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Article

Existence and uniqueness theorems in the inverse problem of recovering surface fluxes from pointwise measurements

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Abstract: Inverse problems of recovering surface fluxes on the boundary of a domain from pointwise observations are considered. Sharp conditions on the data ensuring existence and uniqueness of solutions in Sobolev classes are exposed. They are smoothness conditions on the data, geometric conditions on the location of measurement points, and the boundary of a domain. The proof relies on asymptotics of fundamental solutions to the corresponding elliptic problems and the Laplace transform. The problem is reduced to a linear algebraic system with a nondegerate matrix.

Keywords: inverse problem; surface flux; convection-diffusion equation; heat and mass transfer; pointwise measurements

MSC: 35R30; 35K20;80A20

0. Introduction

Under consideration is the parabolic equation

$$Mu = u_t + Lu = f(t,x), \ \ (t,x) \in Q = (0,T)\times G, \ T \le \infty, \tag{1}$$

where $Lu = -\Delta u + \sum_{i=1}^{n} a_i(x)u_{x_i} + a_0(x)u$, G is a domain in $\mathbb{R}^n$ with boundary $\Gamma \in C^2$, and $n = 2,3$. The equation (1) is furnished with the initial-boundary conditions

$$Bu|_S = g(t,x) \ \ (S = (0,T)\times\Gamma), \ \ u|_{t=0} = u_0(x), \tag{2}$$

where $Bu = \frac{\partial u}{\partial \nu} + \sigma(x)u$, with ν the outward unit normal to Γ, and, respectively, with the overdetermination conditions

$$u(t,b_i) = \psi_i(t) \ (i = 1,2,\dots,r), \tag{3}$$

where $\{b_i\}_{i=1}^{r}$ is a collection of points lying in G. The problem is to find a solution to the equation (1) satisfying (2)-(3) and an unknown function $g(t,x) = \sum_{j=1}^{r} \alpha_i(t)\Phi_i(x)$, where the functions $\Phi_i(x)$ are given and α_i are unknowns.

Inverse problems of recovering the boundary regimes are classical. The arise in many different problems of mathematical physics, in particular, in the heat and mass transfer theory, diffusion, filtration (see [1], [2]), and ecology [3]-[7].

A particular attention is payed to numerical solving the problems (1)-(3) and close to them. Most of the methods are based on reducing the problems to optimal control problems and minimization of the corresponding quadratic functionals (see, for instance, [8–14]). But the problem is that these functionals can have several local minima (see Sect. 3.3 in [15]). First, we describe some articles, where pointwise measurements are employed as additional

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data. Numerical determination of constant fluxes in the case of $n = 2$ is described in [9]. Similar results are presented in [16] for $n = 1$. The three-dimensional problem of recovering constant fluxes of green house gases is discussed in [3]. But numerical results are presented only in the one-dimensional case. In [4] (see also [5]) the method of recovering a constant surface flux relying on the approach developed in [17] is described, where special solutions to the adjoint problem are employed (see also [6,7]). The surface fluxes depending on t are recovered in [12,18–20] in the case of $n = 1$, and in [11,21,22] in the case of $n > 1$. The flux depending on time and spatial variables is reconstructed in [14,23].

It is sometimes the case when additional Diriclet data are given on a part of the boundary and the flux is reconstructed with the use of this data on another part of the boundary (see [24]). The article [13] is devoted to recovering of the flux $h(t,x)f(x)$ (the function $f(x)$ is unknown) with the use of final or integral overdetermination data. The existence and uniqueness theorems for solutions to the inverse problems of recovering the surface flux with the use of integral data are presented in [25,26].

There is a limited number of theoretical results devoted to the problem (1)-(3). We refer the reader to the article [27] (see also [28]), where, in the case of $Mu = u_t - \Delta u$, $r = 1$, and $b_1 \in \Gamma$, the existence and uniqueness theorems of classical solutions to the problem (1)-(3) are established. In contrast to our case, the problem is well-posed in the Hadamard sense. If the points $\{b_i\}_{i=1}^r$ are interior points of G then the problem becomes ill-posed and this fact was observed in many articles. In this article we describe a new approach to the existence theory of solutions to this problem and establish the corresponding existence and uniqueness theorems. We hope that these results can be used in developing new numerical algorithms for solving the problem.

1. Preliminaries

Let E be a Banach space. By $L_p(G;E)$ (G is a domain in $\mathbb{R}^n$), we mean the space of E-valued measurable functions such that $\|\|u(x)\|_E\|_{L_p(G)} < \infty$ [29]. The symbols $W_p^s(G;E)$ and $W_p^s(Q;E)$ stand for the Sobolev spaces (see the definitions in [29], [30]). If $E = \mathbb{R}$ or $E = \mathbb{R}^n$ then the latter spaces is denoted by $W_p^s(Q)$. The definitions of the Hölder spaces $C^{\alpha,\beta}(\overline{Q}), C^{\alpha,\beta}(\overline{S})$ can be found in [31]. By the norm of a vector, we mean the sum of the norms of coordinates. Given an interval $J = (0,T)$, put $W_p^{s,r}(Q) = W_p^s(J;L_p(G)) \cap L_p(J;W_p^r(G))$ and, respectively, $W_p^{s,r}(S) = W_p^s(J;L_p(\Gamma)) \cap L_p(J;W_p^r(\Gamma))$. Denote by $(u,v)_0 = \int_G u(x)\overline{v(x)}dx$ the inner product in $L_2(G)$. Let $\rho(Y,X)$ designate the distance between the sets X,Y. In this case, $\rho(x,\Gamma)$ is the distance from a point x to Γ. Denote by $B_\delta(x)$ the ball of radius δ centered at x.

We say that a boundary Γ of a domain G belongs to C^s, $s \geq 1$ (see the definition in [31, Ch.1]) if, for each point $x_0 \in \Gamma$, there exists a neighborhood Y_{x_0} about x_0 and a coordinate system y (the local coordinate system) obtained from the initial one by the translation of the origin and rotation such that the axis y_n is directed as the interior normal to Γ at x_0 and the equation of the part $Y_{x_0} \cap \Gamma$ of the boundary is of the form $y_n = \gamma(y')$, $\gamma(0) = 0$, $y' = (y_1,\ldots,y_{n-1})$; moreover, $\gamma \in C^s(\overline{B'_\delta(0)})$ ($B'_\delta(0) = \{y' : |y'| < \delta\}$), $G \cap Y_{x_0} = \{y : |y'| < \delta, 0 < y_n - \gamma(y') < \delta_1\}$, and $(\mathbb{R}^n \setminus G) \cap Y_{x_0} = \{y : |y'| < \delta, -\delta_1 < y_n - \gamma(y') < 0\}$. The smoothness of $\Gamma_0 \subset \Gamma$, with Γ_0 an open subset of Γ, is defined similarly. The numbers δ, δ_1 for a given G are fixed and we can assume without loss of generality that $\delta_1 > (2M+1)\delta$, with M the Lipschitz constant of the function γ. We employ the straightening of the boundary, i. e., the transformation $z_n = y_n - \gamma(y')$, $z' = y'$, $y = y(x)$, with y the local coordinate system at a given point b.

Below, we assume that $G = \mathbb{R}^n_+ = \{x \in \mathbb{R}^n : \ x_n > 0\}$ or G is a domain with compact boundary of the class C^2. The coefficients of the equation (1) are assumed to be real. We consider an elliptic operator L, i. e., there exists a constant $\delta_0 > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \geq \delta_0|\xi|^2 \ \forall \xi \in \mathbb{R}^n, \ \forall x \in G.$$

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Assign $\vec{a} = (a_1, a_2)$ for $n = 2$ and $\vec{a} = (a_1, a_2, a_3)$ for $n = 3$. The symbol $(\cdot, \cdot)$ stands for an inner product in $\mathbb{R}^n$. Let

$$\varphi_j(x) = \frac{1}{2}\int_0^1 (\vec{a}(b_j + \tau(x - b_j)), (x - b_j))\, d\tau \tag{4}$$

and assume that

$$a_i \in W^2_\infty(G)\ (i = 1, \dots, n),\ \nabla\varphi_j, \Delta\varphi_j\ (j = 1, \dots, r), a_0 \in L_\infty(G),\ \sigma \in C^1(\Gamma). \tag{5}$$

Moreover, we suppose that the functions a_i admits extensions to the whole $\mathbb{R}^n$ such that the condition (5) is valid in $G = \mathbb{R}^n$. If G is a domain with compact boundary of the class C^2 such an extension always exists (see, Theorem 1 in Subsect. 4.3.6 of Sect. Remarks in [32]). Consider the equation

$$L^*u + \overline{\lambda}u = \delta(x - b_j),\ \ x \in \mathbb{R}^n\ (n = 2, 3),\ j = 1, 2, \dots, r, \tag{6}$$

where the operator L^* is a formally adjoint to L. Its coefficients also satisfy (5). Let $b_j = (b_j^1, \dots, b_j^n)$. Introduce the functions $\lambda^\alpha = |\lambda|^\alpha e^{i\arg\lambda\alpha}$, $|\arg\lambda| < \pi$. It follows from Theorems 3.5 and 3.1 in [33] and Theorem 3.3 in [34] that

Theorem 1. *Assume that $G = \mathbb{R}^n$ $(n = 2, 3)$ and the conditions (5) hold. Fix $\delta_0 \in (0, \pi)$. Then there exists a number $\lambda_1 \geq 0$ such that, for all λ with $|\arg(\lambda - \lambda_1)| \leq \pi - \delta_0$, there exist a unique solution $u_n(x)$ $(n = 2, 3)$ to the equation (6) decreasing at ∞ such that $u_n \in W^1_p(G)$ for all $p \in (1, n/(n-1))$, and $u_n \in W^2_2(G_\varepsilon)$ for all $\varepsilon > 0$, $G_\varepsilon = \{x \in G : |x - b_j| > \varepsilon\}$. In every domain $0 < \varepsilon < |x - b_j| < R$ a solution u_n admits the representation*

$$u_2(x) = \frac{1}{2\sqrt{2\pi|x - b_j|}\lambda^{1/4}} e^{-\varphi_j(x) - \sqrt{\lambda}|x - b_j|}\Big(1 + O\Big(\frac{1}{\sqrt{|\lambda|}}\Big)\Big); \tag{7}$$

$$u_{2x_i}(x) = \frac{-\lambda^{1/4} e^{-\varphi_j(x) - \sqrt{\lambda}|x - b_j|}}{2\sqrt{2\pi|x - b_j|}}\Big(\frac{x_i - b_j^i}{|x - b_j|} + O(\frac{1}{\sqrt{|\lambda|}})\Big); \tag{8}$$

$$u_3(x) = \frac{1}{4\pi|x - b_j|} e^{-\varphi_j(x) - \sqrt{\lambda}|x - b_j|}\Big(1 + O\Big(\frac{1}{\sqrt{|\lambda|}}\Big)\Big); \tag{9}$$

$$u_{3x_i}(x) = \frac{-\sqrt{\lambda} e^{-\varphi_j(x) - \sqrt{\lambda}|x - b_j|}}{4\pi|x - b_j|}\Big(\frac{x_i - b_j^i}{|x - b_j|} + O(\frac{1}{\sqrt{|\lambda|}})\Big). \tag{10}$$

In what follows, we denote by $v_j(x)$ a solution u_n obtained in Theorem 1 for a given j. Consider the problem

$$Lw + \lambda w = f(x)\ (x \in G),\ \ Bw|_S = g, \ , \tag{11}$$

where $G = \mathbb{R}^n$ or $G = \mathbb{R}^n_+$ or G is a domain with compact boundary of the class C^2.

Theorem 2. *Let $a_i \in L_\infty(G)$ $(i = 0, 1, \dots, n)$, $f \in L_p(G)$, $\sigma \in C^1(\Gamma)$, and $g \in W_p^{2-1/p}(\Gamma)$ $(p > 1)$. Then there exists a number $\lambda_0 \geq 0$ such that, for all λ with $Re\,\lambda \geq \lambda_0$, there exists a unique solution $w \in W^2_p(G)$ to the problem (11).*

The theorem results from Theorem 5.7 for $G = \mathbb{R}^n$, Theorem 7.11 for $G = \mathbb{R}^n_+$ and Theorem 8.2 in the case of a domain with compact boundary in [30].

The following Green formula holds.

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Lemma 1. *Let the conditions* (5) *hold and let* $Re\,\lambda \geq \lambda_0$*, where* λ_0 *is chosen so that Theorem* 2 *is valid for* $p = 2$*. If* $w \in W_2^2(G)$ *is a solution to the problem* (11) *with* $f = 0$ *from the class specified in Theorem* 2 *then*

$$\int_\Gamma (-\frac{\partial w}{\partial \nu} - \sigma w)\overline{v_j} + w\overline{(\frac{\partial v_j}{\partial \nu} + \sigma^* v_j)} + w(b_j) = 0,\ \sigma^* = \sigma + \sum_{i=1}^n a_i \nu_i. \tag{12}$$

If $\varphi(x) \in C_0^\infty(\mathbb{R}^n)$ *and* $\varphi = 1$ *in some neighborhood about* b_j*, then*

$$\int_\Gamma (-\frac{\partial w}{\partial \nu} - \sigma w)\overline{\varphi v_j} + w\overline{(\frac{\partial \varphi v_j}{\partial \nu} + \sigma^* \varphi v_j)} + w(b_j) = \int_G 2\nabla\varphi\nabla v_j + \Delta\varphi v_j + \sum_{i=1}^n a_i \varphi_{x_i} v\, dx. \tag{13}$$

Proof. The proof is conventional. It suffices to approximate the functions w, v_j by sequences of smooth functions in the corresponding norms, to write out the above formulas (12), (13) for these approximations, and pass to the limit. □

Assume that $G = \mathbb{R}^n_+$ or G is a domain with compact boundary of the class C^2. Given a collection of points $b_j \in G$ ($j = 1, 2, \dots, r$), construct points $b \in \Gamma$ such that $\delta_j = \rho(b_j, \Gamma) = |b - b_j|$. Denote by K_j the set of these points. Let $b \in K_j$. Take $n = 3$. There exists a local coordinate system y such that the axes y_1, y_2 agree with the principal directions on the surface Γ at $y = 0$, in this case, $\sum_{i,j=1}^2 \gamma_{y_i y_j}(0) y_i y_j = \gamma_{y_1y_1}(0)y_1^2 + \gamma_{y_2y_2}(0)y_2^2$, $\gamma_{y_1,y_2}(0) = 0$, where $\kappa_i = \gamma_{y_i y_i}(0)$ are the principal curvatures of the surface $y_3 = \gamma(y')$ ($y' = (y_1, y_2)$) at 0. In the case of $n = 2$, the equation of the boundary in some neighborhood about b is of the form $y_2 = \gamma(y_1)$ and $\kappa = \gamma''(0)$ is the curvature of the curve γ at b.

Lemma 2. *Assume that, for every* $j = 1, 2, \dots, r$*, the set* K_j *consists of finitely many points and, for every* $b \in K_j$*, we have*

$$\max(\kappa_1, \kappa_2) < 1/\delta_j, \text{ where } n = 3,\ \kappa < 1/\delta_j, \text{ where } n = 2, \tag{14}$$

where κ_i *are principal curvatures of* Γ *at* b^* *for* $n = 3$ *and* κ *is the curvature of* Γ *for* $n = 2$*. Then there are constants* $c_0, c_1 > 0$, $0 < \varepsilon_1 \leq \delta$ *such that* $c_0|x - b|^2 \leq |x - b_j| - \delta_j \leq c_1|x - b|^2$ *for every* $b \in K_j$ *and all* $x \in B_{\varepsilon_1}(b) \cap \Gamma$, $j = 1, 2, \dots, r$.

Remark 1. *For* $n = 3$*, the condition* (14) *can be reformulated as follows. There exists a constant* $q_0 \in (0, 1)$ *such that* $\sum_{k,l=1}^2 \gamma_{y_k y_l}(0) y_k y_l \leq q_0 |y'|^2/\delta_j\ \forall y' \in \mathbb{R}^2$, $j = 1, 2, \dots, r$*, where* y *is a local coordinate system at* $b \in K_j$*. The claim follows from the fact that there exists an orthogonal transformation of coordinates such that the new axes* $\tilde{y}_1, \tilde{y}_2$ *agree with the principal directions on the surface* Γ *at* $y = 0$.

Proof. Take $b \in K_j$. We prove the claim in the case of $n = 3$. If $n = 2$ then the proof is simpler and we omit it. Let y be a local coordinate system at b. Since $y = y(x)$ is a superposition of an orthogonal transformation and a translation, the distances between points and their images are the same. We have $b = 0$, $b_j = (0, 0, y_{3j})$, $x = (y', \gamma(y'))$ ($y' = (y_1, y_2)$), $|x - b| = \sqrt{|y'|^2 + \gamma^2(y')}$, $|b_j - b| = |y_{3j}| = \delta_j$, $|x - b_j| = \sqrt{|y'|^2 + |\gamma(y') - y_{3j}|^2}$, and

$$|x - b_j| - \delta_j = \frac{|x - b_j|^2 - \delta_j^2}{|x - b_j| + \delta_j} = \frac{|y'|^2 + \gamma^2 - 2\gamma y_{3j}}{|x - b_j| + \delta_j} = J.$$

Remark 1 implies that

$$\gamma(y') = \frac{1}{2}\sum_{i,j=1}^{n-1} \gamma_{y_i y_j}(0) y_i y_j + o(|y'|^2) \leq q_0|y'|^2/2\delta_j + o(|y'|^2)$$

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in some neighborhood about 0. Fix a parameter $\varepsilon_0 > 0$ such that $\varepsilon_0 + q_0 < 1$. In this case there exists $\delta_1 \le \delta$ such that

$$q_0|y'|^2/2\delta_j + o(|y'|^2) \le (\varepsilon_0 + q_0)|y'|^2/2\delta_j$$

for $|y'| \le \delta_1$. Therefore, we obtain

$$J \ge \frac{|y'|^2(1-(q_0+\varepsilon_0)) + \gamma^2}{|x-b_j| + \delta_j} \ge c_0|x-b|^2,\ c_0 > 0.$$

The converse inequality follows directly from the definition of the quantity J.

Below, we preserve the notations of Lemma 2. Take $b \in K_j$. We can define the transformations $y = y(x)$ and $x = x(y)$. For $n = 3$, put $c_j(b) = 1/\sqrt{1-\delta_j\kappa_1}$, $d_j(b) = 1/\sqrt{1-\delta_j\kappa_2}$, $I_j(b) = c_j(b)d_j(b)$, $c_j^*(b) = 1/\sqrt{1+\delta_j\kappa_1}$, $d_j^*(b) = 1/\sqrt{1+\delta_j\kappa_2}$, $I_j^*(b) = c_j^*(b)d_j^*(b)$, $B_{j\lambda}(b) = \{x \in \Gamma : y_1^2(x)/c_j^2(b) + y_2^2(x)/d_j^2(b) \le |\lambda|^{-1/2+\varepsilon_0}\}$, $\tilde{B}_{j\lambda}(b) = \{y \in \mathbb{R}^2 : y_1^2/c_j^2(b) + y_2^2/d_j^2(b) \le |\lambda|^{-1/2+\varepsilon_0}\}$, $B_{j\lambda}^*(b) = \{x \in \Gamma : y_1^2(x)/c_j^{*2}(b) + y_2^2(x)/d_j^{*2}(b) \le |\lambda|^{-1/2+\varepsilon_0}\}$, $\tilde{B}_{j\lambda}^*(b) = \{y \in \mathbb{R}^2 :\ y_1^2/c_j^{*2}(b) + y_2^2/d_j^{*2}(b) \le |\lambda|^{-1/2+\varepsilon_0}\}$, where the parameter $\varepsilon_0 \in (0,1/4)$ is chosen below. The map $y = y(x)$ takes $B_\lambda(b)$ onto $\tilde{B}_\lambda(b)$. Similar notations are used in the case of $n = 2$, i. e., $I_j(b) = c_j(b) = 1/\sqrt{1-\delta_j\kappa}$, $I_j^*(b) = c_j^*(b) = 1/\sqrt{1+\delta_j\kappa}$, $B_{j\lambda}(b) = \{x \in \Gamma : |y_1(x)|/c_j(b) \le |\lambda|^{-1/4+\varepsilon_0/2}\}$, $\tilde{B}_{j\lambda}(b) = \{y_1 \in \mathbb{R} :\ |y_1|/c_j(b) \le |\lambda|^{-1/4+\varepsilon_0/2}\}$, $B_{j\lambda}^*(b) = \{x \in \Gamma : |y_1(x)|/c_j(b) \le |\lambda|^{-1/4+\varepsilon_0/2}\}$, $\tilde{B}_{j\lambda}^*(b) = \{y_1 \in \mathbb{R} :\ |y_1|/c_j^*(b) \le |\lambda|^{-1/4+\varepsilon_0/2}\}$. Below, we assume that, for every $j = 1,2,\ldots,r$, the set K_j consists of finitely many points and

$$\forall j = 1,2,\ldots,r,\ \forall b \in K_j\ |\kappa_i|\delta_j < 1\ (i = 1,2) \text{ for } n = 3,\ |\kappa|\delta_j < 1 \text{ for } n = 2, \tag{15}$$

where κ_i are the curvatures of Γ for $n = 3$ and, respectively, κ is the curvature of Γ for $n = 2$.

Let v_j be a solution to the equation (6). Given $b \in K_j$, construct the point b_j^b lying on the straight line joining b_j and b and such that $\delta_j = |b_j - b| = |b - b_j^b|$, $|b_j - b_j^b| = 2\delta_j$. The point b_j^b is symmetric to b_j with respect to the surface Γ. Let v_j^b be a solution to the equation (6), where the point b_j is replaced with b_j^b. Denote by φ_j^b the functions defined by the equality (4), where b_j is replaced with b_j^b. In what follows, we assume that the closures of coordinate neighborhoods about the points $b \in K_j$ are disjoint, otherwise, we can always decrease them. Fix a point $b \in K_j$. The quantity $\min_{b' \in K_j, b' \ne b} |b' - b_j^b| - \delta_j$ is positive (it depends on δ_j and the angles between the vectors $\overrightarrow{bb_j}$ and $\overrightarrow{b'b_j}$). Let $X_b = \overline{Y_b \cap \Gamma}$ (Y_b is the coordinate neighborhood about b). Without loss of generality, we can also assume that the constant $\min_{b' \in K_j, b' \ne b} \rho(X_{b'}, b_j^b) - \delta_j$ is positive for all $b \in K_j$ and all j, otherwise, we decrease the parameter δ of the coordinate neighborhoods Y_b. Denote by $\varepsilon^0 > 0$ a constant smaller than the minimum of these constants. Theorem 1 for $b \ne b'$ and $b \in K_j$ yields

$$|\sqrt{\lambda} v_j^b e^{\delta_j\sqrt{\lambda}}| + |e^{\delta_j\sqrt{\lambda}}|\nabla v_j^b|| \le c_1 e^{-\varepsilon^0\sqrt{|\lambda|/2}}\ \forall x \in X_{b'}, \tag{16}$$

where $c_2 > 0$ and $q_0 \in (0,1)$ are constants independent of j, $b \in K_j$, and λ such that $Re\,\lambda \ge \lambda_1$ (see Theorem 1). $\square$

Lemma 3. *Assume that the conditions* (5) *and* (15) *hold,* $b \in K_j$ $(j = 1,2,\ldots,r)$, *and*

$$\Phi(x) \in C^{\alpha_0}(X_b),\ X_b = \overline{Y_b \cap \Gamma} \tag{17}$$

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for some $\alpha_0 \in (0,1]$. Then there exists a number $\lambda_0 > 0$ such that, for $Re\,\lambda \geq \lambda_0$, we have the representation

$$\Phi_j = \sqrt{\lambda} e^{\delta_j \sqrt{\lambda}} \int_{X_b} \Phi(x) \overline{v_j(x)}\, d\Gamma = \frac{\Phi(b) e^{-\varphi_j(b)}}{2I_j(b)} (1 + O(|\lambda|^{-\beta})),\ \beta = \alpha_0/4, \tag{18}$$

$$\Phi_j^* = \sqrt{\lambda} e^{\delta_j \sqrt{\lambda}} \int_{X_b} \Phi(x) \overline{v_j^b(x)}\, d\Gamma = \frac{\Phi(b) e^{-\varphi_j^b(b)}}{2I_j^*(b)} (1 + O(|\lambda|^{-\beta}), \tag{19}$$

$$e^{\delta_j \sqrt{\lambda}} \int_{X_b} \overline{v_j}\, d\Gamma = \frac{e^{-\varphi_j(b)}}{2I_j(b)\sqrt{\lambda}} (1 + O(|\lambda|^{-1/4})),\ e^{\delta_j \sqrt{\lambda}} \int_{X_b} \frac{\partial \overline{v_j}}{\partial \nu}\, d\Gamma = \frac{-e^{-\varphi_j(b)}}{2I_j(b)} (1 + O(|\lambda|^{-1/4})), \tag{20}$$

$$e^{\delta_j \sqrt{\lambda}} \int_{X_b} \overline{v_j^b}\, d\Gamma = \frac{e^{-\varphi_j^b(b)}}{2I_j^*(b)\sqrt{\lambda}} (1 + O(|\lambda|^{-1/4})),\ e^{\delta_j \sqrt{\lambda}} \int_{X_b} \frac{\partial \overline{v_j^b}}{\partial \nu}\, d\Gamma = \frac{e^{-\varphi_j^b(b)}}{2I_j^*(b)} (1 + O(|\lambda|^{-1/4})). \tag{21}$$

Proof. Consider the case of $n = 3$. We have

$$I = \int_{X_b} \Phi(x) \overline{v_j(x)}\, d\Gamma = \int_{B_\lambda(b)} \Phi(x) \overline{v_j(x)}\, d\Gamma + \int_{X_b \setminus B_\lambda(b)} \Phi(x) \overline{v_j(x)}\, d\Gamma. \tag{22}$$

Theorem 1 implies that

$$\overline{v_j(x)} = \frac{1}{4\pi |x - b_j|} e^{-\varphi_j(x) - \sqrt{\lambda}|x - b_j|} (1 + O(\frac{1}{\sqrt{|\lambda|}})),\ x \in \Gamma,$$

where $Re\,\lambda \geq \lambda_1$. We can assume that $|O(\frac{1}{\sqrt{|\lambda|}})| \leq 1/2$ for all such λ and j. Estimate the second integral J_2 on the right-hand side of (22) from above. We derive that

$$|J_2 e^{\sqrt{\lambda}\delta_j}| \leq c \int_{X_b \setminus B_\lambda(b)} |e^{\sqrt{\lambda}\delta_j}||v_j(x)|\, d\Gamma \leq c_1 \int_{X_b \setminus B_\lambda(b)} e^{-Re\,\sqrt{\lambda}(|x - b_j| - \delta_j)}\, d\Gamma. \tag{23}$$

In view of the definitions, there exists a constant $\varepsilon_2 > 0$ such that $|x - b_j| - \delta_j \geq \varepsilon_2 |\lambda|^{-1/2+\varepsilon_0}$ for all $x \in X_b \setminus B_\lambda(b)$ and, thereby,

$$|J_2 e^{\sqrt{\lambda}\delta_j}| \leq c_4 e^{-\varepsilon_4 |\lambda|^{\varepsilon_0}} \tag{24}$$

for some constant $\varepsilon_4 > 0$. For the first summand J_1 on the right-hand side of (22), we have

$$e^{\sqrt{\lambda}\delta_j} J_1 = e^{\sqrt{\lambda}\delta_j} \Big[\int_{B_\lambda(b)} (e^{-\varphi_j(x)} \Phi(x) - e^{-\varphi_j(b)} \Phi(b)) e^{\varphi_j(x)} \overline{v_j(x)}\, d\Gamma + \Phi(b) e^{-\varphi_j(b)} \int_{B_\lambda(b)} e^{\varphi_j(x)} \overline{v_j(x)}\, d\Gamma \Big]. \tag{25}$$

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Consider the last integral in (25) multiplied by $e^{\sqrt{\lambda}\delta_j}$. This quantity is written as

$$J_2 = \Phi(b)e^{-\varphi_j(b)} \int_{B_\lambda(b)} \frac{e^{-\sqrt{\lambda}(|x-b_j|-\delta_j)}}{4\pi|x-b_j|}(1+O(\frac{1}{\sqrt{|\lambda|}}))\, d\Gamma =$$
$$\Phi(b)e^{-\varphi_j(b)} \int_{\tilde{B}_\lambda(b)} \frac{e^{-\sqrt{\lambda}(|y-\tilde{b}_j|-\delta_j)}}{4\pi|y-\tilde{b}_j|}\sqrt{1+|\nabla\gamma(y)|^2}(1+O(\frac{1}{\sqrt{|\lambda|}}))\, dy' =$$
$$\frac{\Phi(b)e^{-\varphi_j(b)}}{4\pi\delta_j}\int_{\tilde{B}_\lambda(b)} e^{-\sqrt{\lambda}(|y-\tilde{b}_j|-\delta_j)}\, dy' +$$
$$\Phi(b)e^{-\varphi_j(b)} \int_{\tilde{B}_\lambda(b)} e^{-\sqrt{\lambda}(|y-\tilde{b}_j|-\delta_j)}\psi_0(y)O(\frac{1}{\sqrt{|\lambda|}})\, dy' +$$
$$\Phi(b)e^{-\varphi_j(b)} \int_{\tilde{B}_\lambda(b)} e^{-\sqrt{\lambda}(|y-\tilde{b}_j|-\delta_j)}(\psi_0(y)-\psi_0(0))\, dy',\ \psi_0(y) = \frac{\sqrt{1+|\nabla\gamma(y)|^2}}{4\pi|y-\tilde{b}_j|}, \quad (26)$$

where $\tilde{b}_j$ is the point b_j written in the coordinate system y. Consider the integral $I_0 = \int_{\tilde{B}_\lambda(b)} e^{-\sqrt{\lambda}(|y-\tilde{b}_j|-\delta_j)}dy'$. We can assume that the axes of the local coordinate system y are directed as the principal directions on Γ at b. In this case (see Lemma 2) we obtain that

$$|y-b_j|-\delta_j = \frac{y_1^2(1-\kappa_1\delta_j)+y_2^2(1-\kappa_2\delta_j)+o(|y'|^2)}{|x(y)-b_j|+\delta_j} = \frac{y_1^2(1-\kappa_1\delta_j)}{2\delta_j} + \frac{y_2^2(1-\kappa_2\delta_j)}{2\delta_j} + o(|y'|^2),$$

where $o(|y'|^2)$ is a C^2-function in some neighborhood about 0. Make the change of variables $y_i = \tau_i\sqrt{2\delta_j/(1-\kappa_i\delta_j)}$ in I_0. We obtain that

$$I_0 = \frac{2\delta_j}{c_j(b)d_j(b)}\int_{|\tau|\le r_0} e^{-\sqrt{\lambda}(|\tau|^2+o(|\tau|^2))}\, d\tau,\ r_0 = |\lambda|^{-1/4+\varepsilon_0/2}/\sqrt{2\delta_j}.$$

Introducing the polar coordinate system, we arrive at the expression

$$I_0 = \frac{2\delta_j}{c_j(b)d_j(b)}\int_0^{2\pi}\int_0^{r_0} e^{-\sqrt{\lambda}\varphi_0(r,\psi)} r\, dr d\psi,\ \varphi_0(r,\psi) = r^2 + o(r^2).$$

Integrating by parts yields

$$I_0 = \frac{-2\delta_j}{\sqrt{\lambda}c_j(b)d_j(b)}\int_0^{2\pi} e^{-\sqrt{\lambda}\varphi_0(r,\psi)}\frac{r}{\varphi_{0r}(r,\psi)}\Big|_{r=0}^{r_0} d\psi +$$
$$\frac{2\delta_j}{\sqrt{\lambda}c_j(b)d_j(b)}\int_0^{2\pi}\int_0^{r_0} e^{-\sqrt{\lambda}\varphi_0(r,\psi)}\Big(\frac{r}{\varphi_{0r}(r,\psi)}\Big)' dr d\psi =$$
$$\frac{2\delta_j\pi}{\sqrt{\lambda}c_j(b)d_j(b)} - \frac{2\delta_j}{\sqrt{\lambda}c_j(b)d_j(b)}\int_0^{2\pi} e^{-\sqrt{\lambda}\varphi_0(r_0,\psi)}\frac{r_0}{\varphi_{0r}(r_0,\psi)} d\psi +$$
$$\frac{2\delta_j}{\sqrt{\lambda}c_j(b)d_j(b)}\int_0^{2\pi}\int_0^{r_0} e^{-\sqrt{\lambda}\varphi_0(r,\psi)}\Big(\frac{r}{\varphi_{0r}(r,\psi)}\Big)' dr d\psi.$$

The last integral here admits the estimate

$$\Big|\int_0^{2\pi}\int_0^{r_0} e^{-\sqrt{\lambda}\varphi_0(r,\psi)}\Big(\frac{r}{\varphi_{0r}(r,\psi)}\Big)' dr d\psi\Big| \le c_7\int_0^{2\pi}\int_0^{r_0} e^{-Re\sqrt{\lambda}c_0 r^2} dr d\psi \le c_8|\lambda|^{-1/4}.$$

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The second integral on the right-hand side is estimated as

$$\Big|\frac{2\delta_j}{\sqrt{\lambda}c_j(b)d_j(b)}\int_0^{2\pi} e^{-\sqrt{\lambda}\varphi_0(r_0,\psi)}\frac{r_0}{\varphi_{0r}(r_0,\psi)}\,d\psi\Big| \le c_9 e^{-\varepsilon_5|\lambda|^{\varepsilon_0}},$$

where ε_5 is a positive constant. Thus, we establish the representation

$$I_0 = \frac{2\delta_j\pi}{\sqrt{\lambda}c_j(b)d_j(b)}(1+O(|\lambda|^{-1/4})). \tag{27}$$

Consider the integral

$$I_0' = \int_{\tilde{B}_\lambda(b)} |y'|^{\beta_0} e^{-Re\sqrt{\lambda}(|y-\tilde{b}_j|-\delta_j)}\,dy' \le c_0\int_{|\tau|\le r_0} |\tau|^{\beta_0} e^{-Re\sqrt{\lambda}(|\tau|^2+o(|\tau|^2))}\,d\tau.$$

Introducing the polar coordinate system, we infer

$$I_0' \le c_0\int_0^{2\pi}\int_0^{r_0} e^{-Re\sqrt{\lambda}\varphi_0(r,\psi)} r^{1+\beta_0}\,dr d\psi,\ \ \varphi_0(r,\psi) = r^2 + o(r^2).$$

Making the change of variables $r = t/|Re\sqrt{\lambda}|^{1/2}$, we obtain the estimate

$$I_0' \le c_0|Re\sqrt{\lambda}|^{-1-\beta_0/2}\int_0^{2\pi}\int_0^{r_0|Re\sqrt{\lambda}|^{1/2}} e^{-t^2(1+\frac{Re\sqrt{\lambda}}{t^2}o(\frac{t^2}{Re\sqrt{\lambda}}))} t^{1+\beta_0}\,dt d\psi \le c_1|\lambda|^{-1/2-\beta_0/4}.$$

This inequality and (24) imply that

$$I_0' \le c_1|\lambda|^{-1/2-\beta_0/4}, \tag{28}$$

where the constant c_1 is independent of λ. In this case the last integral I_1 on the right-hand side of (26) admits the estimate

$$|I_1| \le \Big|\Phi_i(b)e^{-\varphi_j(b)}\int_{\tilde{B}_\lambda(b)} e^{-\sqrt{\lambda}(|y-\tilde{b}_j|-\delta_j)}(\psi_0(y)-\psi_0(0))\,dy'\Big| \le$$
$$c_{10}\int_{\tilde{B}_\lambda(b)} e^{-Re\sqrt{\lambda}(|y-\tilde{b}_j|-\delta_j)}|y'|^2\,dy' \le c_{11}|\lambda|^{-1}.$$

In view of (28), the previous integral I_2 in (26) ($\beta_0 = 1$) is estimated as follows: $|I_2| \le c_{12}/|\lambda|$. Finally, the second summand on the right-hand side of (25) is representable as

$$J_2 = \frac{\Phi(b)e^{-\varphi_j(b)}}{2\sqrt{\lambda}c_j(b)d_j(b)}(1+O(|\lambda|^{-1/4})). \tag{29}$$

In view of our conditions on the coefficients, $\varphi_j \in W^2_\infty(K)$ for every compact set $K \in \overline{G}$ and, thereby, $|\varphi_j(x)-\varphi_j(b)| \le c|x-b| = c\sqrt{|y'|^2+|\gamma(y')|^2} \le c_1|y'|$. Involving the condition of the lemma and (28), we can estimate the integral $J_3 = e^{\sqrt{\lambda}\delta_j}\int_{B_\lambda(b)}(\Phi(x)-\Phi(b))\overline{v_j(x)} + \Phi(b)(1-e^{-\varphi_j(b)+\varphi_j(x)})\overline{v_j(x)}\,d\Gamma$ on the right-hand side of (25) by

$$|J_3| \le c_2\int_{\tilde{B}_\lambda(b)} |y'|^{\alpha_0} e^{-Re\sqrt{\lambda}(|y-\tilde{b}_j|-\delta_j)}dy' \le c_4|\lambda|^{-1/2-\alpha_0/4}. \tag{30}$$

The representation (29) and the estimate (30) validate the equality (18). The equality (19) is proven by analogy and the former equalities in (20), (21) are consequences of (18) and (19). The proof in the case of $n = 2$ is simpler. Display the asymptotics of the main integral

$$I = \Phi(b)e^{-\varphi_j(b)}I_0,\ \ I_0 = e^{\sqrt{\lambda}\delta_j}\int_{X_b} e^{\varphi_j(x)}\overline{v_j(x)}\,d\Gamma,$$

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where $X_b = \{x(y) \in \Gamma : |y_1| \leq \delta\}$, $y = (y_1, y_2)$ is the local coordinate system at b, and $y_2 = \gamma(y_1)$ is the equation of the curve Γ. To reduce arguments, we take $\delta \leq \varepsilon_1$, where the parameter ε_1 is defined in lemma 2. Theorem 1 implies that

$$I_0 = \frac{\lambda^{-1/4}}{2\sqrt{2\pi}} \int_{X_b} e^{-\sqrt{\lambda}(|x-b_j|-\delta_j)} \frac{1}{|x-b_j|^{1/2}} (1 + O(\frac{1}{\sqrt{|\lambda|}}))\, d\Gamma = \frac{\lambda^{-1/4}}{2\sqrt{2\pi}} \int_{-\delta}^{\delta} e^{-\sqrt{\lambda}(|y-\tilde{b}_j|-\delta_j)} \frac{\sqrt{1+(\gamma'(y_1))^2}}{|y-\tilde{b}_j|^{1/2}}\, dy_1 + \frac{\lambda^{-1/4}}{2\sqrt{2\pi}} \int_{-\delta}^{\delta} e^{-\sqrt{\lambda}(|y-\tilde{b}_j|-\delta_j)} \frac{\sqrt{1+(\gamma'(y_1))^2}}{|y-\tilde{b}_j|^{1/2}} O(\frac{1}{\sqrt{|\lambda|}})\, dy_1. \quad (31)$$

As before, we have $|y - b_j| - \delta_j = \frac{y_1^2(1-\kappa\delta_j)-2\delta_j(\gamma-\kappa y_1^2/2)+\gamma^2}{\sqrt{y_1^2+(\gamma-\delta_j)^2}+\delta_j}$ $(\kappa = \kappa(b) = \gamma''(0))$. We have the asymptotic formula (see §1, Ch. 2 in [41])

$$\int_a^b e^{\lambda S(x)} f(x)\, dx = \sqrt{\frac{-2\pi}{\lambda S''(x_0)}} f(x_0) + O(1/|\lambda|^{3/2}),$$

where $x_0 \in (a, b)$ is a point in which S reaches its maximum. Applying this formula to the first integral on the right-hand side of (31) and estimating the second integral by $c/|\lambda|^{3/4}$, we obtain the formula

$$I_0 = \frac{\lambda^{-1/2}}{2\sqrt{1-\kappa\delta_j}} + O(1/|\lambda|^{3/4}).$$

All other arguments are similar. The proof in the case of $G = \mathbb{R}^n_+$ is even simpler and we omit it.

It remains to prove the latter inequalities in (19), (20). As before, take $n = 3$. The asymptotics from Theorem 1, ensures that

$$e^{\sqrt{\lambda}\delta_j} \frac{\partial \overline{v_j}}{\partial \nu} = \frac{-\sqrt{\lambda} e^{\varphi_j(x(y))} e^{-\sqrt{\lambda}(|y-\tilde{b}_j|-\delta_j)}}{4\pi|y-\tilde{b}_j|} [\frac{(y-\tilde{b}_j, \nu)}{|y-\tilde{b}_j|} + O(|\lambda|^{-1/2})],$$

$\nu = \frac{1}{\sqrt{1+|\nabla\gamma|^2}}(\gamma_{y_1}, \gamma_{y_2}, -1)$. If $y_n = \gamma(y')$ then we have

$$\frac{(y-\tilde{b}_j, \nu)}{|y-\tilde{b}_j|} = \frac{y_1\gamma_{y_1} + y_2\gamma_{y_2} - \gamma(y) + \delta_j}{|y-\tilde{b}_j|} = 1 + O(|y'|^2),\ \tilde{b}_j = (0, 0, \delta_j).$$

Thus, we obtain that

$$e^{\sqrt{\lambda}\delta_j} \frac{\partial \overline{v_j}}{\partial \nu} = -\sqrt{\lambda} e^{\delta_j\sqrt{\lambda}} \overline{v_j(x(y))} (1 + O(|y'|^2) + O(|\lambda|^{-1/2})). \quad (32)$$

This equality and the previous arguments validate the claim. □

Remark 2. *Let $G = B_R(x_0)$. Then the condition (15) holds if $b_j \neq x_0$ for all j.*

Consider the problem (11), where $f = 0$, i. e., the problem

$$Lw + \lambda w = 0,\ x \in G, \quad (33)$$

$$Bw|_S = g, \quad (34)$$

and obtain some estimates of its solution. Fix j and take $b \in K_j$. In Lemma 4 below, we use functions $\varphi(y) \in C_0^\infty(\mathbb{R}^n)$ such that $\varphi(y) = 1$ on the set $U_{3\delta/4} = \{y : |y'| \leq$

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$3\delta/4, |y_n| \leq M\delta + 3\delta/4\}$ and $supp\,\varphi \subset U_\delta = \{y : |y'| < \delta, |y_n| < (M+1)\delta\}$. The condition $\delta_1 \geq (2M+1)\delta$ ensures the inclusion $U_\delta \subset \overline{Y_b}$. The map $z_n = y_n - \gamma(y')$, $z' = y'$ takes a neighborhood $Y_b \cap G$ onto the set $U = \{z : |z'| < \delta, 0 < z_n < \delta_1\}$. Denote $B'_\delta(0) = \{z' : |z'| < \delta\}$ and $\Gamma_\delta = (\cup_{j=1}^r \cup_{b\in K_j} Y_b) \cap \Gamma$.

Lemma 4. *Assume that the conditions (5) hold, $b \in K_j$ $(j = 1,2,\ldots,r)$, and $g \in W_2^{1/2}(\Gamma) \cap W_2^1(X_b)$. Then there exists a number $\lambda_0 > 0$ such that, for $Re\,\lambda \geq \lambda_0$, there exists a unique a solution to the problem (33), (34) from the space $W_2^2(G)$ satisfying the estimates*

$$\int_G |\nabla w|^2 + |\lambda||w|^2\,dx \leq c_0\|g\|^2_{L_2(\Gamma)}|\lambda|^{-1/2+2\varepsilon_7},$$

$$\|w\|_{W_2^\alpha(\Gamma)} \leq c_1\|g\|_{L_2(\Gamma)}|\lambda|^{\alpha/2-1/2+\varepsilon_7},\ \alpha \in (0,1/2). \tag{35}$$

If $v = \varphi w$, with φ from the above-described class of functions, then there exist constants $c_2, c_3 > 0$ such that

$$\int_U \sum_{k,l=1}^{n-1} |v_{z_k z_l}|^2 + \sum_{k=1}^{n-1} |v_{z_n z_k}|^2 + |\lambda||\nabla_{z'} v|^2\,dx \leq c_2(\|g\|^2_{W_2^1(X_b)} + \|g\|^2_{L_2(\Gamma)})|\lambda|^{-1/2+2\varepsilon_7}, \tag{36}$$

$$\|v\|_{W_2^{1+\alpha}(B'_\delta(0))} \leq c_3(\|g\|_{W_2^1(X_b)} + \|g\|_{L_2(\Gamma)})|\lambda|^{\alpha/2-1/2+\varepsilon_7},\ \alpha \in (0,1/2), \tag{37}$$

where $\varepsilon_7 > 0$ is arbitrarily small constant. If additionally $g \in W_2^2(X_b)$ and

$$a_0 \in W^1_\infty(\cup_{b\in K_j}(Y_b \cap G)),\ \Gamma_\delta \in C^3,\ \sigma \in C^{3/2+\varepsilon}(\Gamma_\delta)\ (\varepsilon > 0), \tag{38}$$

then $\varphi w \in W_2^3(Y_b \cap G)$ for any φ and there exist constants $c_4, c_5 > 0$ such that

$$\int_U \sum_{i,j,k=1}^{n-1} |v_{z_i z_j z_k}|^2 + \sum_{k,i=1}^{n-1} |v_{z_i z_k z_n}|^2 + |\lambda| \sum_{i,k=1}^{n-1} v_{z_i z_k}|^2\,dx \leq c_4(\|g\|^2_{W_2^2(X_b)} + \|g\|^2_{L_2(\Gamma)})|\lambda|^{\frac{-1}{2}+2\varepsilon_7}, \tag{39}$$

$$\|v\|_{W_2^{2+\alpha}(B'_\delta(0))} \leq c_5(\|g\|_{W_2^2(X_b)} + \|g\|_{L_2(\Gamma)})|\lambda|^{\alpha/2-1/2+\varepsilon_7},\ \alpha \in (0,1/2). \tag{40}$$

Proof. Theorem 2 for $p = 2$ ensures the existence and uniqueness of solutions provided that $Re\,\lambda \geq \lambda_0$ for some $\lambda_0 > 0$. Multiply the equation (33) by a function $\overline{w}$ and integrate the result over G. Integrating by parts, we infer

$$\int_G |\nabla w|^2 + l_0(w)\overline{w} + \lambda|w|^2 = \int_\Gamma g\overline{w} - \sigma|w|^2\,d\Gamma,\ l_0(w) = \sum_{i=1}^n a_i w_{x_i} + a_0 w.$$

Separating the real and imaginery parts, we obtain

$$\int_G |\nabla w|^2 + Re\lambda|w|^2\,dx = Re\int_\Gamma g\overline{w} - \sigma|w|^2\,d\Gamma - Re\int_G l_0(w)\overline{w}\,dx. \tag{41}$$

$$Im\,\lambda\int_G |w|^2\,dx = Im\int_\Gamma g\overline{w} - \sigma|w|^2\,d\Gamma - Im\int_G l_0(w)\overline{w}\,dx.$$

The last equality yields

$$|Im\,\lambda|\int_G |w|^2\,dx \leq |Im\int_\Gamma g\overline{w} - \sigma|w|^2\,d\Gamma| + |Im\int_G l_0 w\overline{w}\,dx|. \tag{42}$$

Summing (42) and (41) and estimating the modules of the right-hand sides

$$\int_G |\nabla w|^2 + |\lambda||w|^2\,dx \leq c_0(|\int_\Gamma g\overline{w} - \sigma|w|^2\,d\Gamma| + |\int_G l_0(w)\overline{w}\,dx|). \tag{43}$$

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Below, we use the inequality

$$|ab| \leq \varepsilon|a|^p/p + \varepsilon^{-q/p}|b|^q/q,\ p \in (1,\infty),\ q = p/(p-1),\ \varepsilon > 0.$$

The last integral is estimated by

$$|\int_G l_0(w)\overline{w}\,dx| \leq \|\nabla w\|_{L_2(G)}\|w\|_{L_2(G)} + \|w\|^2_{L_2(G)} \leq \frac{1}{4}\|\nabla w\|^2_{L_2(G)} + c_1\|w\|^2_{L_2(G)}. \tag{44}$$

Similarly, we have

$$|\int_\Gamma g\overline{w} - \sigma|w|^2\,d\Gamma| \leq \|g\|_{L_2(\Gamma)}\|w\|_{L_2(\Gamma)} + c_2\|w\|^2_{L_2(\Gamma)} \leq$$
$$c(\varepsilon)\|g\|^2_{L_2(\Gamma)}|\lambda|^{-1/2+2\varepsilon_7} + \varepsilon|\lambda|^{1/2-2\varepsilon_7}\|w\|^2_{L_2(\Gamma)} + c_2\|w\|^2_{L_2(\Gamma)},$$

where ε and ε_7 are arbitrary positive constants. The embedding theorems and interpolation inequalities (see [29]) imply that

$$|\lambda|^{1/2-2\varepsilon_7}\|w\|^2_{L_2(\Gamma)} \leq c_3|\lambda|^{1/2-2\varepsilon_7}\|w\|^2_{W_2^{1/2+2\varepsilon_7}(G)} \leq$$
$$c_5|\lambda|^{1/2-2\varepsilon_7}\|w\|^{2(1/2+2\varepsilon_7)}_{W_2^1(G)}\|w\|^{2(1/2-2\varepsilon_7)}_{L_2(G)} \leq \|\nabla w\|^2 + c_6\|w\|^2_{L_2(G)}|\lambda|.$$

Similarly,

$$c_2\|w\|^2_{L_2(\Gamma)} \leq \frac{1}{4}\|\nabla w\|^2 + c_7\|w\|^2_{L_2(G)}. \tag{45}$$

Estimating the right-hand side of (43) with the use of (44)-(45), we arrive at the inequality

$$\int_G |\nabla w|^2 + |\lambda||w|^2\,dx \leq c_8\|g\|^2_{L_2(\Gamma)}|\lambda|^{-1/2+2\varepsilon_7} +$$
$$\varepsilon c_6|\lambda|\|w\|^2_{L_2(G)} + (\varepsilon + 1/2)\|\nabla w\|^2_{L_2(G)} + c_9\|w\|^2_{L_2(G)}.$$

Choosing sufficiently small ε and increasing λ_0, if necessary, we derive that

$$\int_G |\nabla w|^2 + |\lambda||w|^2\,dx \leq c_9\|g\|^2_{L_2(\Gamma)}|\lambda|^{-1/2+2\varepsilon_7}, \tag{46}$$

where the constant c_9 is independent of λ with $Re\,\lambda \geq \lambda_0$ and $\varepsilon_7 > 0$ can be taken arbitrarily small). Using (46) and interpolation inequalities we obtain that

$$\|w\|_{W_2^\alpha(\Gamma)} \leq c\|w\|_{W_2^{1/2+\alpha}(G)} \leq c_{10}\|w\|^{1/2+\alpha}_{W_2^1(G)}\|w\|^{1/2-\alpha}_{L_2(G)} \leq c_{11}|\lambda|^{\alpha/2-1/2+\varepsilon_7}\|g\|_{L_2(\Gamma)}.$$

Rewriting (33) in the coordinate system y, we obtain the problem

$$-\Delta w + \sum_{i=1}^n \tilde{a}_i w_{y_i} + a_0 w + \lambda w = 0,\ Bw|_\Gamma = g. \tag{47}$$

Multiply the equation (47) by $\varphi(y)$. The result is the problem

$$-\Delta v + \sum_{i=1}^n \tilde{a}_i v_{y_i} + a_0 v + \lambda v = -2\nabla w\nabla\varphi - w\Delta\varphi + \sum_{i=1}^n a_i\varphi_{y_i}w = f_0,\ v = w\varphi. \tag{48}$$

$$Bv|_\Gamma = \varphi g - \varphi_\nu w. \tag{49}$$

Introduce the coordinate system z, with $z' = y'$, $z_n = y_n - \gamma(y')$. In this case, the function $v = w\varphi$ is a solution to the problem

$$-\Delta_{z'} v + 2\sum_{i=1}^{n-1} \gamma_{z_i} v_{z_n z_i} - \sigma_0(z') v_{z_n z_n} + \sum_{i=1}^{n} c_i v_{z_i} + a_0 v + \lambda v = f_0,\ \sigma_0(z') = (1 + |\nabla_{z'}\gamma|^2). \tag{50}$$

$$-v_{z_n}\sigma_0(z') + \sum_{i=1}^{n-1} v_{z_i}\gamma_{z_i} + \sigma(x(z))\sigma_0(z') v|_{z_n=0} = (\varphi g - \varphi_v w)|_{z_n=0}\sigma_0(z').$$

Multiplying the equation (50) by $-\Delta_{z'} v$ and integrating the result over U, we obtain that

$$\int_U |\Delta_{z'} v|^2 - \sum_{i=1}^{n-1} \gamma_{z_i} v_{z_n z_i} \overline{\Delta_{z'} v} + \partial_{z_n}(-\sum_{i=1}^{n-1} \gamma_{z_i} v_{z_i} + \sigma_0(z') v_{z_n})\overline{\Delta_{z'} v} - (\sum_{i=1}^{n} c_i v_{z_i} + a_0 v)\overline{\Delta_{z'} v} + \lambda |\nabla_{z'} v|^2\, dz = (f_0, -\Delta_{z'} v)_0. \tag{51}$$

Integrating by parts, we rewrite the first summand in the form

$$\int_U |\Delta_{z'} v|^2\, dz = \sum_{k,l=1}^{n-1} \int_U |v_{z_k z_l}|^2\, dz. \tag{52}$$

Note that $v \in W_2^2(G)$ and integrating by parts we obtain the integrals containing third order derivatives. However, the result of integration is easily justified if we employ smooth approximations of functions in $W_2^2(G)$. Similar arguments can be found, for instance, in the proof of Lemma 7.1 of Ch. 3 in [38]. We also have

$$\int_U \partial_{z_n}(-\sum_{i=1}^{n-1} \gamma_{z_i} v_{z_i} + \sigma_0 v_{z_n})\overline{\Delta_{z'} v}\, dz = -\int_U \partial_{z_n} \nabla_{z'}(-\sum_{i=1}^{n-1} \gamma_{z_i} v_{z_i} + \sigma_0 v_{z_n})\overline{\nabla_{z'} v}\, dz =$$
$$\int_G \nabla_{z'}(-\sum_{i=1}^{n-1} \gamma_{z_i} v_{z_i} + \sigma_0 v_{z_n}) \cdot \overline{\nabla_{z'} v_{z_n}}\, dz + \int_{B'_\delta(0)} (\nabla_{z'}(-\sum_{i=1}^{n-1} \gamma_{z_i} v_{z_i} + \sigma_0 v_{z_n}) \cdot \overline{\nabla_{z'} v}\, dz' =$$
$$\int_U (1 + |\nabla_{z'}\gamma|^2)|\nabla_{z'} v_{z_n}|^2\, dz - \int_U \sum_{i=1}^{n-1} \gamma_{z_i} \nabla_{z'} v_{z_i} \cdot \overline{\nabla_{z'} v_{z_n}}\, dz - \int_U \sum_{i=1}^{n-1} v_{z_i} \nabla_{z'}\gamma_{z_i} \cdot \overline{\nabla_{z'} v_{z_n}}\, dz$$
$$+ \int_U v_{z_n} \nabla_{z'}\sigma_0(z') \overline{\nabla_{z'} v_{z_n}}\, dz - \int_{B'_\delta(0)} \nabla_{z'}((\varphi g - \varphi_v w)\sqrt{\sigma_0(z')}) \cdot \overline{\nabla_{z'} v}\, dz'. \tag{53}$$

Consider the expression

$$\int_U -\sum_{i=1}^{n-1} \gamma_{z_i} v_{z_n z_i} \overline{\Delta_{z'} v}\, dz = \int_U \sum_{i=1}^{n-1} \gamma_{z_i} v_{z_n} \overline{\Delta_{z'} v_{z_i}} + \sum_{i=1}^{n-1} \gamma_{z_i z_i} v_{z_n} \overline{\Delta_{z'} v}\, dz$$
$$= -\int_U \sum_{i,k=1}^{n-1} \gamma_{z_i} v_{z_n z_k} \overline{v_{z_i z_k}}\, dz + \int_U \sum_{i=1}^{n-1} \gamma_{z_i z_i} v_{z_n} \overline{\Delta_{z'} v}\, dz - \int_U \sum_{i=1}^{n-1} \gamma_{z_i z_k} v_{z_n} \overline{v_{z_i z_k}}\, dz \tag{54}$$

Using (52)-(54) in (51), we obtain

$$\int_U \sum_{k,l=1}^{n-1} |v_{z_k z_l}|^2 + \sum_{k=1}^{n-1} \sigma_0 |v_{z_n z_k}|^2 - 2Re \sum_{l,k=1}^{n-1} \gamma_{z_l} v_{z_n z_k} \overline{v_{z_l z_k}} + \lambda |\nabla_{z'} v|^2\, dz = (f_0, -\Delta_{z'} v)_0$$
$$-\int_U \sum_{i=1}^{n-1} \gamma_{z_i z_i} v_{z_n} \overline{\Delta_{z'} v}\, dz + \int_U \sum_{i=1}^{n-1} \gamma_{z_i z_k} v_{z_n} \overline{v_{z_i z_k}}\, dz + \int_U \sum_{i=1}^{n-1} v_{z_i} \nabla_{z'}\gamma_{z_i} \cdot \overline{\nabla_{z'} v_{z_n}}\, dz -$$
$$\int_U v_{z_n} \nabla_{z'}\sigma_0 \overline{\nabla_{z'} v_{z_n}}\, dz - \int_{B'_\delta(0)} (\nabla_{z'}((\varphi g - \varphi_v w)\sqrt{\sigma_0}) \cdot \overline{\nabla_{z'} v}\, dz'. \tag{55}$$

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As is easily seen, the inequality

$$\sum_{k,l=1}^{n-1} |v_{z_k z_l}|^2 + \sum_{k=1}^{n-1} \sigma_0 |v_{z_n z_k}|^2 - 2Re \sum_{l,k=1}^{n-1} \gamma_{z_l} v_{z_n z_k} \overline{v_{z_l z_k}} \geq c_3 (\sum_{k,l=1}^{n-1} |v_{z_k z_l}|^2 + \sum_{k=1}^{n-1} |v_{z_n z_k}|^2),$$

is valid for some constant $c_3 > 0$. Next, we infer

$$|\int_{B'_\delta(0)} \nabla_{z'}(\varphi_\nu w \sqrt{\sigma_0}) \cdot \overline{\nabla_{z'} v}|\, dz' \leq c \|\nabla_{z'}(\varphi_\nu w \sqrt{\sigma_0})\|_{(W_{2,0}^{1/2}(B'_\delta(0)))'} \|\nabla_{z'} v\|_{W_{2,0}^{1/2}(B'_\delta(0))}$$
$$\leq c_1 \|w\|_{W_2^{1/2}(B'_\delta(0))} \|\nabla_{z'} v\|_{W_2^{1/2}(B'_\delta(0))} \leq \varepsilon \|\nabla_{z'} v\|^2_{W_2^1(U)} + c(\varepsilon) \|w\|^2_{W_2^1(U)},$$

where $W_{2,0}^{1/2}(B'_\delta(0))$ is the space with the norm $\|v\|^2 = \|v\|^2_{W_2^{1/2}(B'_\delta(0))} + \int_{B'_\delta(0)} |v|^2 \frac{dz'}{\rho(z')}$, $\rho(z') = \rho(z', \partial B'_\delta(0))$, $\varepsilon > 0$ is arbitrary, and the last summand is estimated by $c\|g\|_{L_2(\Gamma)}$ (see (46)). Here we rely on the conventional theorems on pointwise multipliers and Proposition 12.1 of Ch. 1 in [39]. Next, we repeat the arguments of the proof of the estimate (46). We conclude that

$$\int_U \sum_{k,l=1}^{n-1} |v_{z_k z_l}|^2 + \sum_{k=1}^{n-1} |v_{z_n z_k}|^2 + |\lambda| |\nabla_{z'} v|^2 \, dz \leq c_0 (\|g\|^2_{W_2^1(X_b)} + \|g\|^2_{L_2(\Gamma)}) |\lambda|^{-1/2 + 2\varepsilon_7},$$

To establish (37), it suffices to prove the estimate

$$\|\nabla_{z'} v\|^2_{W_2^\alpha(\Gamma)} + \|v\|^2_{W_2^\alpha(\Gamma)} \leq c_1 (\|g\|_{W_2^1(X_b)} + \|g\|_{L_2(\Gamma)}) |\lambda|^{\alpha/2 - 1/2 + \varepsilon_7}, \ \alpha \in (0, 1/2),$$

which is justified by repeating of the proof of (35). To validate the second part of the claim, we first demonstrate the smoothness of a solution w. Take an arbitrary point $b \in K_j$ and the set Y_b. Construct a function $\varphi(y) \in C_0^\infty(\mathbb{R}^n)$ such that $supp\, \varphi \subset U_\delta$. The function $w_0 = w\varphi$ is a solution to the equation (48) from the space $W_2^2(Y_b \cap G)$ satisfying (49) on $\Gamma \cap Y_b$ and

$$-\Delta w_0 = -\sum_{i=1}^{n} a_i w_{0x_i} - a_0 w_0 - 2\nabla w \nabla \varphi - \Delta \varphi w \in W_2^1(Y_b),$$

$$\frac{\partial w_0}{\partial \nu}|_\Gamma = -\sigma w_0 - \varphi_\nu w + g\varphi \in W_2^{3/2}(\Gamma \cap Y_b).$$

Using the conventional theorems on extension of boundary data inside the domain [29] and Theorem §3 of Ch. 4 in [42], we conclude that $w_0 \in W_2^3(Y_b \cap G)$.

Consider the equation (50). Multiply (50) by $\Delta^2_{z'} v$ and integrate the result over U. The same arguments as those of the proof of the estimate (36) can be applied to justify (37), (39). The calculations are rather cumbersome and we omit them. □

Assume that the conditions (5) and (15) hold. In this case, for every j and $b \in K_j$, we can consruct the balls $B_j = B_{\delta_j}(b_j)$ and $B_j^b = B_{\delta_j}(b_j^b)$. Let $Y_{b,\varepsilon} = \{y \in Y_b : |y'| \leq \varepsilon\}$ $(\varepsilon \leq \delta)$.

Lemma 5. *Let the conditions* (5) *and* (15) *hold. Then, for every* $j = 1, 2, \ldots, r$, *there exists a function* $\varphi_j \in C_0^\infty(\mathbb{R}^n)$ *and constants* $\varepsilon_0, \rho \in (0, \delta/8)$ *such that* $\varphi_j(x) = 1$ *for* $x \in U_\rho = B_{\delta_j + \rho}(b_j) \cup \cup_{b \in K_j} Y_{b, \varepsilon_0/2 + \rho}$, $\varphi_j(x) = 0$ *for* $x \notin U_{3\rho}$, *and* $\rho(supp\, |\nabla \varphi_j| \cap G, B_j^b) > 0$ *for all* $b \in K_j$.

Proof. In view of (15), it is not difficult to establish that there exists a parameter $\varepsilon_0 < \delta/8$ such that $\rho(B_j \setminus \cup_{b \in K_j} Y_{b,\delta}, \Gamma) = \delta_0(\delta) > 0$ for all $\delta \leq \varepsilon_0$ and $\overline{B_j^b} \cap \cup_{b \in K_j} (Y_{b,\varepsilon_0} \cap \Gamma) = \{b\}$ for all $b \in K_j$. Put $\delta_0 = \min_{\delta \in [\varepsilon_0/2, \varepsilon_0]} \delta_0(\delta)$. Obviously, $\delta_0 > 0$. Take $\rho = \min(\varepsilon_0/8, \delta_0/8)$.

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Construct a nonnegative function $\omega(\xi) \in C_0^\infty(\mathbb{R}^n)$ such that $supp\, \omega \subset B_1(0)$, $\int_{\mathbb{R}^n} \omega(\xi)\, d\xi = 1$ and the averaged function

$$\varphi_j(x) = \frac{1}{\rho^n} \int_{\mathbb{R}^n} \omega(\frac{\xi - x}{\rho}) \chi_{U_{2\rho}}(\xi)\, d\xi,$$

where $\chi_{U_{2\rho}}(\xi)$ is the characteristic function of the set $U_{2\rho}$. By construction, $\varphi_j(x) = 1$ for $x \in U_\rho$ and $\varphi_j(x) = 0$ for $x \notin U_{3\rho}$. This function satisfies our conditions. □

Let

$$u_0(x) \in W_2^1(G),\ e^{-\lambda t} f \in L_2(Q),\ e^{-\lambda t}\alpha_i \in W_2^{1/4}(0,T),\ \Phi_i(x) \in W_2^{1/2}(\Gamma). \tag{56}$$

The following theorem results from Theorem 7.11 for $G = \mathbb{R}^n_+$ and Theorems 8.2 in the case of the domain with compact boundary in [37].

Theorem 3. *Assume that $T = \infty$ and $a_i(x) \in L_\infty(G)$ $(i = 0,1,\dots,n)$. Then there exists a constant $\lambda_0 \geq 0$ such that if $\lambda \geq \lambda_0$ and the condition (56) holds then there exists a unique solution to the problem (1)-(2) such that $e^{-\lambda t} u \in W_2^{1,2}(Q)$ and*

$$\|e^{-\lambda t} u\|_{W_2^{1,2}(Q)} \leq C_0(\|u_0\|_{W_2^1(G)} + \|e^{-\lambda t} f\|_{L_2(Q)} + \|e^{-\lambda t} g\|_{W_p^{1/4,1/2}(S)}). \tag{57}$$

Let E be a Hilbert space. Denote by $\tilde{W}^s_{2,\gamma_0}(0,\infty;E)$ the space of functions u defined on $(0,\infty)$ whose zero extensions $\tilde{u}(t)$ to the negative semiaxis belong to $W^s_{2,loc}(\mathbb{R};E)$ and

$$\|e^{-\gamma_0 t}\tilde{u}(t)\|_{W_2^s(\mathbb{R};E)} = \|u(t)\|_{s,\gamma_0} < \infty.$$

The Laplace transform $\mathcal{L}$ is an isomorphism of this space $\tilde{W}^s_{2,\gamma_0}(\mathbb{R}_+;E)$ onto the space E_{s,γ_0} of analytic functions in the domain $Re\, p > \gamma_0 \geq 0$ such that

$$\|U(p)\|^2_{s,\gamma_0} = \sup_{\gamma > \gamma_0} \int_{-\infty}^{\infty} \|U(\gamma + i\tau)\|^2_E (1 + |\gamma + i\tau|^{2s})\, d\tau < \infty.$$

If $E = \mathbb{C}$ or $E = L_2(G)$ or $E = W_2^s(G)$ (G is a domain in $\mathbb{R}^n$) then these properties of the Laplace transform can be found in [35] (see Theorem 7.1 and §8). For $T < \infty$, we similarly define the space $\tilde{W}_2^s(0,T)$ as the subspace of functions in $W_2^s(0,T)$ admitting the zero extensions for $t < 0$ of the same class. This space coincides with $W_2^s(0,T)$ for $s < 1/2$ and with the space of functions $u \in W_2^s(0,T)$ such that $u(0) = 0$ for $s > 1/2$. For $s = 1/2$, it coincides with the space of functions in $W_2^{1/2}(0,T)$ such that $ut^{-1/2} \in L_2(0,T)$ [35].

2. Basic results

We assume here that the conditions (5), (15), (17) are fulfilled. Let Ψ be the matrix with entries $\Psi_{ji} = \sum_{b\in K_j} \frac{\Phi_i(b)e^{-\varphi_j(b)}}{I_j(b)}$ $(i,j = 1,2,\dots,r)$. We assume that

$$det\, \Psi \neq 0,\ \Phi_i(x) \in W_2^{1/2}(\Gamma), \tag{58}$$

$$\Phi_i(x) \in W_2^1(X_b) \text{ for } n = 2,\ \Phi_i(x) \in W_2^2(X_b) \text{ for } n = 3,\ b \in \cup_{j=1}^r K_j. \tag{59}$$

Fix a parameter $\lambda_0 > 0$ greater than the maximum of the parameters defined in Theorem 1 with $\delta_0 = \pi/2$, Theorem 2 with $p = 2$, and Theorem 3. We assume that

$$u_0(x) \in W_2^1(G),\ e^{-\gamma_0 t} f \in L_2(Q). \tag{60}$$

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By Theorem 3, if the condition (60) holds for some $\gamma_0 \geq \lambda_0$ then there exists a unique solution w_0 to the problem (1)-(2), where $g = 0$, such that $e^{-\gamma_0 t} w_0 \in W_2^{1,2}(Q)$. Consider the problem (1)-(3). Changing the variables $w = u - w_0$, we obtain the simpler problem

$$w_t + Lw = 0,\ Bw|_S = g(t,x),\ \ w|_{t=0} = 0, \tag{61}$$

$$w(b_j,t) = \psi_j(t) - w_0(t,b_j) = \tilde{\psi}_j(t),\ j = 1,2,\dots,r. \tag{62}$$

We assume that $\tilde{\psi}_j(t) \in L_2(0,T)$ and

$$\tilde{\psi}_j(t) = \int_0^t V_{\delta_j}(t-\tau)\psi_{0j}(\tau)d\tau,\ \psi_{0j}e^{-\gamma_0 t} \in \tilde{W}_2^{n/4}(0,T)\ (n = 2,3), \tag{63}$$

where $V_\gamma(t) = \frac{e^{-\gamma^2/4t}}{4\pi t}$ for $n = 2$ and $V_\gamma = \frac{\gamma e^{-\gamma^2/4t}}{2\sqrt{\pi}t^{3/2}}$ for $n = 3$. For $T = \infty$, the condition (63) can be rewritten as

$$\sup_{\sigma > \gamma_0} \int_{-\infty}^{\infty} |\sigma + is|^{n/2} e^{Re\sqrt{p}\delta_j} |\mathcal{L}(\tilde{\psi}_j)(\sigma + is)|^2\, ds < \infty,\ p = \sigma + is. \tag{64}$$

For a finite T, the condition (63) can be stated as follows: there exists an extension of $\tilde{\psi}_j$ on $(0,\infty)$ satisfying (64). We have $\hat{V}_\gamma(\lambda) = \frac{i}{4}H_0^{(1)}(i\sqrt{\lambda}\gamma) = \frac{1}{2\sqrt{2\pi\gamma}\lambda^{1/4}} e^{-\sqrt{\lambda}\gamma}\Big(1 + O\Big(\frac{1}{\sqrt{|\lambda|}}\Big)\Big)$ for $n = 2$ and $\hat{V}_\gamma(\lambda) = e^{-\sqrt{\lambda}\gamma}$ for $n = 3$. Here $H_0^{(1)}$ is the Hankel function. The latter equality is derived in Lemma 1.6.7 in [43]. The former can be easily obtained if we use the Poisson formula for a solution to the Cauchy problem for the heat equation with the right-hand side equal to the Dirac delta function.

Theorem 4. *Assume that $T = \infty$ and the conditions (5), (15), (58), (59), and (38) for $n = 3$ hold. Then there exists $\lambda_1 \geq \lambda_0$ such that if $Re\,\lambda = \gamma_0 \geq \lambda_1$ and the conditions (60), (63) are fulfilled then there exists a unique solution to the problem (1)-(3) such that $e^{-\gamma_0 t} u \in W_2^{1,2}(Q)$, $e^{-\gamma_0 t}\alpha_i(t) \in W_2^{1/4}(0,T)$ $(i = 1,2,\dots,r)$.*

Proof. Consider the equivalent problem (61)-(62). Assuming that $w \in W_2^{1,2}(Q)$ and applying the Laplace transform to (61), we arrive at the problem

$$L_0\hat{w} = \lambda\hat{w} + L\hat{w} = 0,\ B\hat{w}|_\Gamma = \sum_{i=1}^r \hat{\alpha}_i \Phi_i(x) = \hat{g}, \tag{65}$$

$$\hat{w}(b_j) = \hat{\tilde{\psi}}_j,\ j = 1,2,\dots,r. \tag{66}$$

Next, we use the functions v_j, v_j^b constructed before Lemma 3. Theorem 1 yields $v_j \in W_2^2(G_j(\varepsilon))$, $G_j(\varepsilon) = \{x \in G :\ |x - b_j| \geq \varepsilon\}$ for all $j = 1,\dots,r$, $\varepsilon > 0$. Construct the functions $w_j = \varphi_j(v_j + \sum_{b\in K_j} v_j^b D_j)$, $D_j = \frac{I_j^*(b)e^{\varphi_j^b(b)-\varphi_j(b)}}{I_j(b)}$, where the functions φ_j are defined in Lemma 5. The properties of the functions v_j^b imply that $\varphi_j \sum_{b\in K_j} v_j^b D_j \in W_2^2(G)$. Lemma 1 imply that

$$\sum_{i=1}^r \hat{\alpha}_i \int_\Gamma \Phi_i(x)\overline{w_j}\, d\Gamma = \int_\Gamma \hat{w}\overline{(\frac{\partial w_j}{\partial \nu} + \sigma^* w_j)}\, d\Gamma - 2\int_G \hat{w}\nabla\varphi_j \overline{\nabla w_j}\, dx$$
$$+ \int_G -\hat{w}\sum_{i=1}^n a_i \varphi_{jx_i}\overline{w_j}\, dx - \int_G \hat{w}\Delta\varphi_j \overline{w_j}\, dx + \hat{\tilde{\psi}}_j = A_j(\hat{w}) + \hat{\tilde{\psi}}_j, \tag{67}$$

where the function $\hat{w}$ is a solution to the problem (65). Consider the case of $n = 3$. The case of $n = 2$ is considered by analogy. For the integral on the left-hand side, we have

$$\int_\Gamma \Phi_i(x)\overline{w_j}\,d\Gamma = \int_\Gamma \Phi_i(x)\varphi_j\overline{(v_j + \sum_{b\in K_j} v_j^b D_j)}\,d\Gamma = \sum_{b'\in K_j}\int_{X_{b'}} \Phi_i(x)\varphi_j\overline{(v_j + \sum_{b\in K_j} v_j^b D_j)}\,d\Gamma.$$

However, only two summands with v_j and $v_j^{b'}$ are essential on the set $X_{b'}$. Indeed, in view of (16), for $b \neq b'$ and $b \in K_j$, we infer

$$|\sqrt{\lambda}v_j^b e^{\delta_j\sqrt{\lambda}}| + |e^{\delta_j\sqrt{\lambda}}|\nabla v_j^b|| \leq c_1 e^{-q_0\sqrt{|\lambda|}}\ \forall x \in X_{b'},$$

where $q_0 > 0$ is a constant independent of λ. This inequality implies that the remaining integrals decay exponentially. By Lemma 3, we have

$$e^{\delta_j\sqrt{\lambda}}\sqrt{\lambda}\int_\Gamma \Phi_i(x)\overline{w_j}\,d\Gamma = \sum_{b'\in K_j}\frac{\Phi_i(b')e^{-\varphi_j(b')}}{I_j(b')}(1+O(|\lambda|^{-\beta})) = \Psi_{ji}(1+O(|\lambda|^{-\beta})). \quad (68)$$

Consider the right-hand side in (67). The integrals over the domain are estimated by means of Lemma 5. On the support of $|\nabla\varphi_j|$, Theorem 1 and Lemma 5 ensure the estimate

$$|e^{\delta_j\sqrt{\lambda}}|(|v_j| + |\nabla v_j| + \sum_{b\in K_j}(|v_j^b| + |\nabla v_j^b|) \leq c_2 e^{-\varepsilon_{13}\sqrt{|\lambda|}},$$

where the constants $c_2, \varepsilon_{13} > 0$ are independent of λ. The Hölder inequality yields

$$e^{\delta_j Re\sqrt{\lambda}}\Big|-2\int_G \hat{w}\nabla\varphi_j\overline{\nabla(v_j + \sum_{b\in K_j} v_j^b D_j)}\,dx - \int_G \hat{w}\sum_{i=1}^n a_i\varphi_{jx_i}\overline{(v_j + \sum_{b\in K_j} v_j^b D_j)}\,dx$$
$$- \int_G \hat{w}\Delta\varphi_j\overline{(v_j + \sum_{b\in K_j} v_j^b D_j)}\,dx\Big| \leq c_3\|\hat{w}\|_{L_2(G)}e^{-\varepsilon_{13}\sqrt{|\lambda|}/2}. \quad (69)$$

Examine the integrals over Γ in the right-hand side of (67). We have

$$\int_\Gamma \hat{w}\overline{(\frac{\partial w_j}{\partial \nu} + \sigma^* w_j)}\,d\Gamma = \sum_{b\in K_j}\Big(\int_{X_b}\hat{w}\varphi_j\frac{\partial}{\partial\nu}\overline{(v_j + v_j^b D_j)}\,d\Gamma + \int_{X_b}\hat{w}\frac{\partial\varphi_j}{\partial\nu}\overline{(v_j + v_j^b D_j)}\,d\Gamma +$$
$$\int_{X_b}\hat{w}\sigma^*\varphi_j\overline{(v_j + v_j^b D_j)}\,d\Gamma + \int_{X_b}\hat{w}\sum_{b'\in K_j, b'\neq b}\overline{\frac{\partial(\varphi_j v_j^b D_j)}{\partial\nu}}\,d\Gamma + \int_{X_b}\hat{w}\sigma^*\varphi_j\overline{\sum_{b'\in K_j, b'\neq b} v_j^b D_j}\,d\Gamma\Big). \quad (70)$$

As in the estimate (69), the last two integrals are estimated by

$$\Big|\int_{X_b}\hat{w}\sum_{b'\in K_j, b'\neq b}\overline{\frac{\partial(\varphi_j v_j^b D_j)}{\partial\nu}}\,d\Gamma + \int_{X_b}\hat{w}\sigma^*\varphi_j\overline{\sum_{b'\in K_j, b'\neq b} v_j^b D_j}\,d\Gamma\Big)\Big| \leq c_4\|\hat{w}\|_{L_2(\Gamma)}e^{-\varepsilon_{12}\sqrt{|\lambda|}/2}$$

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in view of (16). Estimate the second and third integrals. In view of Theorem 3 and estimates of Lemma 5 (see (27)), they admit the estimates

$$\Big|\int_{X_b} \hat{w}\frac{\partial\varphi_j}{\partial\nu}\overline{(v_j+v_j^bD_j)}\,d\Gamma+\int_{X_b}\hat{w}\sigma^*\varphi_j\overline{(v_j+v_j^bD_j)}\,d\Gamma\Big|\leq$$
$$c_5\|w(x(z',0))\|_{L_\infty(B'_{\delta/2}(0))}ce^{-\delta_j\sqrt{Re\,\lambda}}\int_{B'_\delta(0)}e^{-Re\,\sqrt{\lambda}(|y-b_j|-\delta_j)}+e^{-Re\,\sqrt{\lambda}(|y-b_j^b|-\delta_j)}dy'\leq$$
$$c_6\|\hat{w}(x_b(z))\|_{L_\infty(B'_{\delta/2}(0))}e^{-\delta_jRe\,\sqrt{\lambda}}/\sqrt{|\lambda|},$$

where $x=x_b(z)$ is the straightening of the boundary in X_b. It remains to consider the first integral

$$I_b=\int_{X_b}\hat{w}\varphi_j\frac{\overline{\partial(v_j+v_j^bD_j)}}{\partial\nu}\,d\Gamma=\hat{w}\varphi_j(b)\int_{X_b}\frac{\overline{\partial(v_j+v_j^bD_j)}}{\partial\nu}\,d\Gamma+$$
$$\int_{X_b}(\hat{w}\varphi_j(x)-\hat{w}\varphi_j(b))\frac{\overline{\partial(v_j+v_j^bD_j)}}{\partial\nu}\,d\Gamma \quad (71)$$

Note that $\varphi_j(b)=1$. Lemma 5 ensures the following representation for the first integral I_1 on the right-hand side of (71):

$$I_1=e^{\sqrt{\lambda}\delta_j}\Big(\int_{X_b}\frac{\overline{\partial v_j}}{\partial\nu}\,d\Gamma+\int_{X_b}\frac{\overline{\partial v_j^bD_j}}{\partial\nu}\,d\Gamma\Big)=$$
$$\frac{-e^{-\varphi_j(b)}}{2I_j(b)}(1+O(|\lambda|^{-1/4}))+D_j\frac{e^{-\varphi_j^b(b)}}{2I_j^*(b)}(1+O(|\lambda|^{-1/4}))=O(\frac{1}{|\lambda|^{1/4}}).$$

The second integral on the right-hand side of (71) in view of Lemma 5, (28), and (32) is estimated as follows:

$$\Big|\int_{X_b}(\hat{w}\varphi_j(y)-\hat{w}\varphi_j(b))\frac{\overline{\partial(v_j+v_j^bD_j)}}{\partial\nu}\,d\Gamma\Big|\leq$$
$$c_1\|\hat{w}(x_b(z))\|_{C^1(B'_{\delta/2}(0))}\int_{\tilde{X}_b}|y'|\sqrt{|\lambda|}(e^{-Re\sqrt{\lambda}(|y-b|+\delta_j)}+e^{-Re\sqrt{\lambda}(|y-b_j^b|+\delta_j)})dy'\leq$$
$$c_2\|\hat{w}(x_b(z))\|_{C^1(B'_{\delta/2}(0))}|\lambda|^{-1/4}. \quad (72)$$

Thus, in view of (70)-(72), we have the inequality

$$|e^{\sqrt{\lambda}\delta_j}A_j(\hat{w})|\leq c_3\Big(\sum_{b\in K_j}(\|\hat{w}(x_b(z))\|_{C^1(B'_{\delta/2}(0))}|\lambda|^{-1/4}+(\|w\|_{L_2(G)}+\|w\|_{L_2(\Gamma)}),e^{-\varepsilon_{14}Re\,\sqrt{\lambda}}\Big)$$

with some constant $\varepsilon_{14}>0$. Next, we employ Lemma 4. The embedding theorems for $n=3$ and Lemma 4 imply that

$$\|\hat{w}(x_b(z))\|_{C^1(B'_{\delta/2}(0))}\leq c\|\hat{w}(x_b(z))\|_{W_2^2(B'_{\delta/2}(0))}\leq(\|g\|_{W_2^2(B'_\delta)}+\|g\|_{L_2(\Gamma)})\lambda|^{-1/2+\varepsilon},$$

where ε is an arbitrarily small constant. Similarly, Lemma 4 ensures that

$$\|\hat{w}\|_{L_2(G)}+\|\hat{w}\|_{L_2(\Gamma)}\leq c|\lambda|^{-1/2+\varepsilon}\|g\|_{L_2(\Gamma)}.$$

In view of the conditions on the functions Φ_j, there exists a constant c_2 such that

$$\|g\|_{W_2^2(B'_\delta)}+\|g\|_{L_2(\Gamma)}\leq c_2|\vec{\alpha}|.$$

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Therefore, we have the estimate

$$|e^{\sqrt{\lambda}\delta_j} A_j(\hat{w})| \leq c_3 |\vec{\hat{\alpha}}| \lambda|^{-1/4-1/2+\varepsilon}, \tag{73}$$

where $\varepsilon > 0$ is an arbitrarily small constant. We can rewrite (67) in the form

$$e^{\sqrt{\lambda}\delta_j}\sqrt{\lambda}\sum_{i=1}^{r}\hat{\alpha}_i\int_{\Gamma}\Phi_i(x)\overline{v_j}\,d\Gamma = e^{\sqrt{\lambda}\delta_j}\sqrt{\lambda}A_j(\hat{w}) + \hat{\tilde{\psi}}_j e^{\sqrt{\lambda}\delta_j}\sqrt{\lambda},\;\; j = 1,2,\ldots,r.$$

The left-hand side of this equality is written as $\tilde{\Psi}(\lambda)\vec{\hat{\alpha}}$, where the entries the matrix $\tilde{\Psi}(\lambda)$ are of the form $\Psi_{ij}(1 + O(|\lambda|^{-\beta}))$. The right-hand side is written in the form

$$A(\lambda)\vec{\hat{\alpha}} = \vec{\beta} + S_0(\vec{\hat{\alpha}}),\; \vec{\hat{\alpha}} = (\hat{\alpha}_1, \ldots, \hat{\alpha}_r), \tag{74}$$

where the coordinates of the vectors $\vec{\beta}$, $S_0(\vec{\hat{\alpha}})$ are as follows:

$$\beta_j = \sqrt{\lambda}e^{\delta_j\sqrt{\lambda}}\hat{\tilde{\psi}}_j,\; S_{0j} = \sqrt{\lambda}e^{\delta_j\sqrt{\lambda}}A_j(\hat{w}),\; j = 1,\ldots,r.$$

It is more convenient to rewrite the system (74) in the form

$$\vec{\hat{\alpha}} = A(\lambda)^{-1}\vec{\beta} + A(\lambda)^{-1}S_0(\vec{\hat{\alpha}}). \tag{75}$$

Choose $\lambda_1 \geq \lambda_0$ so that the matrix $A(\lambda)$ is invertible for $Re\lambda \geq \lambda_1 \geq \lambda_0$ and the norm of the operator $A^{-1} : \mathbb{R}^r \to \mathbb{R}^r$ is bounded by a constant c_0 for all $Re\,\lambda \geq \lambda_1$. Estimate the norm of the operator $A(\lambda)^{-1}S_0(\vec{\hat{\alpha}})$. In view of (73), we have the estimate

$$|A(\lambda)^{-1}S_0(\vec{\hat{\alpha}})| \leq c_0\sum_{j=1}^{r}|S_{0j}| \leq c_1|\vec{\hat{\alpha}}||\lambda|^{-1/4+\varepsilon}. \tag{76}$$

Thus, for $\varepsilon < 1/4$, increasing the parameter λ_1 if necessary, we can assume that $c_1|\lambda|^{-1/4+\varepsilon} \leq 1/2$ for $Re\,\lambda \geq \lambda_1$. The norm of the operator $A(\lambda)^{-1}S_0(\vec{\hat{\alpha}}) : \mathbb{C}^r \to \mathbb{C}^r$ is less than $1/2$ in this case and, thereby, the equation (75) has a unique solution. Constructing a solution $\vec{\hat{\alpha}}$ to the equation (75), we can find a solution $\hat{w} \in W_2^2(G)$ to the problem (61), where $Re\,\lambda \geq \lambda_1$. In view of our conditions, the estimates of Lemma 4 holds. Let $Re\,\lambda \geq \lambda_1$. In view of the equation (75), a solution $\vec{\hat{\alpha}}$ meets the estimates $|\vec{\hat{\alpha}}| \leq 2c_0|\vec{\beta}|$. Hence, we infer

$$\sum_{j=1}^{r}|\hat{\alpha}_i|^2 \leq \sum_{i=1}^{r}|\lambda||e^{2\delta_j\sqrt{\lambda}}|\hat{\tilde{\psi}}_j|^2,$$

where the constant c_1 is independent of λ. The properties of Laplace transform validate the equality $\hat{\tilde{\psi}}_j = \hat{V}_{\delta_j}(\lambda)\hat{\tilde{\psi}}_{0j} = e^{-\sqrt{\lambda}\delta_j}\hat{\psi}_{0j}$ and the previous inequality yields

$$\sup_{\gamma>\lambda}\int_{-\infty}^{\infty}\sum_{i=1}^{r}|\gamma + i\xi|^{1/2}|\hat{\alpha}_i(\gamma + i\xi)|^2\,d\xi \leq$$
$$C\sup_{\gamma>\lambda}\int_{-\infty}^{\infty}\sum_{i=1}^{r}|\gamma + i\xi|^{3/2}\|\hat{\psi}_{0j}(\gamma + i\xi)\|^2\,d\xi \leq C\sum_{j=1}^{r}\|e^{-\lambda t}\psi_{0j}\|^2_{W_2^{3/4}(0,\infty)} < \infty. \tag{77}$$

This inequality ensures that the inverse Laplace transform is defined for the functions $\hat{\alpha}_i$, $\alpha_i(t)e^{-\lambda t} \in W_2^{1/4}(0,\infty)$, and

$$\sum_{i=1}^{r}\|\alpha_i e^{-\lambda t}\|^2_{W_2^{1/4}(0,T)} \leq C\sum_{j=1}^{r}\|e^{-\lambda t}\psi_{0j}\|^2_{W_2^{3/4}(0,\infty)} < \infty. \tag{78}$$

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Note that the additional smoothness of the functions $\tilde{\psi}_i$ ensures the additional smoothness of the functions α_i. Consider the problem (61) with the above constructed functions $\vec{\alpha}$. By Theorem 3, there exists a unique solution to this problem such that $e^{-\lambda t}w \in W_2^{1,2}(Q)$. Demonstrate that this function satisfies (62). Indeed, applying the Laplace transform, we obtain that $\hat{w}$ is a solution to the problem (65). Multiplying the equation in (65) by w_j and integrating by parts we obtain (67) with $\hat{w}(b_j)$ rather than $\hat{\tilde{\psi}}_j$. Since $\hat{\alpha}_j$ satisfy (67) with the functions $\hat{\tilde{\psi}}_j$ on the right-hand side, we obtain $\hat{\tilde{\psi}}_j = \hat{w}(b_j)$.

Consider the case of $n = 2$. The arguments are the same. However, in view of another asymtotics of the function $\hat{V}_{\delta_j}$ the inequality (78) can be rewritten as

$$\sum_{i=1}^{r} \|\alpha_i e^{-\lambda t}\|^2_{W_2^{1/4}(0,T)} \leq C \sum_{j=1}^{r} \|e^{-\lambda t}\psi_{0j}\|^2_{\tilde{W}_2^{1/2}(0,\infty)} < \infty.$$

In view of the above arguments, uniqueness of solutions is obvious. □

State our theorem in the case of a finite interval $(0, T)$. The condition (60) looks as follows:

$$u_0(x) \in W_2^1(G),\ f \in L_2(Q). \tag{79}$$

Theorem 5.. *Assume that $T < \infty$ and the conditions (5), (15), (58), (59), (79), (63), and (38) for $n = 3$ hold. Then there exists a unique solution to the problem (1)-(3) such that $u \in W_2^{1,2}(Q)$, $\alpha_i(t) \in W_2^{1/4}(0,T)$ $(1 = 1,2,\ldots,r)$.*

Proof. Extend the functions ψ_{0j} on $(0,\infty)$ as compactly supported functions of the same class. The conditions (63) are fulfilled for every λ. Extend the function f by zero on $(0,\infty)$. Theorem 4 ensures existence of a solution to the problem (1)-(3). Now we prove uniqueness of solutions. Assume that there are two solutions of the problem from the class pointed out in the statement of the theorem. In this case, their difference $v(t,x) \in W_2^{1,2}(Q)$ is a solution to the problem

$$v_t + Lv = 0,\ (t,x) \in Q.$$

$$Bv|_S = g(t,x) = \sum_{i=1}^{r} \alpha_i \Phi_i,\ \ v|_{t=0} = 0,\ \ v(b_j,t) = 0,\ j = 1,2,\ldots,r.$$

Integrating the equation and the boundary condition with respect to time two times, we obtain that the function $v_0 = \int_0^t \int_0^\tau v(\xi)\, d\xi d\tau$ is a solution to the problem

$$v_{0t} + Lv_0 = 0\ ((t,x) \in Q),\ Bv_0|_S = g_0(t,x) = \sum_{i=1}^{r} \alpha_{0i}\Phi_i, \tag{80}$$

$$v_0|_{t=0} = 0,\ \ v_0(b_j,t) = 0,\ \alpha_{0i} = \int_0^t \int_0^\tau \alpha_i(\xi)\, d\xi d\tau,\ j = 1,2,\ldots,r. \tag{81}$$

Make the change of variables $v_0 = e^{\lambda t}w$ $(Re\,\lambda \geq \lambda_0)$. We have

$$w_t + Lw + \lambda w = 0,\ (t,x) \in Q. \tag{82}$$

$$Bw|_S = e^{-\lambda t}g_0(t,x),\ \ w|_{t=0} = 0,\ \ w(b_j,t) = 0,\ j = 1,2,\ldots,r. \tag{83}$$

Integrating (82) over $(0,T)$, we obtain that

$$L\tilde{w} + \lambda\tilde{w} = -w(T,x),\ B\tilde{w}|_\Gamma = \int_0^T e^{-\lambda t} g_0(t,x)\, dt,\ \tilde{w}(b_j) = 0,\ \tilde{w} = \int_0^T w(\tau,x)\, d\tau. \tag{84}$$

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Let $\hat{\alpha}_i = \int_0^T e^{-\lambda t}\alpha_{0i}(t)\,dt$. Make the change of variables $\tilde{w} = w_0 + w_1$, with w_1 a solution to the problem $(L+\lambda)w_1 = -w(T,x) = e^{-\lambda T}v_0(T,x)$, $Bw_1|_\Gamma = 0$, and, respectively, w_0 is a solution to the problem

$$Lw_0 + \lambda w_0 = 0,\ Bw_0|_\Gamma = \sum_{i=1}^{r}\hat{\alpha}_i\Phi_i(x),\ w_0(b_j) = -w_1(b_j). \tag{85}$$

Note that $w(T,x) \in W_2^1(G)$ and, thereby, $w_1 = -(L+\lambda)^{-1}w(T,x) \in W_2^2(G)$. Since $W_2^2(G) \subset C(\overline{G})$ [29], we have the estimate (see Theorem 7.11 for $G = \mathbb{R}^n_+$ and Theorem 8.2 in the case of a domain with compact boundary in [30])

$$|w_1(b_j)| \le c_0 e^{-Re\,\lambda T}\|v_0(T,x)\|_{L_2(G)} \le c_1 e^{-Re\,\lambda T}. \tag{86}$$

Multiply the equation (85) by the function w_j defined in the proof of the previous theorem and integrate over G. As in the proof of Theorem 4, we obtain the system (see (75))

$$\vec{\alpha} = A(\lambda)^{-1}\vec{\beta} + A(\lambda)^{-1}S_0(\vec{\alpha}), \tag{87}$$

where the coordinates of $\vec{\beta}$ are written as $\beta_j = -\sqrt{\lambda}e^{\sqrt{\lambda}\delta_j}w_1(b_j)$. The system can be rewritten as follows

$$\vec{\alpha} = (I - A(\lambda)^{-1}S_0)^{-1}A(\lambda)^{-1}\vec{\beta}, \tag{88}$$

where the right-hand side is analytic for $Re\,\lambda \ge \lambda_1$ and we have

$$\|(I - A(\lambda)^{-1}S_0)^{-1}A(\lambda)^{-1}\vec{\beta}\|_{\mathbb{C}^r} \le c_2\|\vec{\beta}\|_{\mathbb{C}^r},$$

where c_1 is independent of λ. Thus, every of the quantities $\hat{\alpha}_i$ is estimated by

$$|\hat{\alpha}_i| \le c_3\sum_{j=1}^{r}\sqrt{\lambda}e^{\sqrt{\lambda}\delta_j}e^{-Re\,\lambda T},\ Re\,\lambda \ge \lambda_1. \tag{89}$$

The function $S_i(z) = \int_0^T \alpha_{0i}(t)e^{-\lambda_1 t}e^{-zt}\,dt$ is the Laplace transform of the function $\tilde{s}_i(t) = \alpha_{0i}(t)e^{-\lambda_1 t}$ for $t \le T$ and $\tilde{s}_i(t) = 0$ for $t > T$. Fix $\varepsilon > 0$ and define an additional function $W(z) = ze^{z(T-\varepsilon)}S_i(z)$. It is analytic in the right half-plane and is bounded by some constant C_1 on the real semi-axis $\mathbb{R}^+$. Estimate this function on the on the imaginary axis. Integrating by parts, we have

$$S_i(z) = \frac{-1}{\lambda_1 + z}(\alpha_{0i}(T)e^{-\lambda_1 T}e^{-zT} + \int_0^T \alpha'_{0i}(t)e^{-\lambda_1 t}e^{-zt}\,dt).$$

For $z = iy$, we thus have the estimate

$$|W(z)| \le c_4(|\alpha_{0i}(T)| + \|\alpha'_{0i}\|_{L_1(0,T)}) = c_5\ \forall z = iy,\ y \in \mathbb{R}.$$

In each of the sectors $0 \le argz \le \pi/2$, $-\pi/2 \le argz \le 0$ the function $W(z)$ admits the estimate

$$|W(z)| \le e^{|z|(T-\varepsilon)}c_6(|\alpha_{0i}(T)| + \|\alpha'_{0i}\|_{L_1(0,T)})\ \forall Re\,z \ge 0.$$

Applying the Fragment-Lindelef Theorem (see theorem 5.6.1 in [40]) we obtain that in each of the sectors $0 \le argz \le \pi/2$, $-\pi/2 \le argz \le 0$ the function $W(z)$ admits the estimate

$$|W(z)| \le \max(C_1, c_5) = C_2\ \forall Re\,z \ge 0.$$

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Therefore, $|S_i(z)| = |L(\tilde{s}_i(t))(z)| \leq C_2 e^{-(T-\varepsilon)Re\,z}/|z| \quad \forall Re\,z \geq 0$. We have equality ($\sigma \geq \lambda_1, p = \sigma + i\xi$)

$$\tilde{s}_j(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{pt} L(\tilde{s}_j)(p)\,dp = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\sigma t} e^{i\xi t} L(\tilde{s}_j)(\sigma+i\xi)\,d\xi.$$

and, thereby,

$$\tilde{s}_j(t)e^{-\sigma(t-(T-\varepsilon))} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi t} e^{\sigma(T-\varepsilon)} L(\tilde{s}_j)(\sigma+i\xi)\,d\xi.$$

The Parseval identity yields

$$\|\tilde{s}_j(t)e^{-\sigma(t-(T-\varepsilon))}\|^2_{L_2(\mathbb{R})} = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} e^{2\sigma(T-\varepsilon)}|L(\tilde{s}_i)(\sigma+i\xi)|^2\,d\xi \leq \frac{C_2^2}{2\pi}\int\limits_{-\infty}^{\infty}\frac{1}{\sigma^2+\xi^2}\,d\xi \leq \frac{C_2^2}{2\sigma}.$$

Since this inequality is true for all $\sigma > \lambda_1$, $\tilde{s}_i(t) = 0$ for $t \leq T - \varepsilon$. Since the parameter ε is arbitrary, $\alpha_{0j}(t) = 0$ for $t \leq T$ and $\alpha_j(t) = 0$ for $t \leq T$ and every j and, therefore, $g(t,x) = 0$ which implies that $v = 0$. $\square$

3. Discussion

We consider inverse problems of recovering surface fluxes on the boundary of a domain from pointwise observations. These problems arise in many practical applications but there are not theoretical results concerning with the existence and uniqueness questions. The problems are ill-posed in the Hadamard sense. The results can be used in developing new numerical algorithms and provide new conditions of uniqueness of solutions to these problems. We consider a model case but it is clear what changes should be made in the general case for validating similar results. The main conditions on the date are conventional. The only distinction is the conditions on the data of measurements in the reduced problem which must belong to some special class of infinitely differentiable functions. The proof relies on a asymptotics of fundamental solutions to the corresponding elliptic problems and the Laplace transform.

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