A predictor-corrector algorithm for IVPs in frame of generalized Fractional operator with Mittag-Leffler kernels

Ibrahim Slimane[a,*], Zaid Odibat[b], Dumitru Baleanu[c,d]

[a]*Faculty of Exact Sciences and Informatics, UMAB Abdelhamid Ibn Badis*
P. O. Box 227 University of Mostaganem, 27000 Mostaganem, Algeria
[b]*Department of Mathematics, Faculty of Science, Al-Balqa Applied University, Salt 19117, Jordan*
[c]*Department of Mathematics, Cankaya University, 06530 Balgat, Ankara, Turkey*
[d]*Institute of Space Sciences, Magurele-Bucharest, Romania*

Abstract

This study develops a predictor-corrector algorithm for the numerical simulation of IVPs involving singular generalized fractional derivatives with Mittag-Leffler kernels. The proposed algorithm converts the considered IVP into a Volterra-type integral equation and then uses Trapezoidal rule to obtain approximate solutions. Numerical approximate solutions of some singular generalized fractional derivative with Mittag-Leffler kernels models have been presented to demonstrate the efficiency and accuracy of the proposed algorithm. The algorithm describes the influence of the fractional derivative parameters on the dynamics of the studied models. The suggested method is expected to be effectively employed in the field of simulating generalized fractional derivative models.

Keywords: Fractional differential equation; singular generalized fractional derivative; Mittag-Leffler kernel; predictor-corrector algorithm; numerical solution.
2010 MSC: 26A33; 34K28; 78A70.

1. Introduction

Despite of the fact that the origin of the fractional calculus (FC) has started 326 years back, this extension of meaning still has many new open problems in both theoretical and applied directions. During the history of FC, several definitions of fractional operators have been suggested and indeed many applications to real world problems have been emerged [1-9]. Nowadays, FC is an emerging field in mathemtics and it is subjected under an internal dynamical evolution. Despite of loosing some of the classical properties of calculus, FC become more attractive for researchers trying to utilise it in opening the doors of the complicated dynamical systems. Indeed it become clear that fractional calculus methods

*Corresponding author
Email addresses: slimaneibrahim@outlook.fr (Ibrahim Slimane), odibat@bau.edu.jo (Zaid Odibat), dumitru@cankaya.edu.tr (Dumitru Baleanu)

and techniques can be combined efficiently with artificail intelligence, big data and other complex concepts in order to provide better description of complex systems.

In our opinion, one characteristic of the contemporary FC is that it can not be reduced to a certain set of class of operators, and therefore we do not have a unique operator like the classical calculus case [10]. Besides, as emphasized in the excellent handbook of Polyanin and Manzirov, there are plenty of integral operators that can play a potential role for better description of complex dynamical systems. So, a logical question arises: How can we generalize the existing well known operators such as Riemann-Liouville or Caputo, so that we have an interesting mathematical construction but at the same time to be able to successfully use these generalized operators for real-world applications? We stress on the fact that the generalized integral operators were introduced for the first time by Boltzmann in 1874 [32] and since then we can see various ways of defining a generalized fractional operator [11-21]. A crucial step for validation of the generalised operators is to have powerdul computational methods and techniques.

An important class of operators consists of Prabhakar [20] operators which has the property that it contains both singular and non-singular operators as particular cases. For the operators of Caputo type, especially where the kernel is continuous, we have to be very careful in handling the extra conditions which assures that the problem is well defined. For more details on this topic the reader can see page 3 in the handbook of Polyanin and Manzirov [11]. A natural question appears when we are dealing with the Mittag-Leffler kernels: which fractional operator can be defined in such a way that there are no unnatural constraints on solving the relevant fractional differential equations? As it is expected the solution to this issue is not unique because of the properties of the Mittag-Leffler kernel. Recently, fractional operators with Mittag-Leffler kernels [18] have used in different fields of science and engineering to model many real world applications. The Mittag-Leffler kernel is nonsingular operator [18] and there are several ways to initialise these operarors properly [21, 32]. On the other hand, the predictor-corrector method developed in [23] is one of the most powerful, accurate and effective method that is widely used to provide numerical solutions for IVPs equipped with fractional derivatives of the standard Caputo type. Some modified versions and implementations of this numerical method can be found in [23-28].

Having all these aspects in mind, in this manuscript we suggested a predictor-corrector algorithm to numerically simulate IVPs that involves fractional derivative with Mittag-Leffler kernels containing three parameters. The manuscript is organised as follows:The required defnitions and formulas are reported in Section 2. The developed predictor-corrector algorithm is presented in detal in Section 3. The test examples are depicted in Session 4. Finally, the conclussions are depicted in Section 5.

2. Preliminaries

This section recalls some definitions, characteristics and properties related to the generalized Caputo-type Atangana-Baleanu (ABC) fractional derivative with Mittag-Leffler kernels. The most interesting fractional derivative operators that are used in physical applications with non-local and memory properties are the Riemann-Liouville and Caputo fractional derivatives. Their definition is based on the use of the Riemann-Liouville fractional integral operator which is identified as

Definition 1. *The Riemann-Liouville fractional integral operator, of order $\sigma > 0$, can be expressed as [4]*

$$aI^{\sigma} f(x) = \frac{1}{\Gamma(\sigma)} \int_a^x (x-t)^{\sigma-1} f(t)\, \mathrm{d}t, \quad \mathrm{Re}(\sigma) > 0,\ a < x \leq b. \tag{1}$$

Definition 2. *The MittagLeffler (ML) function of one parameter is defined as [4]*

$$\begin{aligned} E_{\sigma}(\lambda, z) &= E_{\sigma}(\lambda z^{\sigma}), \\ &= \sum_{j=0}^{\infty} \lambda^j \frac{z^{\sigma j}}{\Gamma(\sigma j + 1)}, \quad 0 \neq \lambda \in \mathbb{R},\ z \in \mathbb{C},\ \mathrm{Re}(\sigma) > 0, \end{aligned} \tag{2}$$

and the ML function with two parameters σ and β has the following form

$$\begin{aligned} E_{\sigma,\beta}(\lambda, z) &= z^{\beta-1} E_{\sigma,\beta}(\lambda z^{\sigma}), \\ &= \sum_{j=0}^{\infty} \lambda^j \frac{z^{\sigma j+\beta-1}}{\Gamma(\sigma j + \beta)}, \quad 0 \neq \lambda \in \mathbb{R},\ z \in \mathbb{C},\ \mathrm{Re}(\sigma) > 0. \end{aligned} \tag{3}$$

Moreover, the formula of the modified version of the generalized ML of three parameters is given by

$$E^{\gamma}_{\sigma,\beta}(z) = \sum_{j=0}^{\infty} \lambda^j (\gamma)_j \frac{z^{\sigma j+\beta-1}}{j!\Gamma(\sigma j + \beta)}, \tag{4}$$

where $(\gamma)_j = \gamma(\gamma+1)...(\gamma+j-1)$. Next, we recall the generalized ABC fractional derivative with three parameters ML kernel.

Definition 3. *The generalized ABC fractional derivative with kernel $E^{\gamma}_{\sigma,\mu}(\lambda, t)$ where $0 < \sigma < 1$, $\mathrm{Re}(\mu) > 0$, $\gamma \in \mathbb{R}$ and $\lambda = \frac{-\sigma}{1-\sigma}$ is defined by [21]*

$$\begin{aligned} \left({}^{\mathrm{ABC}}_{a}D^{\sigma,\mu,\gamma} f\right)(x) =& \frac{\mathcal{N}(\sigma)}{1-\alpha} \int_a^x E^{\gamma}_{\sigma,\mu}(\lambda, x-t) f'(t)\, dt, \\ =& \frac{\mathcal{N}(\sigma)}{1-\sigma} E^{\gamma}_{\sigma,\mu}(\lambda, x-a) * f'(x). \end{aligned} \tag{5}$$

For $0 < \sigma < 1$, $\mathrm{Re}(1-\mu) > 0$, $\gamma > 0$ and $\lambda = \frac{-\sigma}{1-\sigma}$ the corresponding AB fractional integral is given by

$$\left({}^{\mathrm{AB}}_{a}I^{\sigma,\mu,\gamma} f\right)(x) = \sum_{i=0}^{\infty} \binom{\gamma}{i} \frac{\sigma^i}{\mathcal{N}(\sigma)(1-\sigma)^{i-1}} \left({}_{a}I^{\sigma i+1-\mu} f\right)(x), \tag{6}$$

where $\mathcal{N}(\sigma) > 0$ is a normalization function such that $\mathcal{N}(0) = \mathcal{N}(1) = 1$.

 doi:10.20944/preprints202204.0045.v1

Lemma 4. *For $0<\sigma<1$, $\mu>0$, $\gamma>0$, $\lambda=\frac{-\sigma}{1-\sigma}$ and f defined on $[a,b]$ we have [21]*

$$ {}_{a}^{\mathrm{AB}}I^{\sigma,\mu,\gamma}\big({}_{a}^{\mathrm{ABC}}D^{\sigma,\mu,\gamma}f\big)(x)=f(x)-f(a). \tag{7}$$

Throughout the rest of this paper we take $\mu>0$, $\mu\neq 1$, $\gamma>0$ and $\lambda=\frac{-\sigma}{1-\sigma}$.

Proposition 5. *Assume that $0<\sigma<1$, $\mu>0$, $\gamma\in I\!R$, $\lambda=\frac{-\sigma}{1-\sigma}$ and $\upsilon>0$. Then*

$$ {}_{a}^{ABC}D^{\sigma,\mu,\gamma}(t-a)^{\upsilon}=\frac{1}{1-\sigma}\Gamma(\upsilon+1)E^{\gamma}_{\sigma,\mu+\upsilon}(\lambda,t-a). \tag{8}$$

Theorem 6. *Assume that $0<\sigma<1$, $0<\mu<1$, $\gamma>0$, $\lambda=\frac{-\sigma}{1-\sigma}$ and $\mathcal{F}\in C[a,T]$. Then the IVP [21]*

$$ {}^{\mathrm{ABC}}D^{\sigma,\mu,\gamma}\mathcal{F}(t)=\Psi(t,\mathcal{F}(t)),\quad a<t\leq T, \tag{9}$$

subject to the initial condition $\mathcal{F}(a)=\mathcal{F}_0$, is equivalent to the integral equation

$$\mathcal{F}(t)=\mathcal{F}_0+{}_{a}^{AB}I^{\sigma,\mu,\gamma}\Psi(t,\mathcal{F}(t)). \tag{10}$$

3. Predictor-corrector algorithm

Several predictor-corrector approaches, which are generalizations of the Adams-Bashforth-Moulton method, have been used successfully to solve numerically some Caputo-types fractional differential equations [30, 31]. In this section, we developed a predictor-corrector algorithm for the numerical solutions of IVPs including generalized fractional derivative with Mittag-Leffler kernels. We introduced the main steps of the suggested algorithm to handle the IVP given in Eq. (9) over the interval $[a,T]$. Initially, if $0<\sigma<1$, $0<\mu<1$, $\gamma>0$, $\lambda=\frac{-\sigma}{1-\sigma}$ and $\mathcal{F}\in C[a,T]$, then the IVP (9), referring to Theorem 6 , is equivalent to the Volterra-type integral equation given in Eq. (10). To describe our algorithm, assuming that the function Ψ is set so that a unique solution for the IVP (9) comes out in the interval $[a,T]$, let's start by defining a uniform grid in the interval $[a,T]$ with $N+1$ equispaced nodes t_r, $r=0,1,\cdots N$, given by $t_r=a+rh$, where $r=0,1,...,N$, $N\in\mathbb{N}^*, h=\frac{T-a}{N}$ is the time step size and $t_N=T$. At the node $r+1$ we have

$$\mathcal{F}(t_{r+1})=\mathcal{F}_0+\sum_{i=0}^{\infty}\mathscr{X}_i^{\sigma,\mu,\gamma}(I_i(\mathcal{F}))(t_{r+1}), \tag{11}$$

where

$$\mathscr{X}_i^{\sigma,\mu,\gamma}=\binom{\gamma}{i}\frac{(\sigma)^i}{\Gamma(\sigma i+1-\mu)\mathcal{N}(\sigma)(1-\sigma)^{i-1}}, \tag{12}$$

and

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$$(I_i(\mathcal{F}))(t) = \int_0^t (t-\tau)^{\sigma i-\mu}\Psi(\tau,\mathcal{F}(\tau))d\tau. \tag{13}$$

In order to progress further along in the construction of our numerical scheme, the integral $(I_i(\mathcal{F}))(t_{r+1})$ is approximating by making use of the trapezoidal quadrature formula. That is

$$\begin{aligned}(I_i(\mathcal{F}))(t_{r+1}) &= \int_0^{t_{r+1}} (t_{r+1}-\tau)^{\sigma i-\mu}\Psi(\tau,\mathcal{F}(\tau))d\tau,\\ &\approx \int_0^{t_{r+1}} (t_{r+1}-\tau)^{\sigma i-\mu}\tilde{\Psi}_{r+1}(\tau)d\tau,\end{aligned}$$

where $\tilde{\Psi}_{r+1}(\tau)$ is the piecewise linear interpolant for the function $\Psi(\tau,\mathcal{F}(\tau))$. Using $\mathcal{F}_r$ to denote the numerical approximation of $\mathcal{F}(t_r)$, $\tilde{\Psi}_{r+1}(\tau)$ is given by

$$\begin{aligned}\tilde{\Psi}_{r+1}(\tau))\big|_{\tau\in[t_j,t_{j+1}]} &= \frac{t_{j+1}-\tau}{t_{j+1}-t_j}\Psi(t_j,\mathcal{F}_j) + \frac{\tau-t_j}{t_{j+1}-t_j}\Psi(t_{j+1},\mathcal{F}_{j+1}),\\ & 0\le j\le r.\end{aligned}$$

Let $m=r+1$, and by employing some algebraic manipulations, the previous integral can be determined as

$$\begin{aligned}I_i^m(\Psi) &= \frac{1}{h}\sum_{j=0}^{m-1}\left[\Psi_{j+1}\int_{t_j}^{t_{j+1}}(t_m-\tau)^{\sigma i-\mu}(\tau-t_j)d\tau - \Psi_j\int_{t_j}^{t_{j+1}}(t_m-\tau)^{\sigma i-\mu}(\tau-t_{j+1})d\tau\right],\\ &= h^{\sigma i-\mu+1}\sum_{j=0}^{m} G_{m,i,j}\Psi_j,\end{aligned}$$

where

$$G_{m,i,j} = \begin{cases} -\tilde{v}_{m,i,0}, & \text{if } j=0,\\ \tilde{w}_{m,i,m-1}, & \text{if } j=m,\\ \tilde{w}_{m,i,j-1}-\tilde{v}_{m,i,j}, & \text{if } 1\le j\le m-1,\end{cases}$$

$$\tilde{w}_{m,i,j} = (m-j)\left[\frac{(m-j)^{\sigma i-\mu+1}-(m-j-1)^{\sigma i-\mu+1}}{\sigma i-\mu+1}\right] - \left[\frac{(m-j)^{\sigma i-\mu+2}-(m-j-1)^{\sigma i-\mu+2}}{\sigma i-\mu+2}\right],$$

$$\tilde{v}_{m,i,j} = (m-j-1)\left[\frac{(m-j)^{\sigma i-\mu+1}-(m-j-1)^{\sigma i-\mu+1}}{\sigma i-\mu+1}\right] - \left[\frac{(m-j)^{\sigma i-\mu+2}-(m-j-1)^{\sigma i-\mu+2}}{\sigma i-\mu+2}\right].$$

 doi:10.20944/preprints202204.0045.v1

Consequently, the corrector formula is described by the rule

$$\mathcal{F}_{r+1} = \mathcal{F}_0 + \Psi(t_{r+1}, \mathcal{F}_{r+1}) + \sum_{j=0}^{r} \mathscr{G}_{r+1,j} \Psi(t_j, \mathcal{F}_j), \tag{14}$$

with

$$\mathscr{G}_{r+1,j} = \sum_{i=0}^{\infty} \binom{\gamma}{i} \frac{(\sigma)^i \times h^{\sigma i+1-\mu}}{\Gamma(\sigma i + 3 - \mu)\mathcal{N}(\sigma)(1-\sigma)^{i-1}} a_{r+1,j}, \tag{15}$$

and

$$a_{r+1,j} = \begin{cases} r^{\sigma i-\mu+2} - (r - (\sigma i - \mu + 1))(r+1)^{\sigma i-\mu+1} & if \ j = 0, \\ (r-j+2)^{\sigma i-\mu+2} + (r-j)^{\sigma i-\mu+2} - 2(r-j+1)^{\sigma i-\mu+2} & if \ 1 \leq j \leq r. \end{cases} \tag{16}$$

The principle step of our algorithm is to replace the value of the term $\mathcal{F}_{r+1}$ appears on the right hand side of (14) by the predictor value $\mathcal{F}^p_{r+1}$. In this case, employing the fractional version of Adams-Bashforth method, replacing the function $\Psi(t, \mathcal{F}(t))$ at each subinterval by the quantity $\Psi(t_j, \mathcal{F}_j)$, we obtain [23]

$$\mathcal{F}^p_{r+1} = \mathcal{F}_0 + \sum_{j=0}^{r} \mathscr{E}_{r+1,j} \Psi(t_j, \mathcal{F}_j), \tag{17}$$

where

$$\mathscr{E}_{r+1,j} = \sum_{i=0}^{\infty} \binom{\gamma}{i} \frac{(\sigma)^i \times h^{\sigma i+1-\mu}}{\Gamma(\sigma i + 2 - \mu)\mathcal{N}(\sigma)(1-\sigma)^{i-1}} b_{r+1,j}, \tag{18}$$

and

$$b_{r+1,j} = (r+1-j)^{\sigma i-\mu+1} - (r-j)^{\sigma i-\mu+1}. \tag{19}$$

The last step of our algorithm is to use the truncation series $\sum_{i=0}^{M}(.)$, where $M \in \mathbb{N}$, as an approximation of the infinite series $\sum_{i=0}^{\infty}(.)$. Thus, the predictor-corrector approach is well described by the following formula, where $\mathcal{F}(j) \approx \mathcal{F}(t_j)$ for $j = 0, 1, \cdots, N$,

$$\begin{cases} \mathcal{F}_{r+1} = \mathcal{F}_0 + \sum_{i=0}^{M} \frac{\Gamma(\gamma+1)}{\Gamma(i+1)\Gamma(\gamma-i+1)} \frac{\sigma^i}{(1-\sigma)^{i-1}} \frac{h^{\sigma i+1-\mu}}{\Gamma(\sigma i-\mu+3)} \sum_{j=0}^{r} a_{r+1,j} \Psi(t_j, \mathcal{F}_j) \\ \quad + \sum_{i=0}^{M} \frac{\Gamma(\gamma+1)}{\Gamma(i+1)\Gamma(\gamma-i+1)} \frac{\sigma^i}{(1-\sigma)^{i-1}} \frac{h^{\sigma i+1-\mu}}{\Gamma(\sigma i-\mu+3)} \Psi(t_{r+1}, \mathcal{F}^P_{r+1}), \end{cases} \tag{20}$$

where

$$\mathcal{F}^P_{r+1} = \mathcal{F}_0 + \sum_{i=0}^{M} \frac{\Gamma(\gamma+1)}{\Gamma(i+1)\Gamma(\gamma-i+1)} \frac{\sigma^i}{(1-\sigma)^{i-1}} \frac{h^{\sigma i+1-\mu}}{\Gamma(\sigma i-\mu+2)} \sum_{j=0}^{r} b_{r+1,j} \Psi(t_j, \mathcal{F}_j). \tag{21}$$

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In the next section, the performance of the suggested algorithm we will be examined. For implementation purposes, we can observe that the behaviour of our algorithm is independent of the value of the parameters σ, μ and γ, and of course the accuracy can be enhanced when h becomes small. However, we can notice that the approximation $\mathcal{F}_{r+1}$ is based on the whole data record $(\mathcal{F}_0, \mathcal{F}_1, ..., \mathcal{F}_r)$. This asserts the realization of the non-local property of the studied fractional operator.

4. Test problems

In this section, we demonstrated the efficiency and accuracy of the suggested predictor-corrector algorithm to deal with IVPs involving generalized fractional derivative with Mittag-Leffler kernels. The first two test problems are identified so that their exact solutions can be found in order to compare the numerical results produced using our algorithm with exact solutions. In the third test problem, we used the suggested algorithm to study the dynamic behaviour of the studied problem.

Example 1. Our first problem considers the IVP

$$\begin{cases} {}_0^{ABC}D^{\sigma,\mu,\gamma}u(t) &= \frac{\gamma}{1-\sigma}\Gamma(\sigma\mu+2)E^{\gamma}_{\sigma,\mu+\sigma\mu+1}(\lambda,t) + u^2(t) - (\gamma t^{\sigma\mu+1})^2, \qquad t>0, \\ u(0) &= 0, \end{cases} \tag{22}$$

where ${}_0^{ABC}D^{\sigma,\mu,\gamma}$ is the generalized ABC fractional derivative with ML kernel presented in Definition 3, $0<\sigma<1$, $0<\mu<1$, $\gamma>0$ and $\lambda = \frac{-\sigma}{1-\sigma}$. The exact solution of the IVP (22) is $u(t) = \gamma t^{\sigma\mu+1}$.

In Tables 1, 2, 3 and 4, we display approximate solutions to the IVP (22) obtained using the proposed predictor-corrector algorithm at $t=1$ and $t=2$ for some specific values of the parameters σ, μ and γ. Through the numerical results shown in these tables, we can simply note that the numerical approximations provided using our algorithm are very close to the exact solutions. Furthermore, we can conclude that when N and M become large, the accuracy becomes better.

Example 2. Our second problem considers the IVP

$$\begin{cases} {}_0^{ABC}D^{\sigma,\mu,\gamma}u(t) &= \frac{1}{1-\sigma}\Big(\Gamma(\gamma\sigma+2)E^{\gamma}_{\sigma,\mu+\gamma\sigma+1}(\lambda,t) - \Gamma(\mu+2)E^{\gamma}_{\sigma,2\mu+1}(\lambda,t)\Big) \\ &- u^3(t) + \left(t^{\gamma\sigma+1} - t^{\mu+1}\right)^3, \qquad t>0, \\ u(0) &= 0, \end{cases} \tag{23}$$

 doi:10.20944/preprints202204.0045.v1

N	$M = 10$	$M = 20$	$M = 40$
10	0.73827970	0.73935162	0.73935182
20	0.74554909	0.74668745	0.74668767
40	0.74779723	0.74895962	0.74895985
80	0.74849741	0.74966789	
160	0.74871891		

Table 1: Numerical solutions to the IVP (22), when $\sigma = 0.8$, $\mu = 0.5$ and $\gamma = 0.75$, at $t = 1$. The exact value of u at $t = 1$ is 0.75.

N	$M = 10$	$M = 20$	$M = 40$
10	0.48059149	0.49603680	0.49605572
20	0.48302228	0.49835776	0.49837588
40	0.48395954	0.49930956	0.49932745
80	0.48432259	0.49970232	
160	0.48446575		

Table 2: Numerical solutions to the IVP (22), when $\sigma = 0.85$, $\mu = 0.75$ and $\gamma = 0.5$, at $t = 1$. The exact value of u at $t = 1$ is 0.5.

N	$M = 10$	$M = 20$	$M = 40$
10	1.21074309	1.32748578	1.32869314
20	1.39839620	1.54717462	1.54886505
40	1.47912231	1.64419345	1.64616852
80	1.51074278	1.68276081	
160	1.52263631		

Table 3: Numerical solutions to the IVP (22), when $\sigma = 0.75$, $\mu = 0.25$ and $\gamma = 0.75$, at $t = 2$. The exact value of u at $t = 2$ is 1.70818295.

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N	$M = 10$	$M = 20$	$M = 40$
10	1.42633610	1.44687990	1.44693776
20	1.71501560	1.74198215	1.74206038
40	1.84058353	1.87112448	1.87121479
80	1.88664452	1.91869928	
160	1.90220804		

Table 4: Numerical solutions to the IVP (22), when $\sigma = 0.7$, $\mu = 0.4$ and $\gamma = 0.8$, at $t = 2$. The exact value of u at $t = 2$ is 1.94271182.

where ${}_{0}^{ABC}D^{\sigma,\mu,\gamma}$ is the generalized ABC fractional derivative with ML kernel presented in Definition 3, $0 < \sigma < 1$, $0 < \mu < 1$, $\gamma > 0$ and $\lambda = \frac{-\sigma}{1-\sigma}$. The exact solution of the IVP (23) is $u(t) = t^{\gamma\sigma+1} - t^{\mu+1}$.

In Tables 5, 6, 7 and 8, we display approximate solutions to the IVP (23) obtained using the proposed predictor-corrector algorithm at $t = 0.5$ and $t = 1.5$ for some specific values of the parameters σ, μ and γ. Through the numerical results shown in these tables, we can simply deduce that the numerical approximations provided using our algorithm are highly compatible with the exact solutions. As shown in the previous example, we can conclude that when N and M become large, the accuracy becomes better. Moreover, by noting the convergence of the numerical results displayed in Tables 1-8, we can observe the stability property of proposed algorithm.

Example 3. Our third problem considers the IVP

$$\begin{cases} {}_{0}^{ABC}D^{\sigma,\mu,\gamma}u(t) & = & u(t) - u^2(t), \qquad t > 0, \\ u(0) & = & u_0, \end{cases} \tag{24}$$

where ${}_{0}^{ABC}D^{\sigma,\mu,\gamma}$ is the generalized ABC fractional derivative with ML kernel presented in Definition 3, $u_0 \in \mathbb{R}$, $0 < \sigma < 1$, $0 < \mu < 1$, $\gamma > 0$ and $\lambda = \frac{-\sigma}{1-\sigma}$.

The solution behavior of the fractional model given in the IVP (24) regarding different parameters of generalized derivative versus the variable t is described in Figs. 1 and 2. These figures display numerical solutions to the IVP ((24)) provided using our predictor-corrector algorithm, when $N = 400$, $M = 30$, $T = 5$ and $u_0 = 0.25$, for some specific values of σ, μ and γ.

N	$M = 10$	$M = 20$	$M = 40$
10	0.14512415	0.14512465	0.14512465
20	0.14545820	0.14545871	0.14545871
40	0.14554839	0.14554891	0.14554891
80	0.14557230	0.14557312	
160	0.14557908		

Table 5: Numerical solutions to the IVP (23), when $\sigma = 0.7$, $\mu = 0.75$ and $\gamma = 0.25$, at $t = 0.5$. The exact value of u at $t = 0.5$ is 0.14558198.

N	$M = 10$	$M = 20$	$M = 40$
10	0.17932800	0.17933581	0.17933581
20	0.17949689	0.17950473	0.17950473
40	0.17954318	0.17955102	0.17955102
80	0.17955605	0.17956389	
160	0.17955972		

Table 6: Numerical solutions to the IVP (23), when $\sigma = 0.8$, $\mu = 0.9$ and $\gamma = 0.2$, at $t = 0.5$. The exact value of u at $t = 0.5$ is 0.17956917.

N	$M = 10$	$M = 20$	$M = 40$
10	0.65005987	0.65092136	0.65092229
20	0.61671127	0.61778343	0.61778469
40	0.60620821	0.60735569	0.60735707
80	0.60302358	0.60419510	
160	0.60207416		

Table 7: Numerical solutions to the IVP (23), when $\sigma = 0.75$, $\mu = 0.4$ and $\gamma = 1.5$, at $t = 1.5$. The exact value of u at $t = 1.5$ is 0.60285785.

N	$M = 10$	$M = 20$	$M = 40$
10	0.38051467	0.38055271	0.38055271
20	0.36465861	0.36470581	0.36470582
40	0.35918591	0.35923675	0.35923676
80	0.35737370	0.35742584	
160	0.35678871		

Table 8: Numerical solutions to the IVP (23), when $\sigma = 0.65$, $\mu = 0.35$ and $\gamma = 1.25$, at $t = 1.5$. The exact value of u at $t = 1.5$ is 0.35657074.

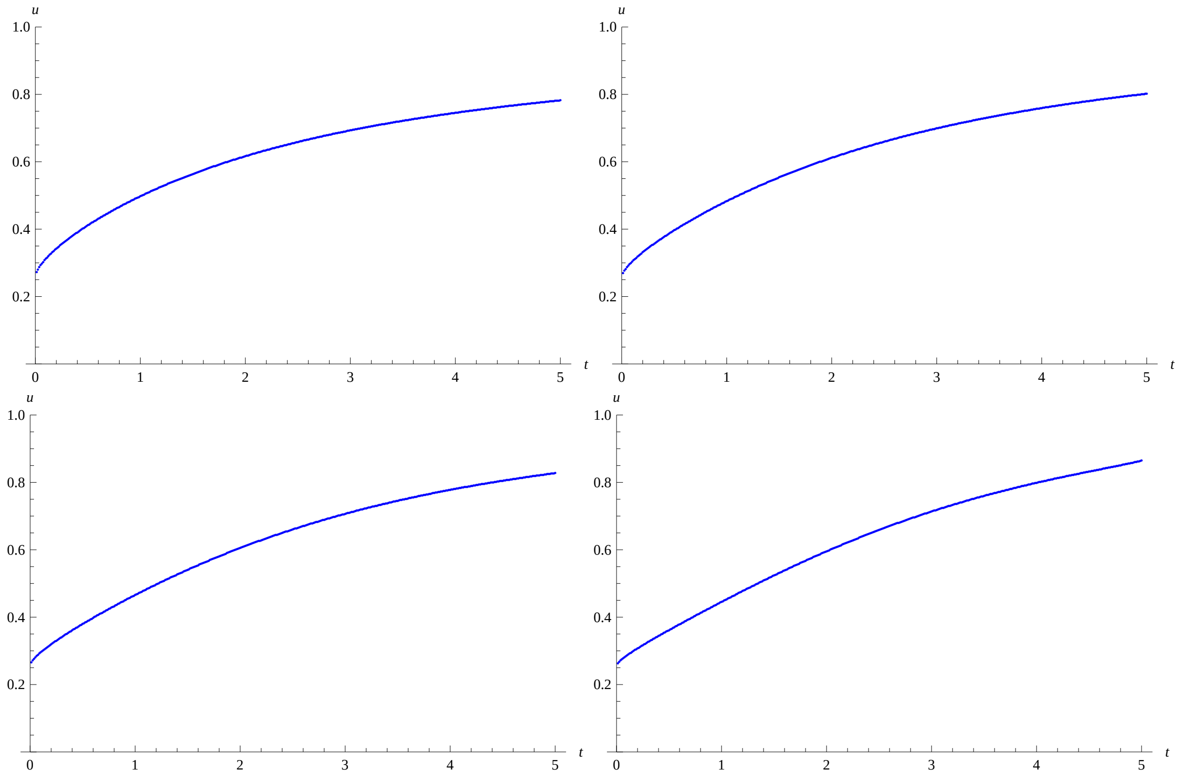

Figure 1: Plots of numerical solutions to the IVP (24), where $\mu = 0.5$ and $\gamma = 0.75$, when $N = 400$, $M = 25$, $u_0 = 0.25$ and $T = 5$.

 doi:10.20944/preprints202204.0045.v1

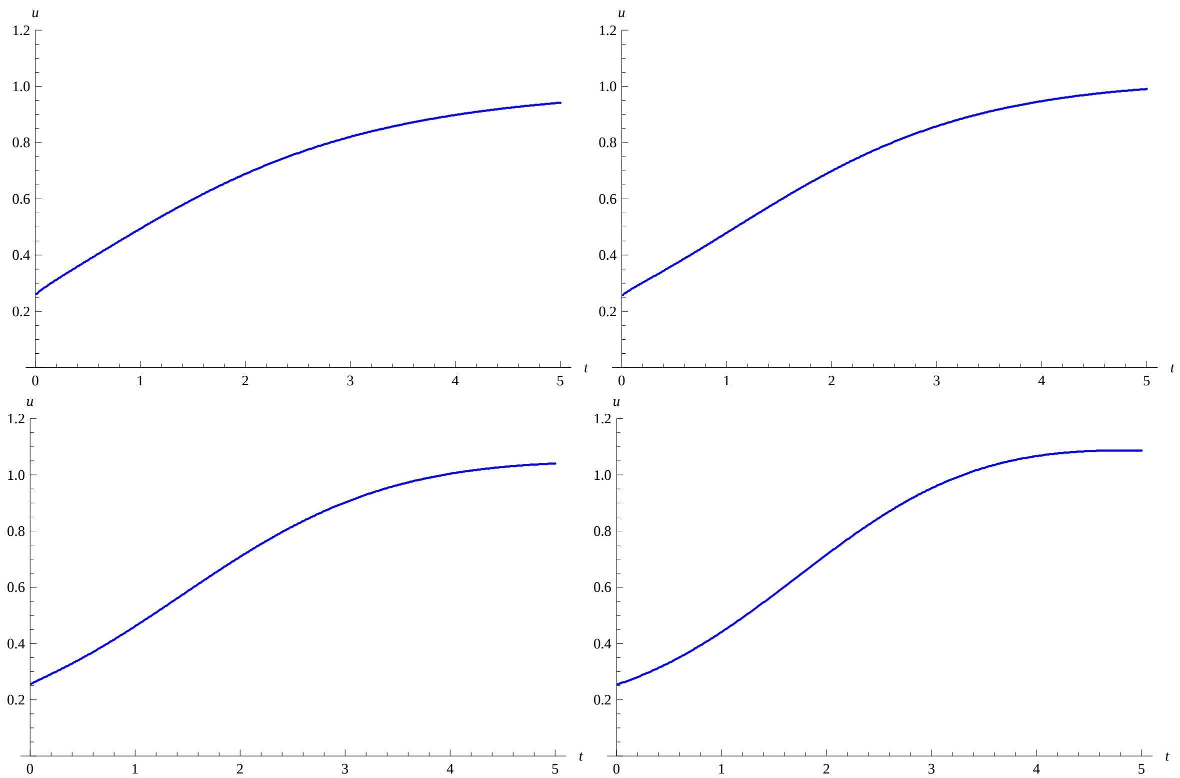

Figure 2: Plots of numerical solutions to the IVP (24), where $\mu = 0.4$ and $\gamma = 1$, when $N = 400$, $M = 25$, $u_0 = 0.25$ and $T = 5$.

5. Conclusion

In this paper, an efficient predictor-corrector algorithm for the numerical solutions of IVPs involving generalized ABC fractional derivatives with three parameters ML kernel is proposed. The developed algorithm was designed to work successfully in handling the corresponding Volterra-type integral equations conveniently and accurately. The results of the first two studied test problems confirm that the precise numerical solutions obtained using the proposed algorithm are very close to the exact solutions. Furthermore, form the convergence of the numerical results provided using the suggested algorithm, which was successfully implemented to demonstrate the solution behavior of the third test problem, we can observe the numerical stability feature of this method. Finally, since the suggested algorithm is the only one that has been developed to deal with the considered fractional models, we believe that it will find useful implementations in providing numerical solutions for many nonlinear fractional models including generalized ABC fractional derivatives with ML kernels.

Authorship contribution

Ibrahim Slimane: Suggested and initiated this work, the algorithm, wrote the first draft of the paper, performed the formal analysis of the investigation, performed the

methodology. **Zaid Odibat:** The software, Test problems, performed the methodology, formal analysis of the investigation and reviewed and edited the paper.
Dumitru Baleanu: Test problems, performed the methodology, reviewed the paper.

Acknowledgments

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