in space with a density

$$\mathbf{s} = \frac{\hbar}{2} \psi^* \boldsymbol{\sigma} \psi \tag{14}$$

while the intrinsic magnetic moment of the electron field is continuously distributed in space with a density

$$\mathfrak{m} = -\frac{e\hbar}{2m_e c} \psi^* \boldsymbol{\sigma} \psi \tag{15}$$

From (14) and (15), it follows

$$\mathfrak{m} = -\frac{e}{m_e c} \mathbf{s} \tag{16}$$

showing that the internal gyromagnetic ratio of the electron field at all points in space is the same and equal to

$$\gamma_e = -\frac{e}{m_e c} \tag{17}$$

In the theory [1-7], the internal angular momentum (spin) and the associated internal magnetic moment are properties of the classical electron field itself and cannot be reduced to motions and, moreover, to rotations of any particles.

The convective (orbital) components of the angular momentum and magnetic moment of the electron field are related by the equation [7]

$$\mathbf{M}_{or} = -\frac{e}{2m_e c} \mathbf{L}_{or} \tag{18}$$

coinciding with the corresponding relation for classical charged matter (for example, classical particles) [8]. Hence it follows that the convective (orbital) gyromagnetic ratio for the electron field is determined by the classical expression

$$\gamma_{or} = -\frac{e}{2m_e c} \tag{19}$$

Thus, the spin (intrinsic) gyromagnetic ratio of the electron field (17) is two times greater than the convective (orbital) gyromagnetic ratio (19). In the classical theory of the Maxwell-Pauli field [7], this fact is a natural property of the electron field, which is an electrically charged magnetic matter continuously distributed in space, and does not lead to paradoxes.

The system of equations (1)-(9) has gauge invariance:

$$\mathbf{A}_\Sigma \to \mathbf{A}_\Sigma + \nabla f, \qquad \varphi_\Sigma \to \varphi_\Sigma - \frac{1}{c}\frac{\partial f}{\partial t}, \qquad \psi \to \psi \exp\left(-\frac{ie}{\hbar c} f\right) \tag{20}$$

where f is an arbitrary function; in this case, the strengths of the electric and magnetic fields, the field $\mathbf{G}$, as well as the densities of the electric charge, current, intrinsic angular momentum and intrinsic magnetic moment of the electron field do not change.

The system of equations (1)-(9) is not relativistically invariant, since equations (1) and (2) do not have relativistic invariance.

The purpose of this work is to develop a relativistically invariant theory of the combined electromagnetic and electron fields based on the Dirac equation.

2. Modified Dirac equation

As is known [9], the Dirac equation admits the generalization proposed by Pauli in the form

$$\left[\gamma^\mu \left(p_\mu + \frac{e}{c} A_\mu^{(\Sigma)}\right) + i\lambda\gamma^\mu\gamma^\nu G_{\mu\nu} - m_e c\right]\Psi = 0 \tag{21}$$

where $\mu, \nu = 0,1,2,3,4$; $p_\mu = i\hbar\partial_\mu$; $\partial_\mu = \partial/\partial x^\mu$; Ψ is the bispinor; $G_{\mu\nu}$ is an arbitrary real-valued antisymmetric 4-tensor; λ is an arbitrary real-valued constant; $A_\mu^{(\Sigma)} = A_\mu + A_\mu^{(e)}$; A_μ is the 4-vector potential of the external electromagnetic field; $A_\mu^{(e)}$ is the 4-vector potential of the electromagnetic field, created by the electron wave; $A^\mu = (\varphi, \mathbf{A})$; φ and $\mathbf{A}$ are the scalar and vector potentials of the electromagnetic field. Note that equation (21) takes into account the sign of the charge of the electron field, i.e. here, $e > 0$.

Equation (21) is the most general relativistically invariant equation, linear with respect to the fields A^μ and $G_{\mu\nu}$.

In particular, Pauli used [9] $G_{\mu\nu} = F_{\mu\nu}$ and $\lambda = -\frac{\mu}{2c}$, where $F_{\mu\nu}$ is the electromagnetic field tensor; μ is a constant having the dimension of a magnetic moment. In this case, the "quantum particle" described by equation (21) has a magnetic moment equal to $(\mu_B + \mu)$, where $\mu_B = \frac{e\hbar}{2m_e c}$ is the Bohr magneton.

We are interested in such a field $G_{\mu\nu}$, which compensates for the own electrostatic field of the electron wave inside the atom, but does not compensate for it outside the atom.

This means that the following condition must be satisfied inside the atom:

$$\frac{e}{c} A_0^{(e)} + i\lambda\gamma^\nu G_{0\nu} = 0 \tag{22}$$

Differentiating equation (22), one obtains $\frac{e}{c}\frac{\partial^2 A_0^{(e)}}{\partial x^\sigma \partial x_\sigma} + i\lambda\gamma^\nu \frac{\partial^2 G_{0\nu}}{\partial x^\sigma \partial x_\sigma} = 0$. Taking into account that, according to the Maxwell equations [8], $\frac{\partial^2 A_0^{(e)}}{\partial x_\sigma \partial x^\sigma} = 4\pi\rho$, where the charge density of the electron field is determined by the relation $\rho = -e\Psi^*\Psi$, one obtains

$$i\lambda\gamma^\nu \frac{\partial^2 G_{0\nu}}{\partial x^\sigma \partial x_\sigma} = 4\pi \frac{e^2}{c}\Psi^*\Psi \tag{23}$$

It follows that the field $G_{\mu\nu}$ must satisfy the equation

$$\frac{\partial^2 G^{\mu\nu}}{\partial x_\sigma \partial x^\sigma} = \varepsilon M^{\mu\nu} \tag{24}$$

where $M^{\mu\nu}$ is an antisymmetric tensor quadratic with respect to Ψ; ε is some constant, which will be defined below.

The field $G^{\mu\nu}$ satisfying equation (24) will be long-range, which raises a number of questions [7]: why the field $G^{\mu\nu}$ has not yet been experimentally discovered, and also how the field $G^{\mu\nu}$ "distinguishes" the own electrostatic field of the electron wave in an atom, which it must compensate, from the electrostatic field of other atoms and ions, which it must not compensate. It was shown in [7] that these questions do not arise if the field $G^{\mu\nu}$ is short-range. Thus, the field described by the equation

$$\frac{\partial^2 G^{\mu\nu}}{\partial x_\sigma \partial x^\sigma} + \varkappa^2 G^{\mu\nu} = \varepsilon M^{\mu\nu} \tag{25}$$

where the constant $\varkappa$ satisfies condition (10), will decay exponentially at distances of the order of a_B from its source and will simultaneously compensate for the own electrostatic field of the electron wave "inside the atom".

As is known [10], using the bispinor Ψ, one can construct a real-valued antisymmetric 4-tensor of the second rank

$$M^{\mu\nu} = -M^{\nu\mu} = i\overline{\Psi}\sigma^{\mu\nu}\Psi \tag{26}$$

where $\sigma^{\mu\nu} = \frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$; $\overline{\Psi} = \Psi^*\gamma^0$.

To find the constants λ and ε, we will analyze the system of equations (21), (25) and (26) in the Pauli approximation and compare the resulting equations with equations (1) and (2). Note that, as follows from equations (21), (25), and (26), the results of the theory under consideration depend only on the product $\lambda\varepsilon$. This means that the constants λ and ε can be chosen arbitrarily, leaving $\lambda\varepsilon$ unchanged.

3. Pauli equation

In the standard representation [10], one writes

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}, \boldsymbol{\alpha} = \gamma^0\boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \beta = \gamma^0;\ \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}$$

As a result, equation (21) for the bispinor $\Psi = \begin{pmatrix} \psi \\ \chi \end{pmatrix}$ takes the form [10]

$$\frac{i\hbar}{c}\frac{\partial \Psi}{\partial t} = \left[\boldsymbol{\alpha}\left(\mathbf{p} + \frac{e}{c}\mathbf{A}_\Sigma\right) - \frac{e}{c}\varphi_\Sigma - i\lambda\gamma^0\gamma^\mu\gamma^\nu G_{\mu\nu} + \beta m_e c\right]\Psi \tag{27}$$

where $\varphi_\Sigma = A_0^{(\Sigma)}$; ψ and χ are the spinors.

After simple transformations (hereinafter, some details of the calculations are given in the Appendix), one obtains

$$\gamma^0\gamma^\mu\gamma^\nu G_{\mu\nu}\Psi = 2G^{0k}\begin{pmatrix} -\sigma_k\chi \\ \sigma_k\psi \end{pmatrix} + G^{ki}\begin{pmatrix} -\sigma_k\sigma_i\psi \\ \sigma_k\sigma_i\chi \end{pmatrix} \tag{28}$$

For the mixed component of the tensor $M^{\mu\nu}$ one obtains

$$M^{0k} = \overline{\Psi}\alpha_k\Psi = i(\psi^*\sigma_k\chi - \chi^*\sigma_k\psi) \tag{29}$$

where $k = 1,2,3$.

Entering a 3-vector

$$K^k = K_k = G^{0k} \tag{30}$$

and taking into account (29), one writes equation (25) in the form

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{K}}{\partial t^2} - \Delta\mathbf{K} + \varkappa^2\mathbf{K} = i\varepsilon(\psi^*\boldsymbol{\sigma}\chi - \chi^*\boldsymbol{\sigma}\psi) \tag{31}$$

Taking into account (30), one rewrites relation (28) in the form

$$\gamma^0\gamma^\mu\gamma^\nu G_{\mu\nu}\Psi = \begin{pmatrix} -2\boldsymbol{\sigma}\mathbf{K}\chi - G^{ki}\sigma_k\sigma_i\psi \\ 2\boldsymbol{\sigma}\mathbf{K}\psi + G^{ki}\sigma_k\sigma_i\chi \end{pmatrix} \tag{32}$$

Let's transform

$$G^{ki}\sigma_k\sigma_i = G^{12}\sigma_1\sigma_2 + G^{13}\sigma_1\sigma_3 + G^{21}\sigma_2\sigma_1 + G^{23}\sigma_2\sigma_3 + G^{31}\sigma_3\sigma_1 + G^{32}\sigma_3\sigma_2$$

$$= G^{12}\sigma_1\sigma_2 + G^{13}\sigma_1\sigma_3 - G^{12}\sigma_2\sigma_1 + G^{23}\sigma_2\sigma_3 - G^{13}\sigma_3\sigma_1 - G^{23}\sigma_3\sigma_2$$

$$= G^{12}(\sigma_x\sigma_y - \sigma_y\sigma_x) + G^{13}(\sigma_x\sigma_z - \sigma_z\sigma_x) + G^{23}(\sigma_y\sigma_z - \sigma_z\sigma_y)$$

For the parameter $G^{ki}\sigma_k\sigma_i$ included in (32), it is easy to obtain

$$G^{ki}\sigma_k\sigma_i = G^{12}(\sigma_x\sigma_y - \sigma_y\sigma_x) + G^{13}(\sigma_x\sigma_z - \sigma_z\sigma_x) + G^{23}(\sigma_y\sigma_z - \sigma_z\sigma_y)$$

Taking into account the properties of Pauli matrices [10]

$$\sigma_y\sigma_z = -\sigma_z\sigma_y = i\sigma_x, \qquad \sigma_z\sigma_x = -\sigma_x\sigma_z = i\sigma_y, \qquad \sigma_x\sigma_y = -\sigma_y\sigma_x = i\sigma_z$$

one obtains

$$G^{ki}\sigma_k\sigma_i = 2iG^{12}\sigma_z - 2iG^{13}\sigma_y + 2iG^{23}\sigma_x \tag{33}$$

Let us introduce a three-dimensional vector **G** with components

$$G_i = G^i = \frac{1}{2}e_{ikl}G^{kl} = (G^{23}, -G^{13}, G^{12}) \tag{34}$$

Taking into account (30) and (34), one writes

$$G^{\mu\nu} = \begin{pmatrix} 0 & K_x & K_y & K_z \\ -K_x & 0 & G_z & -G_y \\ -K_y & -G_z & 0 & G_x \\ -K_z & G_y & -G_x & 0 \end{pmatrix} = (\mathbf{K}, -\mathbf{G}) \tag{35}$$

Then relation (33) can be rewritten as

$$G^{ki}\sigma_k\sigma_i = 2i\boldsymbol{\sigma}\mathbf{G} \tag{36}$$

Taking into account (36), relation (32) takes the form

$$\gamma^0\gamma^\mu\gamma^\nu G_{\mu\nu}\Psi = \begin{pmatrix} -2\boldsymbol{\sigma}\mathbf{K}\chi - 2i\boldsymbol{\sigma}\mathbf{G}\psi \\ 2\boldsymbol{\sigma}\mathbf{K}\psi + 2i\boldsymbol{\sigma}\mathbf{G}\chi \end{pmatrix} = -(2i\gamma^0\boldsymbol{\Sigma}\mathbf{G} + 2\boldsymbol{\gamma}\mathbf{K})\Psi \tag{37}$$

Taking into account (37), the modified Dirac equation (27) takes the form

$$\frac{i\hbar}{c}\frac{\partial\Psi}{\partial t} = \left[\boldsymbol{\alpha}\left(\mathbf{p} + \frac{e}{c}\mathbf{A}_\Sigma\right) - \frac{e}{c}\varphi_\Sigma - 2\lambda\beta\boldsymbol{\Sigma}\mathbf{G} + 2i\lambda\boldsymbol{\gamma}\mathbf{K} + \beta m_e c\right]\Psi \tag{38}$$

For the spatial components of the tensor $G^{\mu\nu}$, equation (25) takes the form

$$\frac{\partial^2 G^{ik}}{\partial x_\sigma \partial x^\sigma} + \varkappa^2 G^{ik} = \varepsilon m_{ik} \tag{39}$$

where $i,k = 1,2,3$; m_{ik} is the antisymmetric 3-tensor

$$m_{ik} = -m_{ki} = M^{ik} \tag{40}$$

Taking into account (26), as a result of simple transformations, one obtains

$$m_{xy} = -m_{yx} = \psi^*\sigma_z\psi - \chi^*\sigma_z\chi \tag{41}$$

$$m_{zx} = -m_{xz} = \psi^* \sigma_y \psi - \chi^* \sigma_y \chi \tag{42}$$

$$m_{yz} = -m_{zy} = \psi^* \sigma_x \psi - \chi^* \sigma_x \chi \tag{43}$$

Taking into account relation (34) and equation (39), one concludes that the vector **G** satisfies the equation

$$\frac{\partial^2 G_i}{\partial x_\sigma \partial x^\sigma} + \varkappa^2 G_i = \frac{1}{2} \varepsilon e_{ikl} m_{kl}$$

which can be rewritten as

$$\frac{\partial^2 \mathbf{G}}{\partial x_\sigma \partial x^\sigma} + \varkappa^2 \mathbf{G} = \varepsilon \mathbf{w} \tag{44}$$

where we introduced the 3-vector

$$w_i = \frac{1}{2} e_{ikl} m_{kl} = (m_{yz}, -m_{xz}, m_{xy}) \tag{45}$$

Substituting into (45) the components of the tensor m_{kl} from relations (41)-(43), one obtains

$$\mathbf{w} = \psi^* \boldsymbol{\sigma} \psi - \chi^* \boldsymbol{\sigma} \chi \tag{46}$$

As a result, equation (44) takes the form

$$\frac{1}{c^2} \frac{\partial^2 \mathbf{G}}{\partial t^2} - \Delta \mathbf{G} + \varkappa^2 \mathbf{G} = \varepsilon (\psi^* \boldsymbol{\sigma} \psi - \chi^* \boldsymbol{\sigma} \chi) \tag{47}$$

Let us consider the nonrelativistic approximation of Eq. (38). To do this, as usual [10], we represent the wave function in the form

$$\Psi = \Psi' \exp(-i m_e c^2 t/\hbar)$$

Then equation (38) with respect to the function Ψ' takes the form

$$\frac{i\hbar}{c} \frac{\partial \Psi'}{\partial t} + m_e c \Psi' = \left[\boldsymbol{\alpha} \left(\mathbf{p} + \frac{e}{c} \mathbf{A}_\Sigma \right) - \frac{e}{c} \varphi_\Sigma + \beta m_e c \right] \Psi' - i\lambda \begin{pmatrix} -2\boldsymbol{\sigma}\mathbf{K}\chi' - 2i\boldsymbol{\sigma}\mathbf{G}\psi' \\ 2\boldsymbol{\sigma}\mathbf{K}\psi' + 2i\boldsymbol{\sigma}\mathbf{G}\chi' \end{pmatrix}$$

where $\Psi' = \begin{pmatrix} \psi' \\ \chi' \end{pmatrix}$. As a result, one obtains two equations for spinors ψ' and χ' (hereinafter we omit the prime symbol of the wave functions to simplify the notation)

$$\left(i\hbar \frac{\partial}{\partial t} + e\varphi_\Sigma + 2c\lambda \boldsymbol{\sigma}\mathbf{G} \right) \psi = c\boldsymbol{\sigma} \left(\mathbf{p} + \frac{e}{c} \mathbf{A}_\Sigma + i2\lambda \mathbf{K} \right) \chi \tag{48}$$

$$\left(i\hbar \frac{\partial}{\partial t} + e\varphi_\Sigma + 2m_e c^2 - 2c\lambda \boldsymbol{\sigma}\mathbf{G} \right) \chi = c\boldsymbol{\sigma} \left(\mathbf{p} + \frac{e}{c} \mathbf{A}_\Sigma + i2\lambda \mathbf{K} \right) \psi - i4c\lambda \boldsymbol{\sigma}\mathbf{K}\psi \tag{49}$$

In the first approximation, as usual [10], we leave only the term $2m_e c^2$ on the left-hand side of equation (49). As a result, one obtains

$$\chi = \frac{1}{2m_e c} \boldsymbol{\sigma} \left(\mathbf{p} + \frac{e}{c} \mathbf{A}_\Sigma + i2\lambda \mathbf{K} \right) \psi - \frac{i2\lambda}{m_e c} \boldsymbol{\sigma}\mathbf{K}\psi \tag{50}$$

Substituting (50) into (48), one obtains

$$\left(i\hbar \frac{\partial}{\partial t} + e\varphi_\Sigma + 2c\lambda \boldsymbol{\sigma}\mathbf{G} \right) \psi = \frac{1}{2m_e} \left(\boldsymbol{\sigma} \left(\mathbf{p} + \frac{e}{c} \mathbf{A}_\Sigma + i2\lambda \mathbf{K} \right) \right)^2 \psi - \frac{i2\lambda}{m_e} \boldsymbol{\sigma} \left(\mathbf{p} + \frac{e}{c} \mathbf{A}_\Sigma + i2\lambda \mathbf{K} \right) (\boldsymbol{\sigma}\mathbf{K}\psi) \tag{51}$$

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Taking into account that the Pauli matrices satisfy the relation [10]

$$(\boldsymbol{\sigma}\mathbf{a})(\boldsymbol{\sigma}\mathbf{b}) = \mathbf{ab} + i\boldsymbol{\sigma}[\mathbf{ab}]$$

after simple transformations, equation (51) is reduced to the form

$$i\hbar\frac{\partial\psi}{\partial t} = \left[\frac{1}{2m_e}\left(\frac{\hbar}{i}\nabla + \frac{e}{c}\mathbf{A}_\Sigma + i2\lambda\mathbf{K}\right)^2 - e\varphi_\Sigma - 2c\lambda\boldsymbol{\sigma}\mathbf{G} + \frac{e\hbar}{2m_e c}\boldsymbol{\sigma}\left(\mathbf{H}_\Sigma - i\frac{2c\lambda}{e}\mathrm{rot}\mathbf{K}\right)\right]\psi - \frac{2\lambda\hbar}{m_e}\nabla(\mathbf{K}\psi) - i\frac{2e\lambda}{m_e c}\left(\mathbf{A}_\Sigma + i\frac{2c\lambda}{e}\mathbf{K}\right)\mathbf{K}\psi + i\frac{2\lambda\hbar}{m_e}\boldsymbol{\sigma}(\mathbf{K}\times\nabla\psi) + \frac{2\lambda e}{m_e c}\boldsymbol{\sigma}(\mathbf{A}_\Sigma\times\mathbf{K})\psi \quad (52)$$

In the same approximation, equation (47) takes the form

$$\frac{1}{c^2}\frac{\partial^2\mathbf{G}}{\partial t^2} - \Delta\mathbf{G} + \varkappa^2\mathbf{G} = \varepsilon\psi^*\boldsymbol{\sigma}\psi \quad (53)$$

Comparing equations (31) and (53), one concludes that, in order of magnitude, $\mathbf{K}\sim(\chi/\psi)\mathbf{G}$. As follows from (50), in the considered nonrelativistic approximation $\chi\sim\alpha\psi$, where $\alpha = \frac{e^2}{\hbar c}$ is the fine structure constant. Thus, we have an order-of-magnitude estimate

$$\mathbf{K}\sim\alpha\mathbf{G} \quad (54)$$

This implies that the terms on the right-hand side of Eq. (52) containing the vector $\mathbf{K}$ are substantially smaller than the term containing the vector $\mathbf{G}$.

Discarding the terms containing the vector $\mathbf{K}$ on the right-hand side of Eq. (52), one reduces Eq. (52) to the form

$$i\hbar\frac{\partial\psi}{\partial t} = \left[\frac{1}{2m_e}\left(\frac{\hbar}{i}\nabla + \frac{e}{c}\mathbf{A}_\Sigma\right)^2 - e\varphi_\Sigma - 2c\lambda\boldsymbol{\sigma}\mathbf{G} + \frac{e\hbar}{2m_e c}\boldsymbol{\sigma}\mathbf{H}_\Sigma\right]\psi \quad (55)$$

Equations (53) and (55) coincide with equations (2) and (1), respectively, when

$$\lambda = \frac{e}{2c}, \varepsilon = 4\pi e \quad (56)$$

The density of the electric charge and the density of the electric current of the electron wave are determined by the relations [10]

$$\rho = -e\Psi^*\Psi = -e(\psi^*\psi + \chi^*\chi) \quad (57)$$

$$\mathbf{j} = -ec\Psi^*\boldsymbol{\alpha}\Psi = -ec(\psi^*\boldsymbol{\sigma}\chi + \chi^*\boldsymbol{\sigma}\psi) \quad (58)$$

In the approximation under consideration, the term $\chi^*\chi$ can be neglected in comparison with the term $\psi^*\psi$; as a result one obtains

$$\rho = -e\psi^*\psi \quad (59)$$

Using (50), one writes (58) as

$$\mathbf{j} = \frac{e\hbar}{2m_e i}[(\nabla\psi^*)\psi - \psi^*\nabla\psi] - \frac{e^2}{m_e c}\mathbf{A}_\Sigma\psi^*\psi - \frac{e\hbar}{2m_e}\mathrm{rot}(\psi^*\boldsymbol{\sigma}\psi) - \frac{e^2}{m_e c}\mathbf{K}\times(\psi^*\boldsymbol{\sigma}\psi) \quad (60)$$

Relation (60) for the electric current density of an electron wave differs from the Pauli current density (9) by an additional term $-\frac{e^2}{m_e c}\mathbf{K}\times\psi^*\boldsymbol{\sigma}\psi$, associated with the vector field $\mathbf{K}$. Taking into account estimate (54), in the considered approximation, this term can be neglected.

Similarly, taking into account (50) and (56), one writes equation (31) as

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{K}}{\partial t^2} - \Delta\mathbf{K} + \left(\varkappa^2 - \frac{4\pi e^2}{m_e c^2}\psi^*\psi\right)\mathbf{K} = 4\pi\frac{e\hbar}{2m_e c}\left(\nabla(\psi^*\psi) + i\psi^*\nabla\times\boldsymbol{\sigma}\psi - i(\nabla\psi^*)\times\boldsymbol{\sigma}\psi - \frac{2e}{\hbar c}\mathbf{A}_\Sigma\times\psi^*\boldsymbol{\sigma}\psi\right) \quad (61)$$

4. Maxwell-Dirac coupling

Taking into account the results of the previous section, one concludes that the system of relativistic equations

$$\left[\gamma^\mu\left(i\hbar\partial_\mu + \frac{e}{c}A_\mu^{(\Sigma)}\right) + i\frac{e}{2c}\gamma^\mu\gamma^\nu G_{\mu\nu} - m_e c\right]\Psi = 0 \quad (62)$$

$$\frac{\partial^2 G^{\mu\nu}}{\partial x_\sigma \partial x^\sigma} + \varkappa^2 G^{\mu\nu} = 4\pi e i\overline{\Psi}\sigma^{\mu\nu}\Psi \quad (63)$$

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} = -\frac{4\pi}{c}j^\mu \quad (64)$$

where $\partial_\mu = \frac{\partial}{\partial x^\mu}$;

$$F_{\mu\nu} = \frac{\partial A_\nu^{(\Sigma)}}{\partial x^\mu} - \frac{\partial A_\mu^{(\Sigma)}}{\partial x^\nu} \quad (65)$$

$$j^\mu = -ec\overline{\Psi}\gamma^\mu\Psi \quad (66)$$

forms a classical self-consistent gauge-invariant theory of electron and electromagnetic fields, and in the non-relativistic limit goes into the system of equations (1)-(9). This means that, at least in the nonrelativistic limit [7], the system of equations (62)–(66) correctly describes the spectra of the hydrogen atom and other basic "quantum effects" [1–6].

Equation (62), which describes the charged field Ψ, implies the law of conservation of the electric charge of the electron wave

$$\frac{\partial j^\mu}{\partial x^\mu} = 0 \quad (67)$$

which agrees with the Maxwell equations (64), (65).

Equations (62)-(66) can be derived from the least action principle

$$\delta S = 0 \quad (68)$$

where

$$S = \frac{1}{c}\int \mathcal{L}d\Omega \quad (69)$$

$$d\Omega = cdtdV$$

$$\mathcal{L} = \frac{c}{2}\overline{\Psi}\gamma^\mu\left(i\hbar\partial_\mu\Psi + \frac{e}{c}A_\mu^{(\Sigma)}\Psi\right) + \frac{c}{2}\left(-i\hbar\partial_\mu\overline{\Psi} + \frac{e}{c}A_\mu^{(\Sigma)}\overline{\Psi}\right)\gamma^\mu\Psi + i\frac{e}{2}\overline{\Psi}\sigma^{\mu\nu}G_{\mu\nu}\Psi - m_e c^2\overline{\Psi}\Psi - \frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu} + \frac{1}{16\pi}\frac{\partial G_{\mu\nu}}{\partial x^\sigma}\frac{\partial G^{\mu\nu}}{\partial x_\sigma} - \frac{\varkappa^2}{16\pi}G_{\mu\nu}G^{\mu\nu} \quad (70)$$

if the functions $\Psi, \overline{\Psi}, G_{\mu\nu}$ and A_μ are considered independent when varying.

Indeed, varying (67)-(69) with respect to $\overline{\Psi}$ leads to equation (62), varying with respect to $G_{\mu\nu}$ leads to equation (63), and varying with respect to A_μ leads to equation (64)-(66).

Varying with respect to Ψ leads to an equation conjugate to equation (62):

$$-i\hbar\partial_\mu(\overline{\Psi}\gamma^\mu)+\frac{e}{c}\overline{\Psi}\gamma^\mu A_\mu^{(\Sigma)}+i\frac{e}{2c}\overline{\Psi}\sigma^{\mu\nu}G_{\mu\nu}-m_e c\overline{\Psi}=0 \tag{71}$$

Lagrangian (70) allows finding the stress-energy tensor $T^{\mu\nu}$ that satisfies the continuity equation [8]

$$\frac{\partial T^{\mu\nu}}{\partial x^\nu}=0 \tag{72}$$

which is a differential record of the energy-momentum conservation law.

Using the general definition of the stress-energy tensor [8], which in the case under consideration takes the form

$$T_\mu^\nu=\overline{\Psi}_{,\mu}\frac{\partial\mathcal{L}}{\partial\overline{\Psi}_{,\nu}}+\frac{\partial\mathcal{L}}{\partial\Psi_{,\nu}}\Psi_{,\mu}+A_{,\mu}^\sigma\frac{\partial\mathcal{L}}{\partial A_{,\nu}^\sigma}+G_{,\mu}^{\sigma\lambda}\frac{\partial\mathcal{L}}{\partial G_{,\nu}^{\sigma\lambda}}-\delta_\mu^\nu\mathcal{L} \tag{73}$$

where $\Psi_{,\mu}=\partial_\mu\Psi$, one obtains (see Appendix)

$$\begin{aligned}T_{\mu\nu}=&\frac{c}{4}\Big[\Big(-i\hbar\big(\partial_\nu\overline{\Psi}\big)+\frac{e}{c}\overline{\Psi}A_\nu^{(\Sigma)}\Big)\gamma_\mu\Psi+\Big(-i\hbar\big(\partial_\mu\overline{\Psi}\big)+\frac{e}{c}\overline{\Psi}A_\mu^{(\Sigma)}\Big)\gamma_\nu\Psi+\overline{\Psi}\gamma_\mu\left(i\hbar\partial_\nu+\frac{e}{c}A_\nu^{(\Sigma)}\right)\Psi\\&+\overline{\Psi}\gamma_\nu\left(i\hbar\partial_\mu+\frac{e}{c}A_\mu^{(\Sigma)}\right)\Psi\Big]-\frac{1}{4\pi}g^{\lambda\sigma}F_{\nu\lambda}F_{\mu\sigma}+\frac{1}{8\pi}\frac{\partial G_{\sigma\lambda}}{\partial x^\nu}\frac{\partial G^{\sigma\lambda}}{\partial x^\mu}\\&-g_{\mu\nu}\Big[\frac{c}{2}\overline{\Psi}\gamma^\sigma\left(i\hbar\partial_\sigma\Psi+\frac{e}{c}A_\sigma^{(\Sigma)}\Psi\right)+\frac{c}{2}\Big(-i\hbar\partial_\sigma\overline{\Psi}+\frac{e}{c}A_\sigma^{(\Sigma)}\overline{\Psi}\Big)\gamma^\sigma\Psi+i\frac{e}{2}G_{\sigma\lambda}\overline{\Psi}\sigma^{\sigma\lambda}\Psi-m_e c^2\overline{\Psi}\Psi\\&-\frac{1}{16\pi}F_{\sigma\lambda}F^{\sigma\lambda}+\frac{1}{16\pi}\frac{\partial G_{\sigma\lambda}}{\partial x^\omega}\frac{\partial G^{\sigma\lambda}}{\partial x_\omega}-\frac{\varkappa^2}{16\pi}G_{\sigma\lambda}G^{\sigma\lambda}\Big]+\frac{1}{8\pi}g^{\lambda\sigma}\frac{\partial}{\partial x_\omega}\left(G_{\mu\sigma}\frac{\partial G_{\lambda\nu}}{\partial x^\omega}-G_{\lambda\nu}\frac{\partial G_{\mu\sigma}}{\partial x^\omega}\right)\end{aligned} \tag{74}$$

The energy density W, the energy flux density $\mathbf{J}$ and the momentum density $\mathbf{P}$ of the field are related to the components of the stress-energy tensor by the relations [8]:

$$W=T^{00},J_k=cT^{0k},P_k=\frac{1}{c}T^{k0} \tag{75}$$

When calculating the parameters (75), the last divergent term in the stress-energy tensor (74) can be discarded, because for a localized electron field (for example, in an atom), when integrated over an infinite volume, it turns into a vanishing surface integral.

As a result, one obtains (see Appendix)

$$\begin{aligned}W=&m_e c^2\overline{\Psi}\Psi+\frac{1}{8\pi}(\mathbf{H}^2+\mathbf{E}^2)-\frac{c}{2}\Big[\Big(-i\hbar\nabla\Psi^*-\frac{e}{c}\mathbf{A}_\Sigma\Psi^*\Big)\boldsymbol{\alpha}\Psi+\Psi^*\boldsymbol{\alpha}\left(i\hbar\nabla\Psi-\frac{e}{c}\mathbf{A}_\Sigma\Psi\right)\Big]-e\mathbf{G}\overline{\Psi}\boldsymbol{\Sigma}\Psi+ei\mathbf{K}\overline{\Psi}\boldsymbol{\alpha}\Psi\\&+\frac{1}{8\pi}\left(\frac{1}{c^2}\frac{\partial\mathbf{G}}{\partial t}\frac{\partial\mathbf{G}}{\partial t}+\frac{\partial\mathbf{G}}{\partial x^k}\frac{\partial\mathbf{G}}{\partial x^k}-\frac{1}{c^2}\frac{\partial\mathbf{K}}{\partial t}\frac{\partial\mathbf{K}}{\partial t}-\frac{\partial\mathbf{K}}{\partial x^k}\frac{\partial\mathbf{K}}{\partial x^k}\right)+\frac{\varkappa^2}{8\pi}(\mathbf{G}^2-\mathbf{K}^2)\end{aligned} \tag{76}$$

$$\begin{aligned}\mathbf{J}=&\frac{c}{4\pi}\mathbf{E}_\Sigma\times\mathbf{H}_\Sigma-\frac{c^2}{4}\Big(i\hbar\Psi^*\nabla\Psi-i\hbar(\nabla\Psi^*)\Psi+i\hbar\Psi^*\boldsymbol{\alpha}\partial_0\Psi-i\hbar(\partial_0\Psi^*)\boldsymbol{\alpha}\Psi+\frac{2e}{c}\mathbf{A}_\Sigma\Psi^*\Psi\\&+\frac{2e}{c}\varphi_\Sigma\Psi^*\boldsymbol{\alpha}\Psi\Big)-\frac{1}{4\pi}\left(\frac{\partial G_k}{\partial t}\nabla G_k-\frac{\partial K_k}{\partial t}\nabla K_k\right)\end{aligned} \tag{77}$$

$$\begin{aligned}\mathbf{P}=&\frac{1}{4\pi c}\mathbf{E}_\Sigma\times\mathbf{H}_\Sigma-\frac{1}{4}\Big(i\hbar\Psi^*\boldsymbol{\alpha}\partial_0\Psi-i\hbar(\partial_0\Psi^*)\boldsymbol{\alpha}\Psi+i\hbar\Psi^*\nabla\Psi-i\hbar(\nabla\Psi^*)\Psi+\frac{2e}{c}\varphi_\Sigma\Psi^*\boldsymbol{\alpha}\Psi\\&+\frac{2e}{c}\mathbf{A}_\Sigma\Psi^*\Psi\Big)-\frac{1}{4\pi c^2}\left(\frac{\partial G_k}{\partial t}\nabla G_k-\frac{\partial K_k}{\partial t}\nabla K_k\right)\end{aligned} \tag{78}$$

It is easy to verify by direct calculations that in the nonrelativistic approximation (50), taking into account (54), relations (76)-(78) go over into the corresponding expressions obtained for the Maxwell-Pauli field [7].

5. Concluding remarks

Thus, we have shown that the Maxwell and Dirac fields can be naturally and consistently combined in the spirit of classical field theory without any quantization.

The self-consistent theory (62)-(66) naturally combines the electromagnetic and electron fields into a single theory, without discarding the non-stationary (radiative) component of the electromagnetic field of the electron wave, which plays a decisive role in the process of rearranging the internal structure of the atom during spontaneous emission [2-6].

In the nonrelativistic limit, this theory naturally transforms into the self-consistent Maxwell-Pauli theory (1)-(9) [7].

Accordingly, all the main conclusions of [7] remain valid for the self-consistent Maxwell-Dirac theory developed in this work. In particular, all basic quantum effects, such as the discrete spectrum of spontaneous emission, the rearrangement of the structure of an atom during spontaneous emission, which is traditionally called the “quantum transition”, the photoelectric effect, the Compton effect, thermal radiation, etc., are trivial results of classical field theory (62)-(66), and do not require quantization and any additional hypotheses [1-6]. In addition, such properties of the electron as its own angular momentum (spin) and associated with it its own magnetic moment, which have no physical explanation in quantum mechanics and are considered special “quantum” properties of the objects of Microworld, have a simple and clear absolutely classical meaning within the framework of the classical theory of the Maxwell-Dirac field developed here and in [1 -7].

The new tensor field $G^{\mu\nu}$ introduced into the theory is short-range (at $\varkappa \neq 0$), which provides compensation for the intrinsic electrostatic field of the electron wave only within the atom, but retains the effect of electrostatic fields from external sources, including other atoms and ions. This feature of the $G^{\mu\nu}$ field apparently makes it impossible, or at least extremely difficult, to directly measure or even simply detect the $G^{\mu\nu}$ field, the existence of which can only be judged from indirect data, such as the compensation of the intrinsic electrostatic field of the electron wave and the observed physical effects directly related to the field $G^{\mu\nu}$, which this theory should predict. These issues will be discussed in future articles in this series.

Note that this theory, considered as a classical field theory, can be easily combined with the general relativity by rewriting equations (62)-(66) in curved space-time. In this case, it seems to us, there should be no fundamental problems and paradoxes that arise in the quantum gravity theory.

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APPENDIX

As usual [10],

$$\gamma^{\mu} = (\gamma^{0}, \boldsymbol{\gamma}), \overline{\Psi} = \Psi^{*}\gamma^{0}$$

Formally, one can write [10]

$$\sigma^{\mu\nu} = (\boldsymbol{\alpha}, i\boldsymbol{\Sigma})$$

Then

$$M^{\mu\nu} = -M^{\nu\mu} = i(\overline{\Psi}\boldsymbol{\alpha}\Psi, i\overline{\Psi}\boldsymbol{\Sigma}\Psi)$$

Let's write in more detail

$$\sigma^{\mu\nu} = \begin{pmatrix} 0 & \alpha_x & \alpha_y & \alpha_z \\ -\alpha_x & 0 & -i\Sigma_z & i\Sigma_y \\ -\alpha_y & i\Sigma_z & 0 & -i\Sigma_x \\ -\alpha_z & -i\Sigma_y & i\Sigma_x & 0 \end{pmatrix}$$

Then

$$M^{\mu\nu} = i\begin{pmatrix} 0 & \overline{\Psi}\alpha_x\Psi & \overline{\Psi}\alpha_y\Psi & \overline{\Psi}\alpha_z\Psi \\ -\overline{\Psi}\alpha_x\Psi & 0 & -i\overline{\Psi}\Sigma_z\Psi & i\overline{\Psi}\Sigma_y\Psi \\ -\overline{\Psi}\alpha_y\Psi & i\overline{\Psi}\Sigma_z\Psi & 0 & -i\overline{\Psi}\Sigma_x\Psi \\ -\overline{\Psi}\alpha_z\Psi & -i\overline{\Psi}\Sigma_y\Psi & i\overline{\Psi}\Sigma_x\Psi & 0 \end{pmatrix}$$

In the standard representation [10], which is used below,

$$\gamma^{0} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}, \boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}$$

$$\Psi = \begin{pmatrix} \psi \\ \chi \end{pmatrix}, \qquad \overline{\Psi} = (\psi^{*} \quad -\chi^{*})$$

$$\overline{\Psi}\boldsymbol{\alpha}\Psi = (\psi^{*} \quad -\chi^{*})\begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}\begin{pmatrix} \psi \\ \chi \end{pmatrix} = (\psi^{*} \quad -\chi^{*})\begin{pmatrix} \boldsymbol{\sigma}\chi \\ \boldsymbol{\sigma}\psi \end{pmatrix} = \psi^{*}\boldsymbol{\sigma}\chi - \chi^{*}\boldsymbol{\sigma}\psi$$

$$\overline{\Psi}\boldsymbol{\Sigma}\Psi = (\psi^{*} \quad -\chi^{*})\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}\begin{pmatrix} \psi \\ \chi \end{pmatrix} = (\psi^{*} \quad -\chi^{*})\begin{pmatrix} \boldsymbol{\sigma}\psi \\ \boldsymbol{\sigma}\chi \end{pmatrix} = \psi^{*}\boldsymbol{\sigma}\psi - \chi^{*}\boldsymbol{\sigma}\chi$$

$$\boldsymbol{\alpha}\beta = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -\boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}$$

$$\alpha_k\beta\alpha_i = \begin{pmatrix} 0 & -\sigma_k \\ \sigma_k & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} = \begin{pmatrix} -\sigma_k\sigma_i & 0 \\ 0 & \sigma_k\sigma_i \end{pmatrix}$$

One can write

$$\gamma^{\mu}A_{\mu} = \gamma^{0}A_{0} - \boldsymbol{\gamma}\mathbf{A}$$

$$\left[\gamma^{0}p_{0} - \boldsymbol{\gamma}\mathbf{p} + \frac{e}{c}\gamma^{0}A_{0} - \frac{e}{c}\boldsymbol{\gamma}\mathbf{A} + i\lambda\gamma^{\mu}\gamma^{\nu}G_{\mu\nu} - m_e c\right]\Psi = 0$$

Multiply by γ^{0}:

$$\left[\gamma^{0}\gamma^{0}p_{0} - \gamma^{0}\boldsymbol{\gamma}\mathbf{p} + \frac{e}{c}\gamma^{0}\gamma^{0}A_{0} - \frac{e}{c}\gamma^{0}\boldsymbol{\gamma}\mathbf{A} + i\lambda\gamma^{0}\gamma^{\mu}\gamma^{\nu}G_{\mu\nu} - \gamma^{0}m_e c\right]\Psi = 0$$

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Taking into account that [10]

$$\gamma^0\gamma^0 = 1$$

and, as usual, denoting

$$\boldsymbol{\alpha} = \gamma^0\boldsymbol{\gamma}, \qquad \beta = \gamma^0$$

one obtains (27).

Let's transform

$$\gamma^0\gamma^\mu\gamma^\nu G_{\mu\nu} = \gamma^0\gamma^0\gamma^\nu G_{0\nu} + \gamma^0\gamma^k\gamma^\nu G_{k\nu} = \gamma^\nu G_{0\nu} + \alpha_k\gamma^\nu G_{k\nu} = \gamma^0 G_{00} + \gamma^k G_{0k} + \alpha_k\gamma^0 G_{k0} + \alpha_k\gamma^i G_{ki}$$
$$= \gamma^k G_{0k} + \alpha_k\gamma^0 G_{k0} + \alpha_k\gamma^i G_{ki} = \gamma^k g_{0\mu}g_{k\nu}G^{\mu\nu} + \alpha_k\gamma^0 g_{k\mu}g_{0\nu}G^{\mu\nu} + \alpha_k\gamma^i g_{k\mu}g_{i\nu}G^{\mu\nu} =$$
$$= -\gamma^k G^{0k} - \alpha_k\gamma^0 G^{k0} + \alpha_k\gamma^i G^{ki} = -\gamma^k G^{0k} + \alpha_k\gamma^0 G^{0k} + \alpha_k\gamma^i G^{ki}$$

Taking into account that

$$\boldsymbol{\gamma} = \gamma^0\boldsymbol{\alpha}$$

one obtains

$$\gamma^0\gamma^\mu\gamma^\nu G_{\mu\nu} = (\alpha_k\beta - \beta\alpha_k)G^{0k} + \alpha_k\beta\alpha_i G^{ki}$$

Taking into account that

$$\alpha_k\beta + \beta\alpha_k = 0$$

one obtains

$$\gamma^0\gamma^\mu\gamma^\nu G_{\mu\nu} = 2\alpha_k\beta G^{0k} + \alpha_k\beta\alpha_i G^{ki}$$

Then

$$\gamma^0\gamma^\mu\gamma^\nu G_{\mu\nu} = 2G^{0k}\begin{pmatrix} 0 & -\sigma_k \\ \sigma_k & 0 \end{pmatrix} + G^{ki}\begin{pmatrix} -\sigma_k\sigma_i & 0 \\ 0 & \sigma_k\sigma_i \end{pmatrix}$$

$$\gamma^0\gamma^\mu\gamma^\nu G_{\mu\nu}\Psi = 2G^{0k}\begin{pmatrix} 0 & -\sigma_k \\ \sigma_k & 0 \end{pmatrix}\begin{pmatrix} \psi \\ \chi \end{pmatrix} + G^{ki}\begin{pmatrix} -\sigma_k\sigma_i & 0 \\ 0 & \sigma_k\sigma_i \end{pmatrix}\begin{pmatrix} \psi \\ \chi \end{pmatrix} = 2G^{0k}\begin{pmatrix} -\sigma_k\chi \\ \sigma_k\psi \end{pmatrix} + G^{ki}\begin{pmatrix} -\sigma_k\sigma_i\psi \\ \sigma_k\sigma_i\chi \end{pmatrix}$$

Obviously,

$$m_{ik} = M^{ik} = \begin{pmatrix} 0 & \overline{\Psi}\Sigma_z\Psi & -\overline{\Psi}\Sigma_y\Psi \\ -\overline{\Psi}\Sigma_z\Psi & 0 & \overline{\Psi}\Sigma_x\Psi \\ \overline{\Psi}\Sigma_y\Psi & -\overline{\Psi}\Sigma_x\Psi & 0 \end{pmatrix}$$

Then one obtains

$$m_{xy} = -m_{yx} = \overline{\Psi}\Sigma_z\Psi = \psi^*\sigma_z\psi - \chi^*\sigma_z\chi$$

$$m_{zx} = -m_{xz} = \overline{\Psi}\Sigma_y\Psi = \psi^*\sigma_y\psi - \chi^*\sigma_y\chi$$

$$m_{yz} = -m_{zy} = \overline{\Psi}\Sigma_x\Psi = \psi^*\sigma_x\psi - \chi^*\sigma_x\chi$$

Let's write

$$G_i = \frac{1}{2} e_{ikl} G^{kl} = (e_{123}G^{23} + e_{132}G^{32}, e_{213}G^{13} + e_{231}G^{31}, e_{312}G^{12} + e_{321}G^{21}) = (G^{23}, -G^{13}, G^{12})$$

Taking into account that the Pauli matrices satisfy the relation [10]

$$(\boldsymbol{\sigma}\mathbf{a})(\boldsymbol{\sigma}\mathbf{b}) = \mathbf{ab} + i\boldsymbol{\sigma}[\mathbf{ab}]$$

after simple transformations, taking into account that $\mathbf{p} = \frac{\hbar}{i}\nabla$, one obtains

$$\left(\boldsymbol{\sigma}\left(\mathbf{p} + \frac{e}{c}\mathbf{A}_\Sigma + i2\lambda\mathbf{K}\right)\right)^2 = \left(\frac{\hbar}{i}\nabla + \frac{e}{c}\mathbf{A}_\Sigma + i2\lambda\mathbf{K}\right)^2 + \frac{e\hbar}{c}\boldsymbol{\sigma}\left(\mathbf{H}_\Sigma + i\frac{2c\lambda}{e}\mathrm{rot}\mathbf{K}\right)$$

$$\left(\boldsymbol{\sigma}\left(\mathbf{p} + \frac{e}{c}\mathbf{A}_\Sigma + i2\lambda\mathbf{K}\right)\right)(\boldsymbol{\sigma}\mathbf{K}\psi) = \left(\mathbf{p} + \frac{e}{c}\mathbf{A}_\Sigma + i2\lambda\mathbf{K}\right)\mathbf{K}\psi + i\boldsymbol{\sigma}\left(\mathbf{p} + \frac{e}{c}\mathbf{A}_\Sigma + i2\lambda\mathbf{K}\right) \times \mathbf{K}\psi$$

$$= \frac{\hbar}{i}\nabla(\mathbf{K}\psi) + \left(\frac{e}{c}\mathbf{A}_\Sigma + i2\lambda\mathbf{K}\right)\mathbf{K}\psi + \hbar\boldsymbol{\sigma}\psi\mathrm{rot}\mathbf{K} + \hbar\boldsymbol{\sigma}\nabla\psi \times \mathbf{K} + i\boldsymbol{\sigma}\left(\frac{e}{c}\mathbf{A}_\Sigma + i2\lambda\mathbf{K}\right) \times \mathbf{K}\psi$$

$$\left(\boldsymbol{\sigma}\left(\mathbf{p} + \frac{e}{c}\mathbf{A}_\Sigma + i2\lambda\mathbf{K}\right)\right)^2 \psi - i4\lambda\boldsymbol{\sigma}\left(\mathbf{p} + \frac{e}{c}\mathbf{A}_\Sigma + i2\lambda\mathbf{K}\right)(\boldsymbol{\sigma}\mathbf{K}\psi)$$

$$= \left(\frac{\hbar}{i}\nabla + \frac{e}{c}\mathbf{A}_\Sigma + i2\lambda\mathbf{K}\right)^2 \psi + \frac{e\hbar}{c}\boldsymbol{\sigma}\left(\mathbf{H}_\Sigma - i\frac{2c\lambda}{e}\mathrm{rot}\mathbf{K}\right)\psi - 4\lambda\hbar\nabla(\mathbf{K}\psi) - i4\lambda\left(\frac{e}{c}\mathbf{A}_\Sigma + i2\lambda\mathbf{K}\right)\mathbf{K}\psi$$

$$+ i4\lambda\hbar\boldsymbol{\sigma}(\mathbf{K} \times \nabla\psi) + 4\lambda\boldsymbol{\sigma}\left(\frac{e}{c}\mathbf{A}_\Sigma + i2\lambda\mathbf{K}\right) \times \mathbf{K}\psi$$

Then equation (51) takes the form

$$\left(i\hbar\frac{\partial}{\partial t} + e\varphi_\Sigma + 2c\lambda\boldsymbol{\sigma}\mathbf{G}\right)\psi$$

$$= \frac{1}{2m_e}\left(\frac{\hbar}{i}\nabla + \frac{e}{c}\mathbf{A}_\Sigma + i2\lambda\mathbf{K}\right)^2 \psi + \frac{e\hbar}{2m_e c}\boldsymbol{\sigma}\left(\mathbf{H}_\Sigma - i\frac{2c\lambda}{e}\mathrm{rot}\mathbf{K}\right)\psi - \frac{2\lambda\hbar}{m_e}\nabla(\mathbf{K}\psi) - i\frac{2e\lambda}{m_e c}\left(\mathbf{A}_\Sigma + i\frac{2c\lambda}{e}\mathbf{K}\right)\mathbf{K}\psi$$

$$+ i\frac{2\lambda\hbar}{m_e}\boldsymbol{\sigma}(\mathbf{K} \times \nabla\psi) + \frac{2\lambda e}{m_e c}\boldsymbol{\sigma}\left(\mathbf{A}_\Sigma + i\frac{2c\lambda}{e}\mathbf{K}\right) \times \mathbf{K}\psi$$

Using

$$\chi^* = \frac{1}{2m_e c}\left(i\hbar\nabla\psi^* + \frac{e}{c}\mathbf{A}_\Sigma\psi^*\right)\boldsymbol{\sigma} + \frac{ie}{2m_e c^2}\psi^*\mathbf{K}\boldsymbol{\sigma}$$

and the properties of the Pauli matrices

$$(\boldsymbol{\sigma}\mathbf{a})\boldsymbol{\sigma} = \mathbf{a} + i[\boldsymbol{\sigma}\mathbf{a}], \qquad \boldsymbol{\sigma}(\boldsymbol{\sigma}\mathbf{a}) = \mathbf{a} + i[\mathbf{a}\boldsymbol{\sigma}]$$

one obtains

$$\psi^*\boldsymbol{\sigma}\chi = \frac{1}{2m_e c}\psi^*\boldsymbol{\sigma}\left(\boldsymbol{\sigma}\left(\mathbf{p} + \frac{e}{c}\mathbf{A}_\Sigma - i\frac{e}{c}\mathbf{K}\right)\psi\right) = \frac{1}{2m_e c}\psi^*\left(\frac{\hbar}{i}\nabla + \frac{e}{c}\mathbf{A}_\Sigma - i\frac{e}{c}\mathbf{K}\right)\psi + i\frac{1}{2m_e c}\psi^*\left(\frac{\hbar}{i}\nabla + \frac{e}{c}\mathbf{A}_\Sigma - i\frac{e}{c}\mathbf{K}\right) \times \boldsymbol{\sigma}\psi$$

$$\chi^*\boldsymbol{\sigma}\psi = \left(\frac{1}{2m_e c}\left(i\hbar\nabla\psi^* + \frac{e}{c}\mathbf{A}_\Sigma\psi^*\right)\boldsymbol{\sigma}\right)\boldsymbol{\sigma}\psi + \left(\frac{ie}{2m_e c^2}\psi^*\mathbf{K}\boldsymbol{\sigma}\right)\boldsymbol{\sigma}\psi$$

$$= \frac{1}{2m_e c}\left(i\hbar\nabla\psi^* + \frac{e}{c}\mathbf{A}_\Sigma\psi^*\right)\psi - i\frac{1}{2m_e c}\left(\left(i\hbar\nabla\psi^* + \frac{e}{c}\mathbf{A}_\Sigma\psi^*\right) \times \boldsymbol{\sigma}\right)\psi + \frac{ie}{2m_e c^2}\mathbf{K}\psi^*\psi + \frac{e}{2m_e c^2}\mathbf{K} \times \psi^*\boldsymbol{\sigma}\psi$$

$$\psi^*\boldsymbol{\sigma}\chi + \chi^*\boldsymbol{\sigma}\psi = -\frac{\hbar}{2m_e ci}[(\nabla\psi^*)\psi - \psi^*\nabla\psi] + \frac{e}{m_e c^2}\mathbf{A}_\Sigma\psi^*\psi + \frac{\hbar}{2m_e c}\mathrm{rot}(\psi^*\boldsymbol{\sigma}\psi) + \frac{e}{m_e c^2}\mathbf{K} \times \psi^*\boldsymbol{\sigma}\psi$$

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$$\psi^*\boldsymbol{\sigma}\chi - \chi^*\boldsymbol{\sigma}\psi = \frac{\hbar}{2m_e ci}(\nabla\psi^*\psi) - i\frac{e}{m_e c^2}\mathbf{K}\psi^*\psi + \frac{\hbar}{2m_e c}[\psi^*\nabla\times\boldsymbol{\sigma}\psi - (\nabla\times\psi^*\boldsymbol{\sigma})\psi] + i\frac{e}{m_e c^2}\mathbf{A}_\Sigma\times\psi^*\boldsymbol{\sigma}\psi$$

$$= \frac{\hbar}{2m_e ci}(\nabla\psi^*\psi) - i\frac{e}{m_e c^2}\mathbf{K}\psi^*\psi + \frac{\hbar}{2m_e c}[-\psi^*\boldsymbol{\sigma}\times\nabla\psi - \nabla\psi^*\times\boldsymbol{\sigma}\psi] + i\frac{e}{m_e c^2}\mathbf{A}_\Sigma\times\psi^*\boldsymbol{\sigma}\psi$$

Consider the stress-energy tensor.

Using (70) and (73), one obtains

$$T_\mu^\nu = -i\hbar\frac{c}{2}(\partial_\mu\overline{\Psi})\gamma^\nu\Psi + i\hbar\frac{c}{2}\overline{\Psi}\gamma^\nu\partial_\mu\Psi - \frac{1}{4\pi}F^{\nu\sigma}\frac{\partial A_\sigma^{(\Sigma)}}{\partial x^\mu} + \frac{1}{8\pi}\frac{\partial G_{\sigma\lambda}}{\partial x_\nu}\frac{\partial G^{\sigma\lambda}}{\partial x^\mu}$$
$$-\delta_\mu^\nu\left[\frac{c}{2}\overline{\Psi}\gamma^\sigma\left(i\hbar\partial_\sigma\Psi + \frac{e}{c}A_\sigma^{(\Sigma)}\Psi\right) + \frac{c}{2}\left(-i\hbar\partial_\sigma\overline{\Psi} + \frac{e}{c}A_\sigma^{(\Sigma)}\overline{\Psi}\right)\gamma^\sigma\Psi + i\frac{e}{2}G_{\sigma\lambda}\overline{\Psi}\sigma^{\sigma\lambda}\Psi - m_e c^2\overline{\Psi}\Psi - \frac{1}{16\pi}F_{\sigma\lambda}F^{\sigma\lambda}\right.$$
$$\left.+\frac{1}{16\pi}\frac{\partial G_{\sigma\lambda}}{\partial x^\omega}\frac{\partial G^{\sigma\lambda}}{\partial x_\omega} - \frac{\varkappa^2}{16\pi}G_{\sigma\nu}G^{\sigma\lambda}\right]$$

Taking into account (65), one writes

$$T_\mu^\nu = -i\hbar\frac{c}{2}(\partial_\mu\overline{\Psi})\gamma^\nu\Psi + i\hbar\frac{c}{2}\overline{\Psi}\gamma^\nu\partial_\mu\Psi - \frac{1}{4\pi}F^{\nu\sigma}F_{\mu\sigma} - \frac{1}{4\pi}F^{\nu\sigma}\frac{\partial A_\mu^{(\Sigma)}}{\partial x^\sigma} + \frac{1}{8\pi}\frac{\partial G_{\sigma\lambda}}{\partial x_\nu}\frac{\partial G^{\sigma\lambda}}{\partial x^\mu}$$
$$-\delta_\mu^\nu\left[\frac{c}{2}\overline{\Psi}\gamma^\sigma\left(i\hbar\partial_\sigma\Psi + \frac{e}{c}A_\sigma^{(\Sigma)}\Psi\right) + \frac{c}{2}\left(-i\hbar\partial_\sigma\overline{\Psi} + \frac{e}{c}A_\sigma^{(\Sigma)}\overline{\Psi}\right)\gamma^\sigma\Psi + i\frac{e}{2}G_{\sigma\lambda}\overline{\Psi}\sigma^{\sigma\lambda}\Psi - m_e c^2\overline{\Psi}\Psi - \frac{1}{16\pi}F_{\sigma\lambda}F^{\sigma\lambda}\right.$$
$$\left.+\frac{1}{16\pi}\frac{\partial G_{\sigma\lambda}}{\partial x^\omega}\frac{\partial G^{\sigma\lambda}}{\partial x_\omega} - \frac{\varkappa^2}{16\pi}G_{\sigma\lambda}G^{\sigma\lambda}\right]$$

Taking into account (64), one writes

$$F^{\nu\sigma}\frac{\partial A_\mu^{(\Sigma)}}{\partial x^\sigma} = \frac{\partial F^{\nu\sigma}A_\mu^{(\Sigma)}}{\partial x^\sigma} - \frac{\partial F^{\nu\sigma}}{\partial x^\sigma}A_\mu^{(\Sigma)} = \frac{\partial F^{\nu\sigma}A_\mu^{(\Sigma)}}{\partial x^\sigma} + \frac{4\pi}{c}j^\nu A_\mu^{(\Sigma)}$$

Considering that the stress-energy tensor satisfies Eq. (72), and due to the antisymmetry of the tensor $F^{\nu\sigma}$

$$\frac{\partial^2 F^{\nu\sigma}A_\mu^{(\Sigma)}}{\partial x^\sigma\partial x^\nu} = 0$$

we conclude that the term $\frac{\partial F^{\nu\sigma}A_\mu^{(\Sigma)}}{\partial x^\sigma}$ in the stress-energy tensor can be discarded.

As a result, one obtains

$$T_\mu^\nu = -i\hbar\frac{c}{2}(\partial_\mu\overline{\Psi})\gamma^\nu\Psi + i\hbar\frac{c}{2}\overline{\Psi}\gamma^\nu\partial_\mu\Psi - \frac{1}{c}j^\nu A_\mu^{(\Sigma)} - \frac{1}{4\pi}F^{\nu\sigma}F_{\mu\sigma} + \frac{1}{8\pi}\frac{\partial G_{\sigma\lambda}}{\partial x_\nu}\frac{\partial G^{\sigma\lambda}}{\partial x^\mu}$$
$$-\delta_\mu^\nu\left[\frac{c}{2}\overline{\Psi}\gamma^\sigma\left(i\hbar\partial_\sigma\Psi + \frac{e}{c}A_\sigma^{(\Sigma)}\Psi\right) + \frac{c}{2}\left(-i\hbar\partial_\sigma\overline{\Psi} + \frac{e}{c}A_\sigma^{(\Sigma)}\overline{\Psi}\right)\gamma^\sigma\Psi + i\frac{e}{2}G_{\sigma\lambda}\overline{\Psi}\sigma^{\sigma\lambda}\Psi - m_e c^2\overline{\Psi}\Psi - \frac{1}{16\pi}F_{\sigma\lambda}F^{\sigma\lambda}\right.$$
$$\left.+\frac{1}{16\pi}\frac{\partial G_{\sigma\lambda}}{\partial x^\omega}\frac{\partial G^{\sigma\lambda}}{\partial x_\omega} - \frac{\varkappa^2}{16\pi}G_{\sigma\lambda}G^{\sigma\lambda}\right]$$

Transformation

$$T_{\mu\nu} \to T_{\mu\nu} + \partial^\lambda f_{\lambda\mu\nu}$$

where $f_{\lambda\mu\nu} = -f_{\nu\mu\lambda}$ does not change the conservation law (72).

We choose the function $f_{\lambda\mu\nu}$ in the form [11]

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$$f_{\lambda\mu\nu} = -\frac{i\hbar c}{8}\overline{\Psi}(\gamma_\lambda\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu\gamma_\lambda)\Psi$$

Then

$$\partial^\lambda f_{\lambda\mu\nu} = -\frac{i\hbar c}{8}\partial^\lambda\overline{\Psi}(\gamma_\lambda\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu\gamma_\lambda)\Psi + \frac{i\hbar c}{8}\overline{\Psi}(\gamma_\lambda\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu\gamma_\lambda)\partial^\lambda\Psi$$

We transform this expression using the properties of Dirac matrices [10]

$$\gamma_\nu\gamma_\lambda + \gamma_\lambda\gamma_\nu = 2g_{\nu\lambda}$$

Then

$$\gamma_\lambda\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu\gamma_\lambda = -\gamma_\lambda(\gamma_\nu\gamma_\mu - \gamma_\mu\gamma_\nu) + 2g_{\lambda\nu}\gamma_\mu - 2g_{\mu\lambda}\gamma_\nu$$

$$\gamma_\lambda\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu\gamma_\lambda = -2g_{\lambda\nu}\gamma_\mu + 2\gamma_\nu g_{\mu\lambda} - (\gamma_\nu\gamma_\mu - \gamma_\mu\gamma_\nu)\gamma_\lambda$$

$$\partial^\lambda f_{\lambda\mu\nu} = \frac{i\hbar c}{8}\left[(\partial^\lambda\overline{\Psi})\gamma_\lambda(\gamma_\nu\gamma_\mu - \gamma_\mu\gamma_\nu)\Psi - 2(\partial_\nu\overline{\Psi})\gamma_\mu\Psi + 2(\partial_\mu\overline{\Psi})\gamma_\nu\Psi + 2\overline{\Psi}\gamma_\mu\partial_\nu\Psi - 2\overline{\Psi}\gamma_\nu\partial_\mu\Psi + \overline{\Psi}(\gamma_\nu\gamma_\mu - \gamma_\mu\gamma_\nu)\gamma_\lambda\partial^\lambda\Psi\right]$$

Taking into account equations (62) and (71), one writes

$$i\hbar(\partial^\lambda\overline{\Psi})\gamma_\lambda(\gamma_\nu\gamma_\mu - \gamma_\mu\gamma_\nu)\Psi = \frac{e}{c}A_\lambda^{(\Sigma)}\overline{\Psi}\gamma^\lambda(\gamma_\nu\gamma_\mu - \gamma_\mu\gamma_\nu)\Psi + i\frac{e}{2c}G_{\lambda\sigma}\overline{\Psi}\gamma^\lambda\gamma^\sigma(\gamma_\nu\gamma_\mu - \gamma_\mu\gamma_\nu)\Psi - m_e c\overline{\Psi}(\gamma_\nu\gamma_\mu - \gamma_\mu\gamma_\nu)\Psi$$

$$i\hbar\overline{\Psi}(\gamma_\nu\gamma_\mu - \gamma_\mu\gamma_\nu)\gamma_\lambda\partial^\lambda\Psi = -\frac{e}{c}A_\lambda^{(\Sigma)}\overline{\Psi}(\gamma_\nu\gamma_\mu - \gamma_\mu\gamma_\nu)\gamma^\lambda\Psi - i\frac{e}{2c}\overline{\Psi}(\gamma_\nu\gamma_\mu - \gamma_\mu\gamma_\nu)\gamma^\lambda\gamma^\sigma G_{\lambda\sigma}\Psi + m_e c\overline{\Psi}(\gamma_\nu\gamma_\mu - \gamma_\mu\gamma_\nu)\Psi$$

Then

$$i\hbar(\partial^\lambda\overline{\Psi})\gamma_\lambda(\gamma_\nu\gamma_\mu - \gamma_\mu\gamma_\nu)\Psi + i\hbar\overline{\Psi}(\gamma_\nu\gamma_\mu - \gamma_\mu\gamma_\nu)\gamma_\lambda\partial^\lambda\Psi$$

$$= \frac{e}{c}A_\lambda^{(\Sigma)}\overline{\Psi}[\gamma^\lambda(\gamma_\nu\gamma_\mu - \gamma_\mu\gamma_\nu) - (\gamma_\nu\gamma_\mu - \gamma_\mu\gamma_\nu)\gamma^\lambda]\Psi + i\frac{e}{2c}G_{\lambda\sigma}\overline{\Psi}[\gamma^\lambda\gamma^\sigma(\gamma_\nu\gamma_\mu - \gamma_\mu\gamma_\nu) - (\gamma_\nu\gamma_\mu - \gamma_\mu\gamma_\nu)\gamma^\lambda\gamma^\sigma]\Psi$$

Let's transform

$$\gamma_\lambda(\gamma_\nu\gamma_\mu - \gamma_\mu\gamma_\nu) - (\gamma_\nu\gamma_\mu - \gamma_\mu\gamma_\nu)\gamma_\lambda = \gamma_\lambda\gamma_\nu\gamma_\mu - \gamma_\lambda\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu\gamma_\lambda + \gamma_\mu\gamma_\nu\gamma_\lambda = -2\gamma_\lambda\gamma_\mu\gamma_\nu + 2\gamma_\mu\gamma_\nu\gamma_\lambda = 4\gamma_\mu g_{\nu\lambda} - 4g_{\mu\lambda}\gamma_\nu$$

$$\gamma_\lambda\gamma_\sigma(\gamma_\nu\gamma_\mu - \gamma_\mu\gamma_\nu) - (\gamma_\nu\gamma_\mu - \gamma_\mu\gamma_\nu)\gamma_\lambda\gamma_\sigma = \gamma_\lambda\gamma_\sigma\gamma_\nu\gamma_\mu - \gamma_\lambda\gamma_\sigma\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu\gamma_\lambda\gamma_\sigma + \gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\sigma$$

$$= 2g_{\mu\lambda}(\gamma_\sigma\gamma_\nu - \gamma_\nu\gamma_\sigma) + 4g_{\nu\lambda}\gamma_\mu\gamma_\sigma - 4g_{\mu\sigma}\gamma_\lambda\gamma_\nu + 2g_{\nu\sigma}(\gamma_\lambda\gamma_\mu - \gamma_\mu\gamma_\lambda)$$

$$= 4g_{\mu\lambda}\gamma_\sigma\gamma_\nu + 4g_{\nu\sigma}\gamma_\lambda\gamma_\mu - 4g_{\mu\lambda}g_{\nu\sigma} + 4g_{\nu\lambda}\gamma_\mu\gamma_\sigma - 4g_{\mu\sigma}\gamma_\lambda\gamma_\nu - 4g_{\nu\sigma}g_{\mu\lambda}$$

Then

$$i\hbar(\partial^\lambda\overline{\Psi})\gamma_\lambda(\gamma_\nu\gamma_\mu - \gamma_\mu\gamma_\nu)\Psi + i\hbar\overline{\Psi}(\gamma_\nu\gamma_\mu - \gamma_\mu\gamma_\nu)\gamma_\lambda\partial^\lambda\Psi = 4\frac{e}{c}\overline{\Psi}(A_\nu^{(\Sigma)}\gamma_\mu - A_\mu^{(\Sigma)}\gamma_\nu)\Psi + 4i\frac{e}{c}\overline{\Psi}[G_{\mu\sigma}\gamma^\sigma\gamma_\nu - G_{\nu\sigma}\gamma^\sigma\gamma_\mu + 2G_{\nu\mu}]\Psi$$

As a result, one obtains

$$\partial^\lambda f_{\lambda\mu\nu} = \frac{c}{2}\left[\frac{e}{c}\overline{\Psi}(A_\nu^{(\Sigma)}\gamma_\mu - A_\mu^{(\Sigma)}\gamma_\nu)\Psi + i\frac{e}{c}\overline{\Psi}[G_{\mu\sigma}\gamma^\sigma\gamma_\nu - G_{\nu\sigma}\gamma^\sigma\gamma_\mu + 2G_{\nu\mu}]\Psi\right]$$

$$+\frac{i\hbar c}{4}\left[-(\partial_\nu\overline{\Psi})\gamma_\mu\Psi + (\partial_\mu\overline{\Psi})\gamma_\nu\Psi + \overline{\Psi}\gamma_\mu\partial_\nu\Psi - \overline{\Psi}\gamma_\nu\partial_\mu\Psi\right]$$

Then, taking into account (66), one writes the stress-energy tensor in the form

$$T_{\mu\nu} = \frac{c}{4}\Big[\Big(-i\hbar(\partial_\nu\bar{\Psi}) + \frac{e}{c}\bar{\Psi}A_\nu^{(\Sigma)}\Big)\gamma_\mu\Psi + \Big(-i\hbar(\partial_\mu\bar{\Psi}) + \frac{e}{c}\bar{\Psi}A_\mu^{(\Sigma)}\Big)\gamma_\nu\Psi + \bar{\Psi}\gamma_\mu\Big(i\hbar\partial_\nu + \frac{e}{c}A_\nu^{(\Sigma)}\Big)\Psi + \bar{\Psi}\gamma_\nu\Big(i\hbar\partial_\mu + \frac{e}{c}A_\mu^{(\Sigma)}\Big)\Psi\Big]$$
$$+ i\frac{e}{2}\bar{\Psi}(G_{\mu\sigma}\gamma^\sigma\gamma_\nu - G_{\nu\sigma}\gamma^\sigma\gamma_\mu + 2G_{\nu\mu})\Psi - \frac{1}{4\pi}g^{\lambda\sigma}F_{\nu\lambda}F_{\mu\sigma} + \frac{1}{8\pi}\frac{\partial G_{\sigma\lambda}}{\partial x^\nu}\frac{\partial G^{\sigma\lambda}}{\partial x^\mu}$$
$$- g_{\mu\nu}\Big[\frac{c}{2}\bar{\Psi}\gamma^\sigma\Big(i\hbar\partial_\sigma\Psi + \frac{e}{c}A_\sigma^{(\Sigma)}\Psi\Big) + \frac{c}{2}\Big(-i\hbar\partial_\sigma\bar{\Psi} + \frac{e}{c}A_\sigma^{(\Sigma)}\bar{\Psi}\Big)\gamma^\sigma\Psi + i\frac{e}{2}G_{\sigma\lambda}\bar{\Psi}\sigma^{\sigma\lambda}\Psi - m_e c^2\bar{\Psi}\Psi - \frac{1}{16\pi}F_{\sigma\lambda}F^{\sigma\lambda}$$
$$+ \frac{1}{16\pi}\frac{\partial G_{\sigma\lambda}}{\partial x^\omega}\frac{\partial G^{\sigma\lambda}}{\partial x_\omega} - \frac{\varkappa^2}{16\pi}G_{\sigma\lambda}G^{\sigma\lambda}\Big]$$

Let's transform

$$i\bar{\Psi}(G_{\mu\sigma}\gamma^\sigma\gamma_\nu - G_{\nu\sigma}\gamma^\sigma\gamma_\mu + 2G_{\nu\mu})\Psi$$
$$= \frac{i}{2}g^{\lambda\sigma}G_{\mu\sigma}\bar{\Psi}(\gamma_\lambda\gamma_\nu + \gamma_\nu\gamma_\lambda)\Psi + \frac{i}{2}g^{\lambda\sigma}G_{\mu\sigma}\bar{\Psi}(\gamma_\lambda\gamma_\nu - \gamma_\nu\gamma_\lambda)\Psi - \frac{i}{2}g^{\lambda\sigma}G_{\nu\sigma}\bar{\Psi}(\gamma_\lambda\gamma_\mu + \gamma_\mu\gamma_\lambda)\Psi$$
$$- \frac{i}{2}g^{\lambda\sigma}G_{\nu\sigma}\bar{\Psi}(\gamma_\lambda\gamma_\mu - \gamma_\mu\gamma_\lambda)\Psi + 2iG_{\nu\mu}\bar{\Psi}\Psi = ig^{\lambda\sigma}G_{\mu\sigma}\bar{\Psi}\sigma_{\lambda\nu}\Psi - ig^{\lambda\sigma}G_{\nu\sigma}\bar{\Psi}\sigma_{\lambda\mu}\Psi$$

Then the stress-energy tensor takes the form

$$T_{\mu\nu} = \frac{c}{4}\Big[\Big(-i\hbar(\partial_\nu\bar{\Psi}) + \frac{e}{c}\bar{\Psi}A_\nu^{(\Sigma)}\Big)\gamma_\mu\Psi + \Big(-i\hbar(\partial_\mu\bar{\Psi}) + \frac{e}{c}\bar{\Psi}A_\mu^{(\Sigma)}\Big)\gamma_\nu\Psi + \bar{\Psi}\gamma_\mu\Big(i\hbar\partial_\nu + \frac{e}{c}A_\nu^{(\Sigma)}\Big)\Psi + \bar{\Psi}\gamma_\nu\Big(i\hbar\partial_\mu + \frac{e}{c}A_\mu^{(\Sigma)}\Big)\Psi\Big]$$
$$+ i\frac{e}{2}g^{\lambda\sigma}G_{\mu\sigma}\bar{\Psi}\sigma_{\lambda\nu}\Psi - i\frac{e}{2}g^{\lambda\sigma}G_{\nu\sigma}\bar{\Psi}\sigma_{\lambda\mu}\Psi - \frac{1}{4\pi}g^{\lambda\sigma}F_{\nu\lambda}F_{\mu\sigma} + \frac{1}{8\pi}\frac{\partial G_{\sigma\lambda}}{\partial x^\nu}\frac{\partial G^{\sigma\lambda}}{\partial x^\mu}$$
$$- g_{\mu\nu}\Big[\frac{c}{2}\bar{\Psi}\gamma^\sigma\Big(i\hbar\partial_\sigma\Psi + \frac{e}{c}A_\sigma^{(\Sigma)}\Psi\Big) + \frac{c}{2}\Big(-i\hbar\partial_\sigma\bar{\Psi} + \frac{e}{c}A_\sigma^{(\Sigma)}\bar{\Psi}\Big)\gamma^\sigma\Psi + i\frac{e}{2}G_{\sigma\lambda}\bar{\Psi}\sigma^{\sigma\lambda}\Psi - m_e c^2\bar{\Psi}\Psi - \frac{1}{16\pi}F_{\sigma\lambda}F^{\sigma\lambda}$$
$$+ \frac{1}{16\pi}\frac{\partial G_{\sigma\lambda}}{\partial x^\omega}\frac{\partial G^{\sigma\lambda}}{\partial x_\omega} - \frac{\varkappa^2}{16\pi}G_{\sigma\lambda}G^{\sigma\lambda}\Big]$$

Using equation (63), one obtains

$$\frac{e}{2}\big(ig^{\lambda\sigma}G_{\mu\sigma}\bar{\Psi}\sigma_{\lambda\nu}\Psi - ig^{\lambda\sigma}G_{\nu\sigma}\bar{\Psi}\sigma_{\lambda\mu}\Psi\big) = \frac{1}{8\pi}g^{\lambda\sigma}\frac{\partial}{\partial x_\omega}\left(G_{\mu\sigma}\frac{\partial G_{\lambda\nu}}{\partial x^\omega} - G_{\lambda\nu}\frac{\partial G_{\mu\sigma}}{\partial x^\omega}\right)$$

Let us calculate some components of the stress-energy tensor.

As a result, one obtains

$$T_{00} = m_e c^2\bar{\Psi}\Psi - \frac{c}{2}\Big[\Big(-i\hbar\nabla\bar{\Psi} - \frac{e}{c}\mathbf{A}_\Sigma\bar{\Psi}\Big)\boldsymbol{\gamma}\Psi + \bar{\Psi}\boldsymbol{\gamma}\Big(i\hbar\nabla\Psi - \frac{e}{c}\mathbf{A}_\Sigma\Psi\Big)\Big] + \frac{1}{8\pi}(\mathbf{H}^2 + \mathbf{E}^2) - i\frac{e}{2}G_{\sigma\lambda}\bar{\Psi}\sigma^{\sigma\lambda}\Psi$$
$$+ \frac{1}{8\pi}\left(\frac{\partial\mathbf{G}}{\partial x^0}\frac{\partial\mathbf{G}}{\partial x^0} + \frac{\partial\mathbf{G}}{\partial x^k}\frac{\partial\mathbf{G}}{\partial x^k} - \frac{\partial\mathbf{K}}{\partial x^0}\frac{\partial\mathbf{K}}{\partial x^0} - \frac{\partial\mathbf{K}}{\partial x^k}\frac{\partial\mathbf{K}}{\partial x^k}\right) + \frac{\varkappa^2}{8\pi}(\mathbf{G}^2 - \mathbf{K}^2)$$

Taking into account (35), by direct calculation one obtains

$$iG_{\sigma\lambda}\bar{\Psi}\sigma^{\sigma\lambda}\Psi = 2\mathbf{G}\bar{\Psi}\boldsymbol{\Sigma}\Psi - 2i\mathbf{K}\bar{\Psi}\boldsymbol{\alpha}\Psi = 2\mathbf{G}(\psi^*\boldsymbol{\sigma}\psi - \chi^*\boldsymbol{\sigma}\chi) - 2i\mathbf{K}(\psi^*\boldsymbol{\sigma}\chi - \chi^*\boldsymbol{\sigma}\psi)$$

As a result

$$T_{00} = m_e c^2\bar{\Psi}\Psi + \frac{1}{8\pi}(\mathbf{H}^2 + \mathbf{E}^2) - \frac{c}{2}\Big[\Big(-i\hbar\nabla\Psi^* - \frac{e}{c}\mathbf{A}_\Sigma\Psi^*\Big)\boldsymbol{\alpha}\Psi + \Psi^*\boldsymbol{\alpha}\Big(i\hbar\nabla\Psi - \frac{e}{c}\mathbf{A}_\Sigma\Psi\Big)\Big] - e\mathbf{G}\bar{\Psi}\boldsymbol{\Sigma}\Psi + ei\mathbf{K}\bar{\Psi}\boldsymbol{\alpha}\Psi$$
$$+ \frac{1}{8\pi}\left(\frac{\partial\mathbf{G}}{\partial x^0}\frac{\partial\mathbf{G}}{\partial x^0} + \frac{\partial\mathbf{G}}{\partial x^k}\frac{\partial\mathbf{G}}{\partial x^k} - \frac{\partial\mathbf{K}}{\partial x^0}\frac{\partial\mathbf{K}}{\partial x^0} - \frac{\partial\mathbf{K}}{\partial x^k}\frac{\partial\mathbf{K}}{\partial x^k}\right) + \frac{\varkappa^2}{8\pi}(\mathbf{G}^2 - \mathbf{K}^2)$$

Similarly,

$$T_{0k}=\frac{c}{4}\left[\left(-i\hbar\partial_k\Psi^*+\frac{e}{c}A_k^{(\Sigma)}\Psi^*\right)\Psi+\left(-i\hbar\partial_0\Psi^*+\frac{e}{c}\varphi_\Sigma\Psi^*\right)\alpha_k\Psi+\Psi^*\left(i\hbar\partial_k\Psi+\frac{e}{c}A_k^{(\Sigma)}\Psi\right)+\Psi^*\alpha_k\left(i\hbar\partial_0\Psi+\frac{e}{c}\varphi_\Sigma\Psi\right)\right]$$
$$-\frac{1}{4\pi}g^{\lambda\sigma}F_{k\lambda}F_{0\sigma}+\frac{1}{8\pi}\frac{\partial G_{\sigma\lambda}}{\partial x^k}\frac{\partial G^{\sigma\lambda}}{\partial x^0}+\frac{1}{8\pi}g^{\lambda\sigma}\frac{\partial}{\partial x_\omega}\left(G_{0\sigma}\frac{\partial G_{\lambda k}}{\partial x^\omega}-G_{\lambda k}\frac{\partial G_{0\sigma}}{\partial x^\omega}\right)$$

$$T_{k0}=\frac{c}{4}\left[\left(-i\hbar\partial_0\Psi^*+\frac{e}{c}\varphi_\Sigma\Psi^*\right)\alpha_k\Psi+\left(-i\hbar\partial_k\Psi^*+\frac{e}{c}A_k^{(\Sigma)}\Psi^*\right)\Psi+\Psi^*\alpha_k\left(i\hbar\partial_0\Psi+\frac{e}{c}\varphi_\Sigma\Psi\right)+\Psi^*\left(i\hbar\partial_k\Psi+\frac{e}{c}A_k^{(\Sigma)}\Psi\right)\right]$$
$$-\frac{1}{4\pi}g^{\lambda\sigma}F_{0\lambda}F_{k\sigma}+\frac{1}{8\pi}\frac{\partial G_{\sigma\lambda}}{\partial x^0}\frac{\partial G^{\sigma\lambda}}{\partial x^k}+\frac{1}{8\pi}g^{\lambda\sigma}\frac{\partial}{\partial x_\omega}\left(G_{k\sigma}\frac{\partial G_{\lambda 0}}{\partial x^\omega}-G_{\lambda 0}\frac{\partial G_{k\sigma}}{\partial x^\omega}\right)$$

After simple transformations, one obtains

$$T_{0k}=-\frac{1}{4\pi}(\mathbf{E}_\Sigma\times\mathbf{H}_\Sigma)_k+\frac{c}{4}\left(i\hbar\Psi^*\partial_k\Psi-i\hbar(\partial_k\Psi^*)\Psi+i\hbar\Psi^*\alpha_k\partial_0\Psi-i\hbar(\partial_0\Psi^*)\alpha_k\Psi+\frac{2e}{c}\varphi_\Sigma\Psi^*\alpha_k\Psi+\frac{2e}{c}A_k^{(\Sigma)}\Psi^*\Psi\right)$$
$$+\frac{1}{4\pi}\left(\frac{\partial\mathbf{G}}{\partial x^k}\frac{\partial\mathbf{G}}{\partial x^0}-\frac{\partial\mathbf{K}}{\partial x^k}\frac{\partial\mathbf{K}}{\partial x^0}\right)+\frac{1}{8\pi}g^{\lambda\sigma}\frac{\partial}{\partial x_\omega}\left(G_{0\sigma}\frac{\partial G_{\lambda k}}{\partial x^\omega}-G_{\lambda k}\frac{\partial G_{0\sigma}}{\partial x^\omega}\right)$$

$$T_{k0}=-\frac{1}{4\pi}(\mathbf{E}_\Sigma\times\mathbf{H}_\Sigma)_k+\frac{c}{4}\left(i\hbar\Psi^*\alpha_k\partial_0\Psi-i\hbar(\partial_0\Psi^*)\alpha_k\Psi+i\hbar\Psi^*\partial_k\Psi-i\hbar(\partial_k\Psi^*)\Psi+\frac{2e}{c}\varphi_\Sigma\Psi^*\alpha_k\Psi+\frac{2e}{c}A_k^{(\Sigma)}\Psi^*\Psi\right)$$
$$+\frac{1}{4\pi}\left(\frac{\partial\mathbf{G}}{\partial x^k}\frac{\partial\mathbf{G}}{\partial x^0}-\frac{\partial\mathbf{K}}{\partial x^k}\frac{\partial\mathbf{K}}{\partial x^0}\right)+\frac{1}{8\pi}g^{\lambda\sigma}\frac{\partial}{\partial x_\omega}\left(G_{k\sigma}\frac{\partial G_{\lambda 0}}{\partial x^\omega}-G_{\lambda 0}\frac{\partial G_{k\sigma}}{\partial x^\omega}\right)$$

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