

Cross-sectional Analysis of Beams Subjected to Saint-Venant Torsion

Article

Cross-sectional Analysis of Beams Subjected to Saint-Venant Torsion Using the Green's Theorem and the Finite Difference Method

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Abstract: This paper presents an approach to the elastic analysis of beams subjected to Saint-Venant torsion using Green's theorem and the finite difference method (FDM). The Saint-Venant torsion of beams, also called free torsion or unrestrained torsion, is characterized by the absence of axial stresses due to torsion; only shear stresses are developed. A solution to this torsion problem consists of finding a stress function that satisfies the governing equation and the boundary conditions. The FDM is an approximate method for solving problems described with differential equations; it does not involve solving differential equations, equations are formulated with values at selected nodes of the structure. In this paper, the beam's cross-section was discretized using a two-dimensional grid and additional nodes were introduced on the boundaries. The introduction of additional nodes allowed us to apply the governing equations at boundary nodes and satisfy the boundary conditions. Beams with solid sections as well as multiply connected cross-sections were analyzed using this model; shear stresses and localized stresses at reentrant corners, torsion constant, and warping displacements were determined. Furthermore, beams with thin-walled closed sections, single-cell or multiple-cell, were analyzed using the Prandtl stress function whereby a linear distribution of the shear stresses over the thickness was considered; closed-form solutions for shear stresses and torsion constant were derived. The results obtained in this study showed good agreement with the exact results for rectangular cross-sections, and the accuracy was increased through a grid refinement. For thin-walled closed sections, the shear stresses obtained at the centerline using the closed-form solutions were in agreement with the values using Bredt's analysis but the maximal values in the cross-section, which did not necessarily occur at the position with the smallest thickness, were higher; in addition, the results using the closed-form solutions were in good agreement with those using FDM.

Keywords: Theory of elasticity; Saint-Venant torsion; Green's theorem; finite difference method; additional nodes; thin-walled sections; stress concentration at reentrant corners; multiply connected cross-section; warping displacement

1. Introduction

This paper describes the application of Fogang's model [1] based on the finite difference method (FDM), used for the homogeneous Euler–Bernoulli beam, to the elastic analysis of beams subjected to Saint-Venant torsion. The Saint-Venant torsion is characterized by the absence of axial stresses due to torsion; only shear stresses are developed. The correct solution of the problem of torsion of prismatical beams was given by Saint-Venant [2]; he made assumptions on

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the deformation of the twisted bar that could satisfy the equations of equilibrium and the boundary conditions. Bredt [3] derived formulas for the torsional analysis of thin-walled bars; these formulas form a basis for the calculation of shear stresses and deformations in components with closed, thin-walled hollow cross-sections under pure torsional loading. Prandtl [4] introduced a stress function in terms of which the shear stresses were defined. Therefore, the solution of the torsion problem consists of finding a stress function which satisfies the governing equation and the boundary conditions. Analytical solutions of the stress function are available for beams with elliptical cross-sections, equilateral triangle and several other shapes; furthermore, the torsional problem was solved for rectangular cross-section using infinite trigonometric series. However, analytical solutions of the Saint-Venant torsion are only suitable for simple cross-sections; therefore, numerical methods to evaluate the torsional behavior of complicated cross-sections are indispensable. Numerical methods such as energy methods were considered by numerous authors i.e. Ritz [5] and Trefftz [6]; the stress function was approximately determined from the minimum condition of the strain energy of the twisted beam. Various studies have focused on the analysis of beams under unrestrained torsion. Pluzsik et al. [7] presented a theory for thin-walled, closed section, orthotropic beams which takes into account the shear deformation in restrained warping induced torque; the analytical (“exact”) solution of simply supported beams subjected to a sinusoidal load was developed for this purpose. Pan et al. [8] presented a matrix stiffness method for the torsion and warping analysis of beam-columns in order to investigate the exact element torsional stiffness considering warping deformations; the equilibrium analysis of an axial-loaded torsion member was conducted, and the torsion-warping problem was solved based on a general solution of the established governing differential equation for the angle of twist. Pavazza et al. [9] presented a novel theory of torsion of thin walled beams (“shear deformable beams”) of arbitrary open cross-sections with influence of shear; the theory is based on the classical Vlasov’s theory of thin-walled beams of open cross-section, as well on the Timoshenko’s beam bending theory. Choi & Kim [10] proposed a higher-order Vlasov torsion theory that not only includes as many torsion-related modes as desired but also provides the explicit forces–kinematic variables–stresses relationships that are fully consistent with those by the Vlasov theory. Amulu & Ezeagu [11] investigated the effect of combined actions of torsion moments, bending moments and shear forces in reinforced concrete beams with concrete compressive strength of 30N/mm²; the ultimate torsion moments, bending moments, and shear forces of the beams were determined experimentally. Tran [12] used isoparametric eight-noded quadrilateral elements in order to improve Gruttmann’s isoparametric four-noded quadrilateral elements; MATLAB was the language for programming the numerical method.

In this paper the torsion problem of beams was solved using FDM. The cross-sections were discretized with a two-dimensional grid and additional nodes were introduced at the boundaries. The introduction of additional nodes allowed us to apply the governing equations at boundary nodes and to satisfy the boundary conditions. Multiply connected cross-sections were also analyzed. So, the values of the stress function at nodes were determined and thereafter the shear stresses and torsion constant. Beams with thin-walled closed sections were analyzed using the stress function whereby the linear distribution of the shear stresses over the thickness was considered; closed form solutions for shear stresses and torsion constant were presented and compared to those of Bredt [3].

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2. Materials and methods

2.1 Linear elasticity of unrestrained torsion of beams

The axis convention in the cross-section is represented in Figure 1, whereas the x-axis is the longitudinal beam's axis.

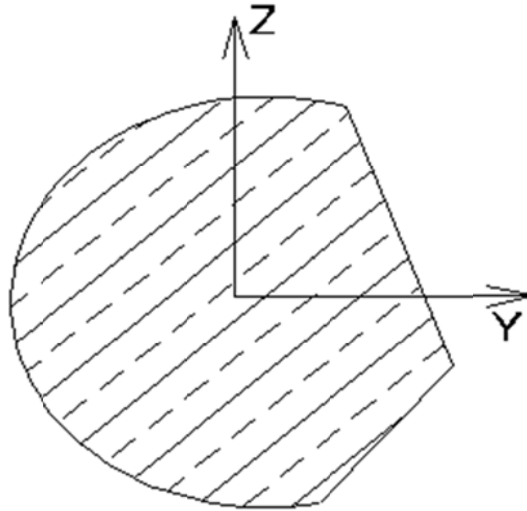


Figure 1. Axis convention within the beam's cross-section

The equations in this section are related to the theory of linear elasticity of beams. The displacements in x-, y-, and z-direction are denoted by $u(x,y,z)$, $v(x,y,z)$, and $w(x,y,z)$, respectively, and the rotation of cross-section or angle of twist (positive anticlockwise) is denoted by $\theta(x)$. The shearing strains γ_{xy} and γ_{xz} are related to the displacements as follows:

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \quad (1a, b)$$

The cross-sections are assumed to rotate about an axis through the center of torsion T (y_T, z_T) (which is equivalent to the shear center). Therefore, the displacements $v(x,y,z)$ and $w(x,y,z)$ are related to the angle of twist $\theta(x)$ as follows

$$v = -(z - z_T)\theta, \quad w = (y - y_T)\theta \quad (2a, b)$$

Combining Equations (1a-b) and (2a-b) yields the stress strains relationships

$$\tau_{xy} = G \left(\frac{\partial u}{\partial y} - (z - z_T) \frac{d\theta}{dx} \right), \quad \tau_{xz} = G \left(\frac{\partial u}{\partial z} + (y - y_T) \frac{d\theta}{dx} \right) \quad (3a, b)$$

where G is the shear modulus. Combining Equations (3a) and (3b) yields following relationship between the shear stresses and the rotation $\theta(x)$

$$\frac{\partial \tau_{xy}}{\partial z} - \frac{\partial \tau_{xz}}{\partial y} = -2G \frac{d\theta}{dx} \quad (4)$$

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Recalling that axial stresses do not occur in the beam subjected to unrestrained torsion, the equilibrium equation in x-direction applied to an infinitesimal beam element with dimensions dx , dy , and dz yields

$$\frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0 \quad (5)$$

Prandtl [4] introduced a stress function $\phi(x,y,z)$, defined in terms of stress components as follows

$$\tau_{xy} = \frac{\partial \phi}{\partial z}, \quad \tau_{xz} = -\frac{\partial \phi}{\partial y} \quad (6)$$

It is noted that Equations (6) satisfy the equilibrium equation (5). Substituting Equations (6) into (4) yields the following condition to be satisfied by the stress function $\phi(x,y,z)$

$$\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = -2G \frac{d\theta}{dx} \quad (7)$$

The analysis is then reduced of determining the stress function.

Let us consider an infinitesimal beam element at the boundary with dimensions dx , $-dy$, and dz . The outer normal $\mathbf{n}$ to the boundary makes an angle α with the $+y$ -axis, as represented in Figure 2.

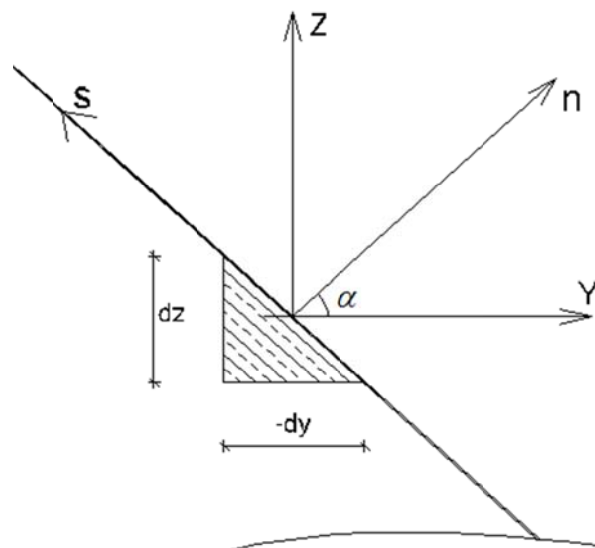


Figure 2 Infinitesimal beam element at boundary

Considering that axial stresses do not occur, the equilibrium equation applied to the beam element along the x-axis combined with Equation (6) yields

$$\begin{aligned} -\tau_{xz}(-dy) \times dx - \tau_{xy}dz \times dx &= 0 \\ \rightarrow \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz &= 0 \rightarrow d\phi = 0 \end{aligned} \quad (8)$$

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This means that the stress function $\phi(x,y,z)$ is constant along the boundary of the cross-section. Observing that $\sin\alpha = -dy/ds$ and $\cos\alpha = dz/ds$, the shearing stress component τ_t at the boundary directed along the tangent is

$$\tau_t = -\tau_{xy} \sin \alpha + \tau_{xz} \cos \alpha = -\frac{\partial \phi}{\partial z} \sin \alpha - \frac{\partial \phi}{\partial y} \cos \alpha = \frac{\partial \phi}{\partial z} \frac{dy}{ds} - \frac{\partial \phi}{\partial y} \frac{dz}{ds} \quad (9a)$$

Likewise, the shearing stress component τ_n at the boundary directed along the outer normal to the boundary is

$$\tau_n = \tau_{xy} \cos \alpha + \tau_{xz} \sin \alpha = \frac{\partial \phi}{\partial z} \cos \alpha - \frac{\partial \phi}{\partial y} \sin \alpha = \frac{\partial \phi}{\partial z} \frac{dz}{ds} + \frac{\partial \phi}{\partial y} \frac{dy}{ds} \quad (9b)$$

Observing Equation (8), the shearing stress component τ_n is zero; this is in agreement with the condition of stress free outer surfaces. Therefore, the resultant shearing stress at the boundary is τ_t . Let us recall the Green's theorem which relates a line integral around a closed curve B to a double integral over the plane region A bounded by B

$$\oint_B [L dy + M dz] = \iint_A \left[\frac{\partial M}{\partial y} - \frac{\partial L}{\partial z} \right] dy dz \quad (10)$$

whereby L and M are functions defined on the region A and the path of integration along B is anticlockwise.

Recalling that the stress function has a constant value ϕ_B at the boundary, the Green's theorem shows as follows that the resultant shear forces are zero

$$\begin{aligned} \iint \tau_{xz} dy dz &= \iint -\frac{\partial \phi}{\partial y} dy dz = -\oint_B \phi dz = -\phi_B \oint_B dz = 0 \\ \iint \tau_{xy} dy dz &= \iint \frac{\partial \phi}{\partial z} dy dz = -\oint_B \phi dy = -\phi_B \oint_B dy = 0 \end{aligned} \quad (11)$$

where the line integrals are taken around the closed boundary B of the cross-section. Using Equations (6) and (11), the torque is then given by

$$M_t = \iint [\tau_{xz} (y - y_T) - \tau_{xy} (z - z_T)] dy dz = \iint \left[-\frac{\partial \phi}{\partial y} y - \frac{\partial \phi}{\partial z} z \right] dy dz \quad (12a)$$

The derivatives of ϕy with respect to y and ϕz with respect to z, respectively, are given by

$$\frac{\partial [\phi y]}{\partial y} = \frac{\partial \phi}{\partial y} y + \phi, \quad \frac{\partial [\phi z]}{\partial z} = \frac{\partial \phi}{\partial z} z + \phi \quad (12b)$$

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Substituting Equations (12b) into (12a) yields

$$M_t = 2 \iint \phi dydz - \iint \left[\frac{\partial [\phi y]}{\partial y} + \frac{\partial [\phi z]}{\partial z} \right] dydz \quad (12c)$$

In the case of solid cross-sections and recalling that the stress function has a constant value ϕ_B at the boundary, the Green's theorem applied to the second term at the right-hand side of Equation (12c) is given by

$$\iint \left[\frac{\partial [\phi y]}{\partial y} + \frac{\partial [\phi z]}{\partial z} \right] dydz = \oint_B [\phi y dz - \phi z dy] = \phi_B \oint_B [y dz - z dy] = 2\phi_B A_B \quad (12d)$$

where A_B is the area enclosed by the outer boundary of the cross-section. Let us express the stress function as follows

$$\phi(x, y, z) = \phi_B + \phi^*(x, y, z) \quad (12e)$$

where the function ϕ^* is zero along the boundary and satisfies Equation (7). Substituting Equations (12d) and (12e) into (12c) yields

$$M_t = 2 \iint \phi^* dydz \quad (12f)$$

Therefore, for a beam with solid cross-section the expression of the torque is independent of the value ϕ_B of the stress function at the boundary. Furthermore, Equations (12a-f) show that half the torque is due to the stress component τ_{xy} and the other half to τ_{xz} . For simplification purpose, a modified stress function $\psi(y, z)$ is defined

$$\phi^*(x, y, z) = G \frac{d\theta}{dx} \times \psi(y, z) \quad (13a)$$

Thus, the function $\psi(y, z)$ is zero along the boundary. Substituting Equations (13a) and (12e) into (7) and (6) yields

$$\frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = -2 \quad (13b)$$

$$\tau_{xy} = G \frac{d\theta}{dx} \times \frac{\partial \psi}{\partial z}, \quad \tau_{xz} = -G \frac{d\theta}{dx} \times \frac{\partial \psi}{\partial y}$$

The torsional stiffness GI_t is related to the torque M_t and the angle of twist $\theta(x)$ as follows

$$M_t = GI_t \frac{d\theta}{dx} \quad (13c)$$

Combining Equations (12f), (13a), and (13c) yields the torsion constant I_t

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$$I_t = 2 \iint \psi dydz \quad (13d)$$

In the case of multiply connected cross-sections with one or many holes Equation (12d) becomes

$$\iint \left[\frac{\partial[\phi y]}{\partial y} + \frac{\partial[\phi z]}{\partial z} \right] dydz = 2\phi_B A_B - \sum_i 2\phi_i A_i \quad (14a)$$

where A_i is the area enclosed by the hole i and ϕ_i the constant value of the stress function along the boundary of the hole i . Combining Equations (12c, e) and (14a) yields

$$M_t = 2 \iint \phi^* dydz + \sum_i 2\phi_i A_i \quad (14b)$$

Observing that $\phi_i = \phi_B + \phi_i^*$, we set ϕ_B to zero and combine Equations (13a, c) and (14b) to obtain the torsion constant

$$I_t = 2 \iint_A \psi dydz + 2 \sum_i \psi_i A_i \quad (14c)$$

Equation (14c) can be found in Dieker [13] and Sadd [14].

2.2 Finite difference approximations

For simplification purpose, the analysis will be conducted in the following with the modified stress function $\psi(y,z)$. The analysis is then governed by Equation (13b-1). This equation has second order derivatives; consequently, the function $\psi(y,z)$ is approximated around the node of interest i as a second degree polynomial in each direction. The unknown at any node is the value ψ_i of the modified stress function.

Given the grid spacings $\Delta y = h$ and $\Delta z = \lambda h$ in y - and z -direction, respectively. The finite difference approximations (FDAs) for the first and second derivatives are then given by

$$\begin{aligned} \left. \frac{\partial^2 \psi}{\partial y^2} \right|_{i-1} &= \left. \frac{\partial^2 \psi}{\partial y^2} \right|_i = \left. \frac{\partial^2 \psi}{\partial y^2} \right|_{i+1} = \frac{\psi_{i-1} - 2\psi_i + \psi_{i+1}}{h^2}, \\ \left. \frac{\partial \psi}{\partial y} \right|_i &= \frac{-\psi_{i-1} + \psi_{i+1}}{2h} \end{aligned} \quad (15a, b)$$

The partial derivatives in z -direction are formulated similarly.

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2.3 Analysis at interior nodes

2.3.1 Uniform grid

Figure 3 shows the node distribution of the beam's cross-section having equidistant nodes with spacings Δy and Δz in y- and z-direction, respectively. The node of interest **k** and the surrounding nodes are represented, whereby n, s, e, and w stand for the directions north, south, east, and west, respectively, according to the directions in the stencil. The node k may even be at the boundary of the beam, however being not at angles.

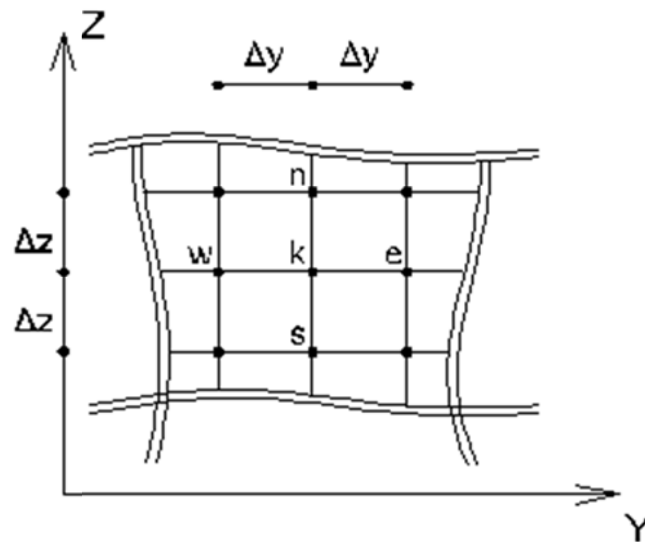


Figure 3 Node of interest **k** and its surrounding nodes for uniform grid

Given the grid spacings $\Delta y = h$ and $\Delta z = \lambda h$. The governing equation (Equation (13b-1)) at a given node can be expressed by means of a stencil using Equations (15a) as follows

$$\frac{1}{h^2} \begin{bmatrix} 1 & \frac{1}{\lambda^2} \\ -2 - \frac{2}{\lambda^2} & 1 \\ \frac{1}{\lambda^2} & 1 \end{bmatrix} \times [\psi] = -2 \quad (16)$$

The shearing stresses are expressed using Equations (13b) and (15b) as follows

$$\tau_{xy} = G \frac{d\theta}{dx} \times \frac{1}{2\lambda h} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \times [\psi], \quad \tau_{xz} = -G \frac{d\theta}{dx} \times \frac{1}{2h} \begin{bmatrix} -1 & 0 & 1 \end{bmatrix} \times [\psi] \quad (17)$$

In the stencil notation the factor associated to the node of interest is in brackets.

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2.3.2 Non-uniform grid

The distances to the node of interest k are represented in Figure 4 below

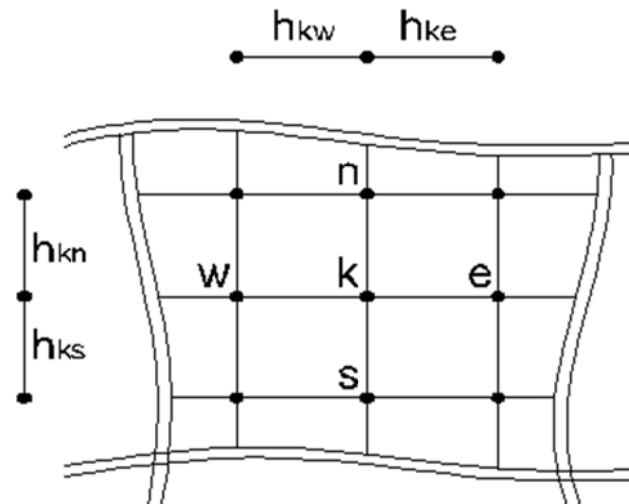


Figure 4 Node of interest k and its surrounding nodes for non-uniform grid

The stress function in y -direction i.e. can be described with values at grid points as follows:

$$\psi_{i-1} \times f_{i-1}(x) + \psi_i \times f_i(x) + \psi_{i+1} \times f_{i+1}(x) \quad (18a)$$

The shape functions $f_j(x)$ ($j = i-1; i; i+$) can be expressed using the following Lagrange interpolating polynomials

$$f_j(x) = \prod_{\substack{k=i-1 \\ k \neq j}}^{i+1} \frac{x - x_k}{x_j - x_k} \quad (18b)$$

The governing equation at node k is derived using Equations (13b-a), (18a), and (18b) as follows

$$\left[\begin{array}{c} \frac{2}{h_{kw}(h_{kw} + h_{ke})} \\ \frac{2}{h_{kn}(h_{kn} + h_{ks})} \\ \left[-\frac{2}{h_{kw}h_{ke}} - \frac{2}{h_{kn}h_{ks}} \right] \\ \frac{2}{h_{ks}(h_{kn} + h_{ks})} \end{array} \right] \times [\psi] = -2 \quad (19)$$

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The shearing stresses are expressed using Equations (13b), (18a), and (18b) as follows

$$\tau_{xy} = G \frac{d\theta}{dx} \times \begin{bmatrix} \frac{h_{ks}}{h_{kn}(h_{kn} + h_{ks})} \\ \left[\frac{1}{h_{ks}} - \frac{1}{h_{kn}} \right] \\ -\frac{h_{kn}}{h_{ks}(h_{kn} + h_{ks})} \end{bmatrix} \times [\psi], \quad (20a-b)$$

$$\tau_{xz} = -G \frac{d\theta}{dx} \times \begin{bmatrix} \frac{-h_{ke}}{h_{kw}(h_{kw} + h_{ke})} \\ \left[\frac{1}{h_{kw}} - \frac{1}{h_{ke}} \right] \\ \frac{h_{kw}}{h_{ke}(h_{kw} + h_{ke})} \end{bmatrix} \times [\psi]$$

2.4 Analysis at skew edges

Figure 5 shows the skew edge of the beam's cross-section with regular and additional nodes. The tangent **t** to the skew edge makes an angle α with the +y-axis. One additional node is associated to each edge node; therefore, the governing equation (Equation (16) or (19)) is applied at any edge node and the boundary condition $\psi = 0$ is set.

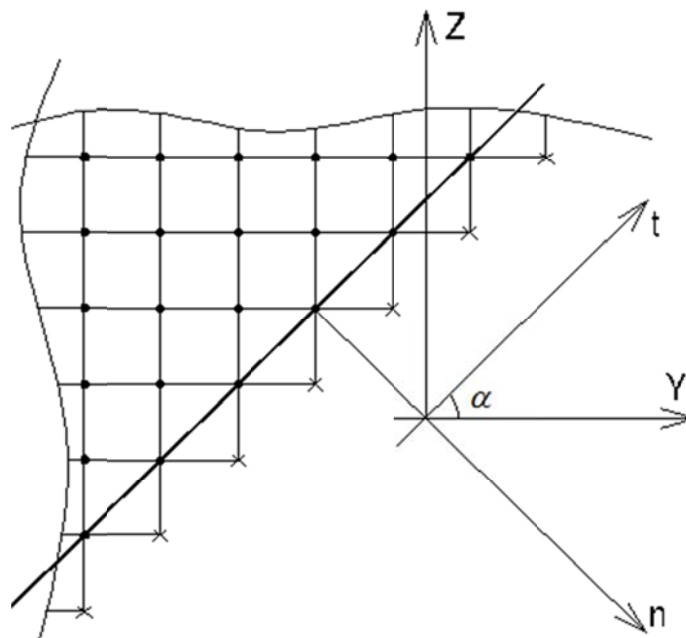


Figure 5 Skew edge of beam's cross-section with regular nodes (•) and additional nodes (x)

It is recalled that the shear stress component τ_t directed along the tangent is calculated using Equations (9a), and the shear stress component τ_{xy} and τ_{xz} using Equations (17) or (20a-b).

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2.5 Analysis at beam angles

The node distribution must be such that at any position three nodes be available in both directions; so the governing equations that are formulated through the second derivatives of the stress function can be set.

2.5.1 Right angles

Figure 6 below represents angles of the beam's cross-section whereby regular nodes (•) and additional nodes (×) are represented. The unknown at each node, regular or additional, is the value of the function $\psi(y, z)$. One additional node is associated to each boundary node, at which the boundary condition $\psi = 0$ is set. The governing equations at angle nodes W, E, S, and SE are modified using Equation (15a) to account for the absence of additional nodes in z-direction; Therefore Equation (21a) is the governing equation at angles W and E, while Equation (21b) is the governing equation at angles S and SE. Alternatively, additional nodes at angles could be considered in z-direction instead. An analysis without additional nodes may be considered whereby the governing equations are not applied at the boundary; it will be shown in the Results Section that the results using additional nodes are more accurate.

$$\frac{1}{h^2} \begin{bmatrix} 1 & \begin{bmatrix} -2 + \frac{1}{\lambda^2} \end{bmatrix} & 1 \\ & -\frac{2}{\lambda^2} & \\ & \frac{1}{\lambda^2} & \end{bmatrix} \times [\psi] = -2, \quad \frac{1}{h^2} \begin{bmatrix} & \frac{1}{\lambda^2} & \\ & -\frac{2}{\lambda^2} & \\ 1 & \begin{bmatrix} -2 + \frac{1}{\lambda^2} \end{bmatrix} & 1 \end{bmatrix} \times [\psi] = -2 \quad (21a-b)$$

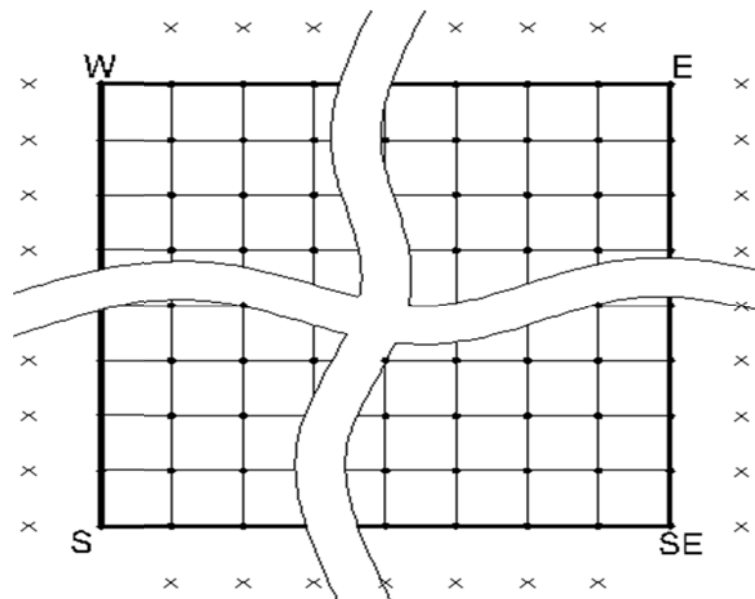


Figure 6 Angles of beam's cross-section with regular nodes (•) and additional nodes (×)

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2.5.2 Various types of angles and shapes of beam's cross-sections

Examples of node distributions at various angles and shapes of beam's cross-sections are represented in Figure 7. The additional node associated to a node k is denoted by ka .

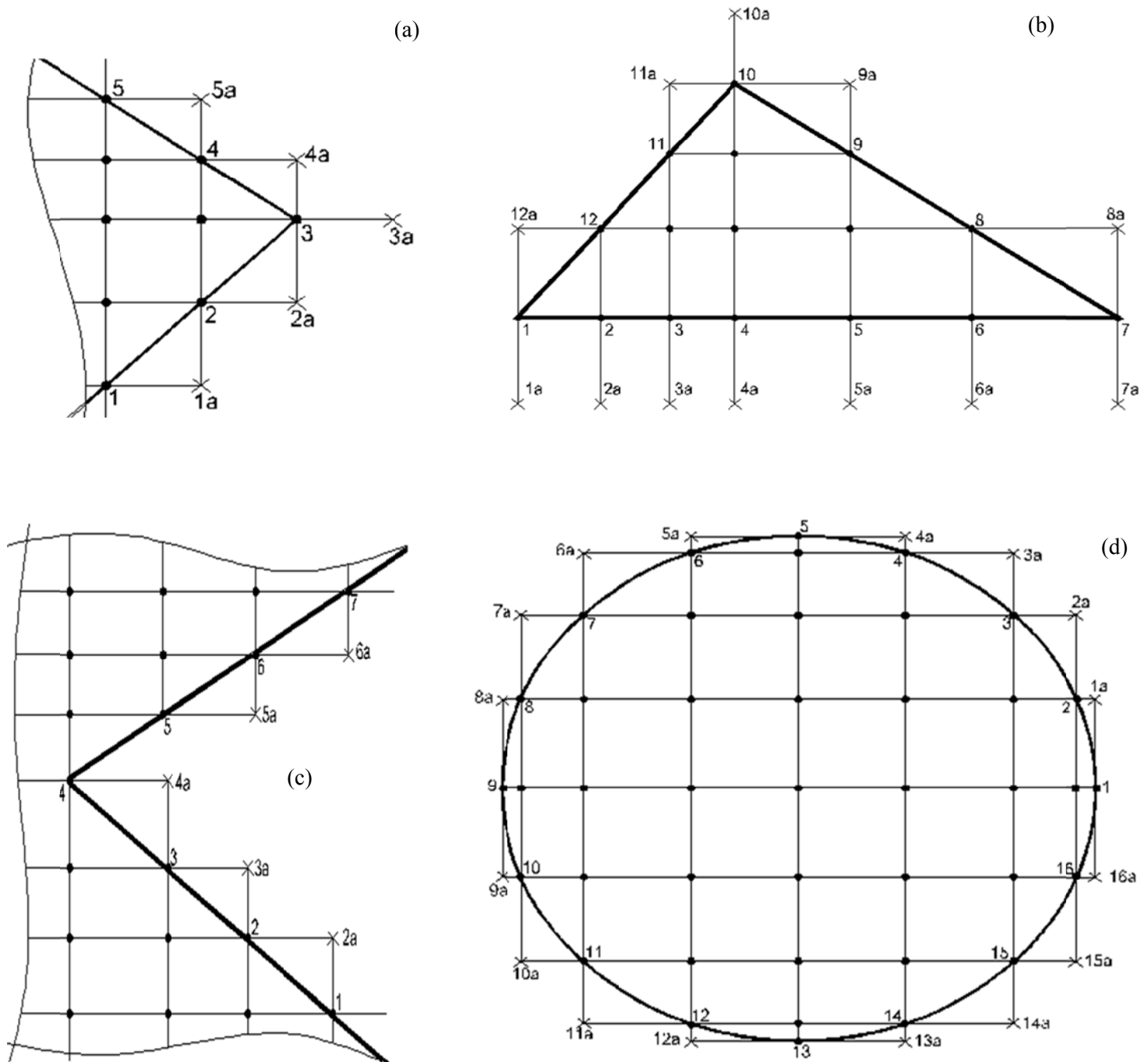


Figure 7 Examples of node distributions at various angles and shapes of beam's cross-sections

In the following the stencils may have to be modified to account for eventual non-uniform grids. Equations (16) or (19), the governing equations, are applied at any boundary node if possible. In Figure 7b, the governing equations at nodes 8

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and 12 are expressed using Equation (21a); furthermore, the governing equations at nodes 1 and 11, and at nodes 7 and 9, respectively, are expressed using Equations (21c-d) as follows

$$\frac{1}{h^2} \begin{bmatrix} \frac{1}{\lambda^2} \\ 1 - \frac{2}{\lambda^2} \\ \frac{1}{\lambda^2} \end{bmatrix} \begin{bmatrix} -2 & 1 \end{bmatrix} \times [\psi] = -2, \quad \frac{1}{h^2} \begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{\lambda^2} \\ 1 - \frac{2}{\lambda^2} \\ \frac{1}{\lambda^2} \end{bmatrix} \times [\psi] = -2 \quad (21c-d)$$

In Figure 7c, the governing equations at node 5 is applied using Equation (21b). In Figure 7d, the governing equations at nodes 9 and 1, respectively, are expressed using Equations (21c-d) and the governing equations at nodes 5 and 13, respectively, are expressed using Equations (21a-b).

2.6 Beams with multiply connected cross-sections

2.6.1 Holes aligned with the Cartesian coordinate system

A beam with multiply connected cross-section is represented in Figure 8 (a), together with the grid around the hole (b) showing the regular nodes (●) and additional nodes (×). Let the hole be aligned with the Cartesian coordinate system. The unknown at regular nodes and additional nodes is the value of the stress function. The additional node associated to a node k is denoted by k_a or k_{aa} .

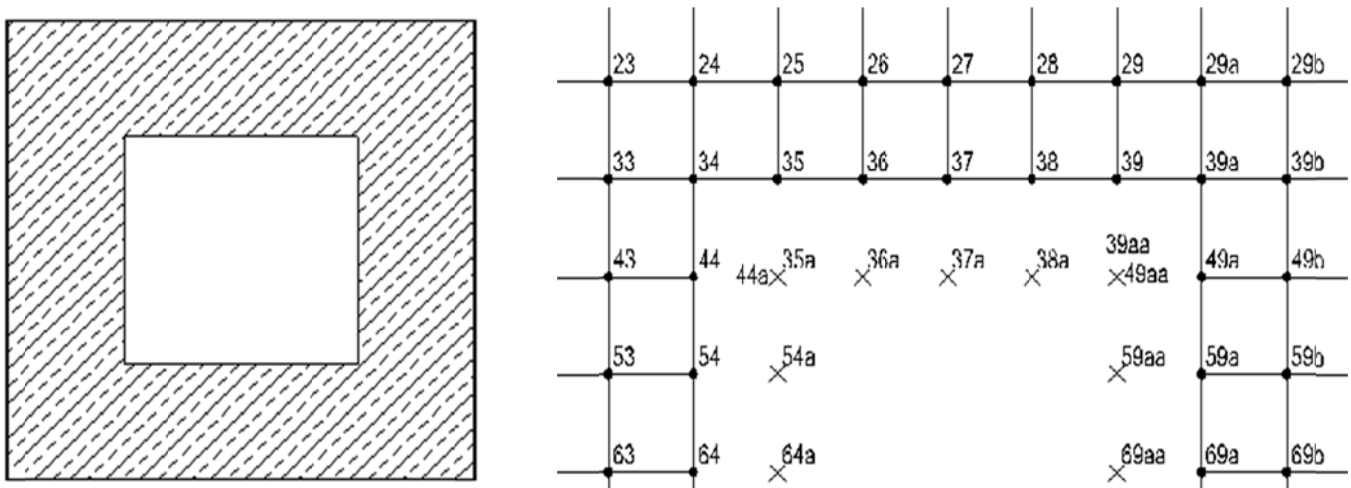


Figure 8 (a) Beam with multiply connected section, (b) grid around the hole with regular nodes (●) and additional nodes (×)

Similarly to Fogang [15] related to the analysis of deep beams, **it is assumed that different nodes may be at the same geometrical position**, i.e. nodes 35a and 44a, 39aa and 49aa in Figure 8. The boundary conditions $\Psi = \text{constant}$ are

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applied to nodes on the hole whereby the constant value ψ_i for the hole i is set at these nodes. The governing equations are applied at nodes which additional nodes are associated to. Therefore, the governing equations will not be applied at angle nodes 34 and 39a since additional nodes are not associated to them. Inspired from Timoshenko [16] and Sadd [14], we perform a line integral (anticlockwise) of the shear stresses along the boundary B_i of the hole i and use Equations (7), (9a), (12e), (13a), and the Green's theorem

$$\oint_{B_i} \tau_t ds = \oint_{B_i} \left[\frac{\partial \phi}{\partial z} dy - \frac{\partial \phi}{\partial y} dz \right] = \iint_{O_i} \left[-\frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \phi}{\partial z^2} \right] dy dz = 2G \frac{d\theta}{dx} A_i$$

$$\rightarrow \oint_{B_i} \left[\frac{\partial \psi}{\partial z} dy - \frac{\partial \psi}{\partial y} dz \right] = 2A_i \quad (22a-b)$$

where A_i is the area enclosed by the hole i . Therefore, the value ψ_i for the hole i must be chosen so that Equation (22b) is satisfied; so, there are as many equations as unknowns. Particular attention must be taken by the formulation of the governing equations and the derivatives $\partial\psi/\partial z$ and $\partial\psi/\partial y$, especially in the vicinity of angle nodes; those equations at any node k involve the node ka .

Equation (22b) can be applied at any inner boundary whatever the shape of the beam's cross-section.

For a hole whose edges are aligned with the Cartesian coordinate system, the integral at the left-hand side of Equation (22b) can be calculated using the Simpson rule i.e. whereby the signs of dy and dz are adjusted in each edge. Let H_k and V_q be horizontal and vertical edges of the hole, respectively; Equation (22b) can be expressed as

$$\oint_{B_i} \left[\frac{\partial \psi}{\partial z} dy - \frac{\partial \psi}{\partial y} dz \right] = \sum_k \int_{H_k} \frac{\partial \psi}{\partial z} dy - \sum_q \int_{V_q} \frac{\partial \psi}{\partial y} dz = 2A_i \quad (22c)$$

Alternatively, the right angles can be chamfered such that the angle nodes 34 and 39a behave like interior nodes, as represented in Figure 9; a modified hole is then obtained.

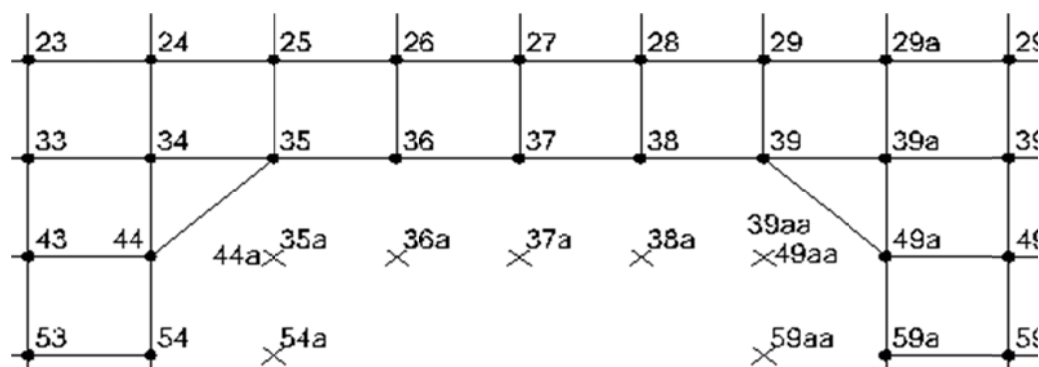


Figure 9 Beam's cross-section having an hole with chamfering of the right angles

Cross-sectional Analysis of Beams Subjected to Saint-Venant Torsion

Governing equations, not boundary conditions, are then applied at nodes 34 and 39a. Furthermore, governing equations are applied at any node of the new hole that have associated additional node, and the constant value Ψ_i is set at these nodes. Equation (22b) is formulated in the skew segments at angles as follows

$$\int_k^{k+1} \left[\frac{\partial \psi}{\partial z} dy - \frac{\partial \psi}{\partial y} dz \right] = \frac{1}{2} \left(\left. \frac{\partial \psi}{\partial z} \right|_k + \left. \frac{\partial \psi}{\partial z} \right|_{k+1} \right) (y_{k+1} - y_k) - \frac{1}{2} \left(\left. \frac{\partial \psi}{\partial y} \right|_k + \left. \frac{\partial \psi}{\partial y} \right|_{k+1} \right) (z_{k+1} - z_k) \quad (22d)$$

2.6.2 Holes of arbitrary shapes

A beam's cross-section having a hole is represented in Figure 10, together with regular nodes (•) and additional nodes (×). The hole of arbitrary shape is not aligned with the Cartesian coordinate system.

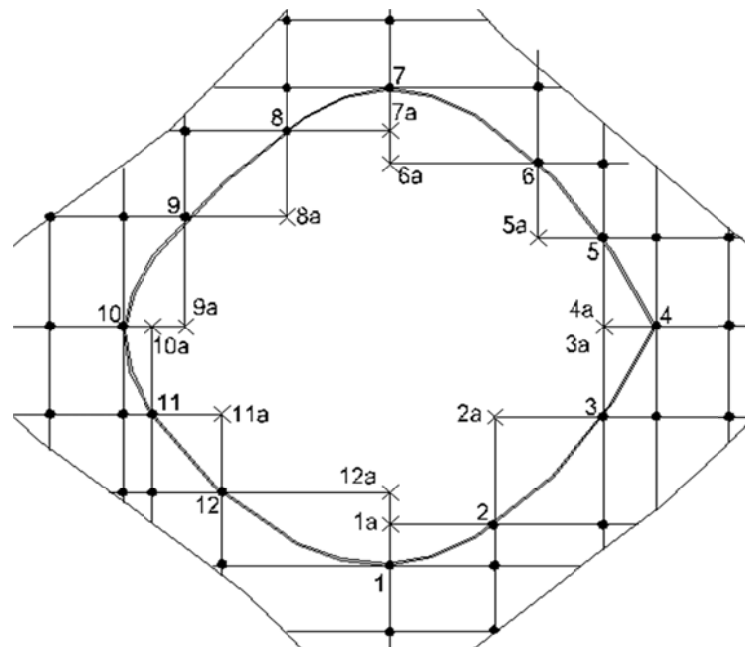


Figure 10 Beam's cross-section having a hole of arbitrary shape

It is further assumed that different nodes may be at the same geometrical position, i.e. nodes 4a and 3a. The governing equations are applied to any node on the hole that have associated additional nodes, and the constant value Ψ_i of the modified stress function is set at any node on the hole. Assuming a linear approximation of the inner boundary and the derivatives $\partial \psi / \partial z$ and $\partial \psi / \partial y$ between neighbor nodes, Equation (22b) is expressed as follows for a hole having n nodes (1, 2 ...n)

$$\frac{1}{2} \sum_{k=1}^{k=n} \left[\left. \frac{\partial \psi}{\partial z} \right|_k (y_{k+1} - y_{k-1}) - \left. \frac{\partial \psi}{\partial y} \right|_k (z_{k+1} - z_{k-1}) \right] = 2A_i \quad (22e)$$

where the nodes 0 and $n + 1$ in Equation (22e) are equivalent to nodes n and 1, respectively.

Cross-sectional Analysis of Beams Subjected to Saint-Venant Torsion

2.7 Beams with thin-walled sections

2.7.1 Fundamentals of thin-walled sections

Given a beam with thin-walled section represented in Figure 11.

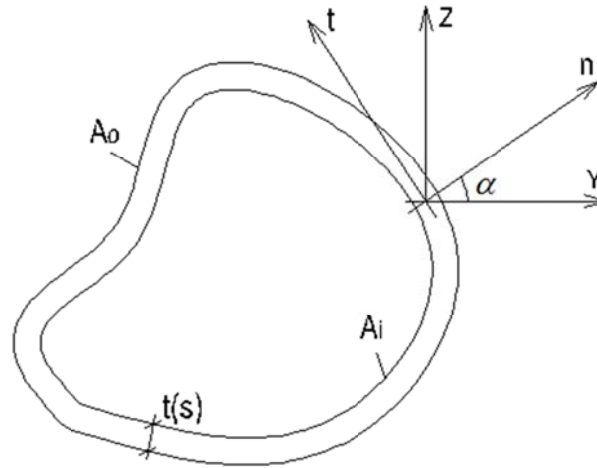


Figure 11 Beam with thin-walled section

On the one hand, the analysis of beams with single-cell thin-walled closed sections i.e. is governed by the following Bredt's [3] formulas for the shear stress $\tau_t(s)$ at an arc coordinate s with thickness $t(s)$ and the torsion constant I_t

$$\tau_t(s) = \frac{M_t}{2A_m t(s)}, \quad I_t = \frac{4A_m^2}{\oint \frac{ds}{t(s)}} \quad (23a-b)$$

where A_m is the area enclosed by the midline of the cross-section. Bredt's formulas are derived under the assumption of a constant shear stress τ_t over the thickness whereby stresses τ_n are neglected.

On the other hand, the analysis of thin-walled open sections is governed by the following formulas

$$\tau_{\max}(s) = \frac{M_t}{I_t} t(s), \quad I_t = \frac{1}{3} \int t(s)^3 ds \quad (23c-d)$$

In the present study, the stress components τ_n and τ_t can be defined as follows using a coordinate transformation and observing Figure 2 and Equations (6),

$$\tau_n = \frac{\partial \phi}{\partial s}, \quad \tau_t = -\frac{\partial \phi}{\partial n} \quad (24a-b)$$

Furthermore, Equations (4), (7), and (13b) are transformed as follows

$$\frac{\partial \tau_n}{\partial s} - \frac{\partial \tau_t}{\partial n} = -2G \frac{d\theta}{dx}, \quad \frac{\partial^2 \phi}{\partial s^2} + \frac{\partial^2 \phi}{\partial n^2} = -2G \frac{d\theta}{dx}, \quad \frac{\partial^2 \psi}{\partial s^2} + \frac{\partial^2 \psi}{\partial n^2} = -2 \quad (25a-c)$$

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Let us consider continuity regions and discontinuity regions in the cross-section defined as follows: continuity regions that are away from segment ends and angles of the cross-section, are characterized by constant stress function (for constant thickness) along the arc coordinate s while discontinuity regions are segment ends and angles.

2.7.2 Thin-walled open sections

At an arc coordinate s the inner and outer node are denoted by s_i and s_o , respectively. Recalling that the stress function is constant along the unique boundary, the stress component τ_s at the midline at position s of thickness $t(s)$ can be determined using Equations (15b) and (24b) as follows

$$\tau_t(s) = -\frac{\partial \phi}{\partial n} = -\frac{\phi_{s_o} - \phi_{s_i}}{t(s)} = 0 \quad (26)$$

The stress component τ_t at the midline is zero: this finding is in agreement with the analysis of thin-walled open sections.

Equation (24a) shows that the stress component τ_n is zero in continuity regions while τ_t that is zero at midline can be expressed as follows using Equation (25a)

$$\tau_t(s, n) = 2G \frac{d\theta}{dx} \times n \quad (27a)$$

The stress τ_t can be formulated using the torsion constant I_t and the torque M_t in Equation (13c) as follows

$$\tau_t(s, n) = 2 \frac{M_t}{I_t} \times n \quad (27b)$$

Equation (27b) shows that the stress component τ_t is linearly distributed over the thickness, zero at midline and maximal at the boundaries as follows

$$\tau_{t,\max}(s) = \pm 2 \frac{M_t}{I_t} \times \frac{t(s)}{2} = \pm \frac{M_t}{I_t} \times t(s) \quad (27c)$$

Consequently, the maximal stresses occur at positions with greatest thickness; this finding is in agreement with the analysis of thin-walled open sections. The modified stress function $\psi(s, n)$ defined in Equation (13a) which is zero at $n = \pm t(s)/2$ and satisfies Equation (25c) is given by

$$\psi(s, n) = \frac{t(s)^2}{4} \left(1 - \frac{4n^2}{t(s)^2} \right) \quad (28a)$$

Then, the torsion constant I_t calculated using Equation (13d) and (28a) is

$$I_t = \frac{1}{3} \int t(s)^3 ds \quad (28b)$$

Equation (28b) is a well-known formula in the analysis of thin-walled open sections.

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2.7.3 Thin-walled closed sections

The analysis of closed thin-walled sections is conducted using the Bredt's [3] formulas (Equations (23a-b)). These formulas are based on the assumptions of constant shear stress τ_t over the thickness and neglecting the stress component τ_n . First, these assumptions do not satisfy the relationship between the shear stresses (Equation (25a)). Second, Bredt's Equation (23a) implies that the total torque is due to the stress component τ_t , and as shown earlier (Equation (12f)), that is inexact. In the following, another approach for the torsional analysis of closed thin-walled sections is presented.

Single-cell thin-walled section: The stress function is taken zero along the outer boundary and ϕ_{si} along the inner one. The stress component τ_t at the midline at position s of thickness $t(s)$ can be determined using Equations (15b) and (24b) as follows

$$\tau_t(s) = -\frac{\partial \phi}{\partial n} = -\frac{\phi_{so} - \phi_{si}}{t(s)} = \frac{\phi_{si}}{t(s)} \quad (29)$$

It is worth mentioning that ϕ_{si} corresponds to the shear flow of the Bredt's analysis. The stress component τ_t can also be determined using Equations (9a) as follows

$$\tau_t = \frac{\partial \phi}{\partial z} \frac{dy}{ds} - \frac{\partial \phi}{\partial y} \frac{dz}{ds} \rightarrow \tau_t ds = \frac{\partial \phi}{\partial z} dy - \frac{\partial \phi}{\partial y} dz \quad (30)$$

Let us perform a line integral along the midline using Equations (7), (13a), and (29-30) and the Green's theorem

$$\begin{aligned} \oint_M \tau_t ds &= \oint_M \frac{\phi_{si}}{t(s)} ds = \oint_B \left[\frac{\partial \phi}{\partial z} dy - \frac{\partial \phi}{\partial y} dz \right] = \iint_A \left[-\frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \phi}{\partial z^2} \right] dy dz = 2G \frac{d\theta}{dx} A_M \\ \rightarrow \oint_M \frac{\psi_{si}}{t(s)} ds &= 2A_M \rightarrow \psi_{si} = \frac{2A_M}{\oint_M \frac{ds}{t(s)}} \end{aligned} \quad (31)$$

where A_M is the area enclosed by the midline of the cross-section. The modified stress function $\psi(s, n)$ defined in Equation (13a) which is zero and ψ_{si} at $n = \pm t(s)/2$, respectively, and satisfies Equation (25c) is given by

$$\psi(s, n) = \psi_{si} \left(\frac{1}{2} - \frac{n}{t(s)} \right) + \frac{t(s)^2}{4} \left(1 - \frac{4n^2}{t(s)^2} \right) \quad (32a)$$

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The torsion constant I_t is then calculated using Equations (14c), (31), and (32a) as follows

$$I_t = \frac{2AA_M}{\oint_M \frac{ds}{t(s)}} + \frac{1}{3} \oint_M t(s)^3 ds + \frac{4A_M A_i}{\oint_M \frac{ds}{t(s)}} \quad (32b)$$

The second term at the right-hand side of Equation (32b) can be neglected; thus Equation (32b) becomes

$$I_t = \frac{2A_M (A_i + A_o)}{\oint_M \frac{ds}{t(s)}} \quad (32c)$$

where A_i and A_o are the areas enclosed by the inner and the outer boundary of the cross-section, respectively, and A is the area of the cross-section material. This expression of the torsion constant looks similar to that of Bredt (Equation (23b), $A_i + A_o$ replacing $2A_M$) and coincides with it for very small thicknesses.

In continuity regions, the stress component τ_n is zero and τ_t is as follows using Equations (25a) and (29).

$$\tau_t(s, n) = 2G \frac{d\theta}{dx} \times n + \frac{\phi_{si}}{t(s)} \quad (33a)$$

Equation (33a) shows that at a position s the stress component τ_t is linearly distributed over the thickness, maximal and minimal at the boundaries. Using Equations (13c) and (31) τ_t is given by

$$\tau_{t, \max, \min}(s) = \frac{M_t}{I_t} \times \left(\pm t(s) + \frac{1}{t(s)} \times \frac{2A_M}{\oint_M \frac{ds}{t(s)}} \right) \quad (33b)$$

Equation (33b) shows that contrarily to Bredt's formulas the maximum shear stress in the cross-section does not necessarily occur at the position with the smallest thickness. The shear stress at the midline is as follows using Equations (13c), (29), (31), and (32c);

$$\tau_t(s) = \frac{M_t}{(A_i + A_o)t(s)}, \quad (33c)$$

it is noted that its maximal value occurs at the position with the smallest thickness; interestingly the expression looks similar to that of Bredt (Equation (23a), $A_i + A_o$ replacing $2A_M$).

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Multiple-cell thin-walled section: The stress function is taken zero along the outer boundary and ϕ_{sk} along the inner boundary of a cell k. For a given cell at a position s the stress function at the inner and outer boundary are denoted by ϕ_{si} and ϕ_{so} , respectively. So, the stress component τ_t at the midline at position s of thickness t(s) is as follows

$$\tau_t(s) = -\frac{\partial \phi}{\partial n} = -\frac{\phi_{so} - \phi_{si}}{t(s)} = \frac{\phi_{si} - \phi_{so}}{t(s)} \quad (34a)$$

Equation (31) is applied at each cell through the midline using Equations (12e), (13a), and (34a); for the cell k it yields

$$\oint_k \frac{\psi_{si} - \psi_{so}}{t_k(s)} ds = 2A_{Mk} \quad (34b)$$

where A_{Mk} is the area enclosed by the midline of the cell k. Let the cell k of thickness $t_k(s)$ be bordered by q cells of thickness $t_q(s)$; Equation (34b) applied at the cell k is

$$\psi_k \oint_k \frac{ds}{t_k(s)} - \sum_q \psi_q \int \frac{ds}{t_q(s)} = 2A_{Mk} \quad (34c)$$

So, the values ψ_k of the modified stress function at the inner boundaries of the cells can be determined from the system of linear equations (34c). $\psi(s, n)$ defined in Equation (13a) with values ψ_{so} and ψ_{si} at $n = \pm t(s)/2$, respectively, which satisfies Equation (25c) is given by

$$\psi(s, n) = \psi_{si} \left(\frac{1}{2} - \frac{n}{t(s)} \right) + \psi_{so} \left(\frac{1}{2} + \frac{n}{t(s)} \right) + \frac{t(s)^2}{4} \left(1 - \frac{4n^2}{t(s)^2} \right) \quad (35a)$$

The torsion constant I_t is then calculated using Equation (14c) and (35a)

$$I_t = \int (\psi_{si} + \psi_{so}) t(s) ds + \frac{1}{3} \int t(s)^3 ds + 2 \sum_k \psi_k A_{ik} \quad (35b)$$

where A_{ik} is the area enclosed by the inner boundary of the cell k. The first integral in Equation (35b) is rewritten as

$$\int (\psi_{si} + \psi_{so}) t(s) ds = \sum_k \psi_k A_k \quad (35c)$$

where A_k is the area of the cross-section material associated to the cell k. The second term at the right-hand side of Equation (35b) can also be neglected; substituting Equation (35c) into (35b) yields

$$I_t = \sum_k \psi_k (A_{ik} + A_{ok}) \quad (35d)$$

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where A_{ok} is the area enclosed by the outer boundary of the cell k . An example of the analysis of a three-cell thin-walled section is presented in Appendix B.

In continuity regions, the stress component τ_n is zero and τ_t is as follows using Equations (25a) and (34a)

$$\tau_t(s, n) = 2G \frac{d\theta}{dx} \times n + \frac{\phi_{si} - \phi_{so}}{t(s)} = \frac{M_t}{I_t} \times \left(2n + \frac{\psi_{si} - \psi_{so}}{t(s)} \right) \quad (36a)$$

Similarly to the single-cell thin-walled section, the stress component τ_t at a position s is linearly distributed over the thickness, maximal and minimal at the boundaries as follows

$$\tau_{t, \max, \min}(s) = \frac{M_t}{I_t} \times \left(\pm t(s) + \frac{\psi_{si} - \psi_{so}}{t(s)} \right) \quad (36b)$$

For a segment of the beam's cross-section bordered by cells j and k and having the thickness t_{jk} , the shear stresses can be calculated using Equation (36b) either from the cell j or from the cell k ; $\tau_{p,i}$ and $\tau_{p,o}$ being the stresses at the inner and outer boundary of the cell p ($p = j, k$), respectively, it yields

$$\begin{aligned} \tau_{j,i} &= \frac{M_t}{I_t} \times \left(-t_{jk} + \frac{\psi_j - \psi_k}{t_{jk}} \right), & \tau_{j,o} &= \frac{M_t}{I_t} \times \left(t_{jk} + \frac{\psi_j - \psi_k}{t_{jk}} \right) \\ \tau_{k,i} &= \frac{M_t}{I_t} \times \left(-t_{jk} + \frac{\psi_k - \psi_j}{t_{jk}} \right), & \tau_{k,o} &= \frac{M_t}{I_t} \times \left(t_{jk} + \frac{\psi_k - \psi_j}{t_{jk}} \right) \end{aligned} \quad (36c)$$

It is noted as expected that $\tau_{j,i} = -\tau_{k,o}$ and $\tau_{j,o} = -\tau_{k,i}$.

2.8 Warping of the cross-section

2.8.1 Warping of beams with solid sections

In the following the cross-sections are assumed free to warp. For a given cross-section and torque the values of the stress function $\psi(y, z)$ were determined according to previous sections. Substituting Equations (3a-b) into (5) yields

$$\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad (37a)$$

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Let the outer normal to the boundary make an angle α with the +y-axis (see Figure 2). The boundary conditions are such that the shearing stress component at the boundary directed along the outer normal is zero from the condition of stress free outer surfaces; this shear stress is expressed using Equations (9b) and (3a-b)

$$\tau_n = G \left(\frac{\partial u}{\partial y} - (z - z_T) \frac{d\theta}{dx} \right) \cos \alpha + G \left(\frac{\partial u}{\partial z} + (y - y_T) \frac{d\theta}{dx} \right) \sin \alpha = 0 \quad (37b)$$

Therefore, two equations are set at cross-section's angles of the outer boundary and consequently two additional nodes should be considered at these positions, as shown in Figure 12.

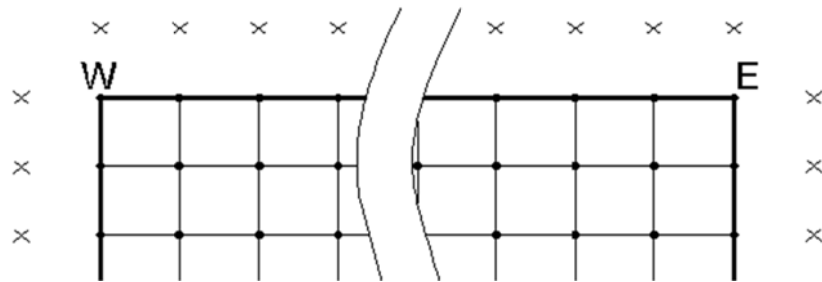


Figure 12 Node distribution for warping calculation with regular nodes (•) and additional nodes (×)

Equation (37b) can be rewritten using (13c)

$$G \left(\frac{\partial u}{\partial y} \cos \alpha + \frac{\partial u}{\partial z} \sin \alpha \right) + \frac{M_t}{I_t} \left[(y - y_T) \sin \alpha - (z - z_T) \cos \alpha \right] = 0 \quad (37c)$$

Equation (37a) is the governing equation which is applied at any node of the cross-section and Equation (37c) is the boundary condition. For convenience the node distribution is taken the same as that for the stress function. So, the governing equation (Equation (37a)) is expressed similar to that of the stress function (Equations (16) and (19)); however, zero is put in place of the right-hand side of the latter equations. A modified warping U is defined as $U = GI_t/M_t \times u \rightarrow u = M_t/GI_t \times U = d\theta/dx \times U$; therefore, Equations (37a) and (37c) become

$$\begin{aligned} \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} &= 0 \\ \frac{\partial U}{\partial y} \cos \alpha + \frac{\partial U}{\partial z} \sin \alpha + (y - y_T) \sin \alpha - (z - z_T) \cos \alpha &= 0 \end{aligned} \quad (38a-b)$$

In Equation (38b) the derivatives of U are expressed using the stencils in Equations (17) and (20a-b). The boundary conditions are expressed as follows for vertical edges (outer normal to the boundary making an angle $\alpha = 0$ or π with the +y-axis) and horizontal edges (outer normal making an angle $\alpha = \pi/2$ or $3\pi/2$ with the +y-axis), respectively

$$U_e - U_w - 2h(z - z_T) = 0, \quad U_n - U_s + 2\lambda h(y - y_T) = 0 \quad (38c-d)$$

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At cross-section's right angles of the outer boundary Equations (38c-d) are both applied.

A kinematic boundary condition is needed for solving the warping problem; therefore, at the center of torsion T (y_T, z_T) (equivalent to the shear center) or at another point suitably chosen (i.e. a point in a symmetry axis) the warping displacement is set to zero and the governing equation is not applied.

Beam with circular cross-section: the center of torsion is at $y_T = z_T = 0$, and the boundary is defined by $y = R\cos\alpha$, and $z = R\sin\alpha$. It results that both governing equation and boundary condition ($\partial U/\partial y \times \cos\alpha + \partial U/\partial z \times \sin\alpha = 0$) are homogeneous differential equations and consequently the warping is zero.

2.8.2 Warping of beams with multiply connected cross-sections

Given a beam with multiply connected cross-sections where the hole is aligned with the Cartesian coordinate system, as represented in Figure 8. The boundary conditions at the outer boundary are defined in terms of stresses, and Equations (38b) and (38c-d) are applied. However at the inner boundary's angle nodes 34 and 39a of Figure 8 for example, the stress component τ_n is not zero; therefore, the governing equation (38a) and not the boundary conditions are applied at those angles. Attention should always be paid at the inner boundary to have as many equations (governing equations and boundary conditions) as nodes (regular nodes and additional nodes).

In the case the hole has an arbitrary shape the boundary condition (38b) can be applied. Generally, governing equations and boundary conditions at the inner boundary are applied such that the number of equations corresponds to the sum of regular nodes and additional nodes. At a given inner boundary node where two directions intersect, as shown in Figure 13, the angle α of Equation (38b) can be taken as the mean value of both directions' angles.

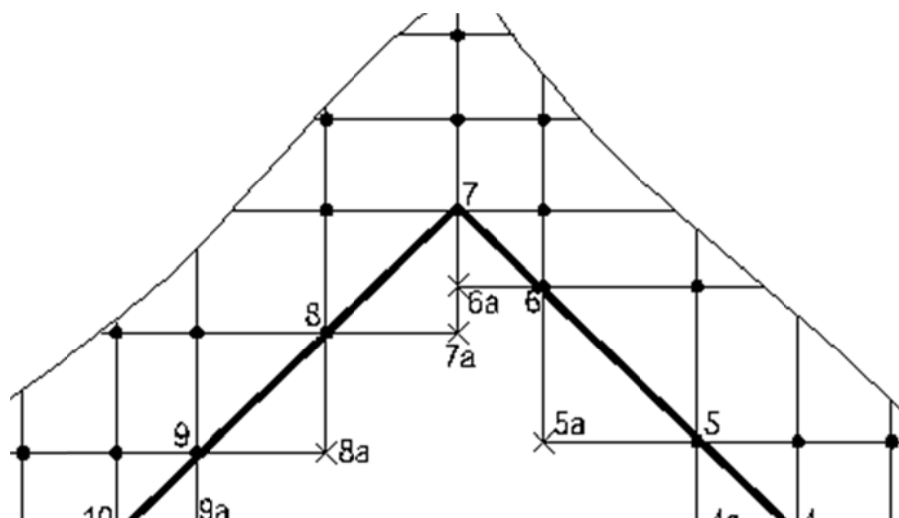


Figure 13 Inner boundary with intersection of two directions

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3 Results and discussions

3.1 Torsion of beam with rectangular cross-section

The elastic torsional behavior of a beam with rectangular cross-section was analyzed. The dimensions of the cross-section in y- and z-directions are denoted by a and b, respectively. The torsion constant $I_t = k_1 b a^3$ and the maximal shear stress $\tau_{max} = k a G d\theta/dx$ were determined, depending on the ratio b/a. A 4 × 4, 8 × 8, and 12 × 12 element mesh were considered. Details of the analysis and results are presented in Appendix A and in the supplementary material “Torsion of beam with rectangular cross-section.” Table 1 lists the exact results obtained with Timoshenko [16] using the membrane analogy and those obtained in the present study (FDM). In this study, the calculation is made considering additional nodes on the one hand and not considering them on the other hand.

Table 1 Coefficients of the torsion constant and maximal shear stress

	b/a =									
	1.00		1.50		2.00		3.00		10.00	
	k ₁	k	k ₁	k	k ₁	k	k ₁	k	k ₁	k
Solution by Timoshenko [16] (exact results)										
	0.141	0.675	0.196	0.848	0.229	0.930	0.263	0.985	0.312	1.000
Present study (Finite Difference Method, with additional nodes)										
4 × 4 elements	0.133	0.688	0.185	0.838	0.215	0.913	0.243	0.971	0.270	1.000
8 × 8 elements	0.140	0.677	0.194	0.844	0.227	0.925	0.260	0.982	0.298	1.000
12 × 12 elements	0.140	0.676	0.195	0.846	0.228	0.928	0.262	0.984	0.305	1.000
Present study (Finite Difference Method, without additional nodes)										
4 × 4 elements	0.133	0.594	0.185	0.790	0.215	0.888	0.243	0.963	0.270	0.999
8 × 8 elements	0.140	0.652	0.194	0.832	0.227	0.919	0.260	0.980	0.298	1.000
12 × 12 elements	0.140	0.665	0.195	0.841	0.228	0.925	0.262	0.983	0.305	1.000

As Table 1 shows, the results using FDM show good agreement with the exact results, and the accuracy is increased through a grid refinement. For high values of b/a, the torsion constant I_t is equal to $1/3 \times b a^3$. In addition, the results of torsion constant are identical regardless of the consideration of additional nodes; it is noted that the governing equations applied at the boundary permit only to determine the values of stress function at additional nodes and these values have no influence on the torsion constant. However, the consideration of additional nodes delivers better results of shear stresses at the boundary than that not considering them but both converge towards the exact results. Without additional nodes the shear stresses at boundary are calculated using following one-sided finite difference approximations

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$$\left.\frac{d\psi}{dx}\right|_i = \frac{-3\psi_i + 4\psi_{i+1} - \psi_{i+2}}{2h}, \quad \left.\frac{d\psi}{dx}\right|_i = \frac{\psi_{i-2} - 4\psi_{i-1} + 3\psi_i}{2h} \tag{39}$$

The warping displacement $u = d\theta/dx \times U$ is calculated for a beam with dimensions $a = b = 1.0\text{m}$, whereby a 12×12 mesh with spacings $\Delta y = \Delta z = h$ is considered. Recalling that $y_T = z_T = 0$ for the rectangular cross-section, the boundary conditions are applied using Equations (38c-d). As kinematic boundary condition the warping displacement is zero at the center of twist where the governing equation is not applied. Detailed results are presented in the above mentioned supplementary material. The coefficient U (for clarity $100 \times U$) of the warping is given in Table 2a.

Table 2a $100 \times$ Coefficient U of the warping displacement for a beam with dimensions $a = b = 1.0\text{m}$

Z =	y =												
	-a/2	-5a/12	-a/3	-a/4	-a/6	-a/12	0.00	a/12	a/6	a/4	a/3	5a/12	a/2
b/2	0.000	2.627	3.566	3.445	2.640	1.419	0.000	-1.419	-2.640	-3.445	-3.566	-2.627	0.000
5b/12	-2.627	0.000	1.317	1.703	1.459	0.823	0.000	-0.823	-1.459	-1.703	-1.317	0.000	2.627
b/3	-3.566	-1.317	0.000	0.593	0.669	0.414	0.000	-0.414	-0.669	-0.593	0.000	1.317	3.566
b/4	-3.445	-1.703	-0.593	0.000	0.209	0.166	0.000	-0.166	-0.209	0.000	0.593	1.703	3.445
b/6	-2.640	-1.459	-0.669	-0.209	0.000	0.042	0.000	-0.042	0.000	0.209	0.669	1.459	2.640
b/12	-1.419	-0.823	-0.414	-0.166	-0.042	0.000	0.000	0.000	0.042	0.166	0.414	0.823	1.419
0.00	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
-b/12	1.419	0.823	0.414	0.166	0.042	0.000	0.000	0.000	-0.042	-0.166	-0.414	-0.823	-1.419
-b/6	2.640	1.459	0.669	0.209	0.000	-0.042	0.000	0.042	0.000	-0.209	-0.669	-1.459	-2.640
-b/4	3.445	1.703	0.593	0.000	-0.209	-0.166	0.000	0.166	0.209	0.000	-0.593	-1.703	-3.445
-b/3	3.566	1.317	0.000	-0.593	-0.669	-0.414	0.000	0.414	0.669	0.593	0.000	-1.317	-3.566
-5b/12	2.627	0.000	-1.317	-1.703	-1.459	-0.823	0.000	0.823	1.459	1.703	1.317	0.000	-2.627
-b/2	0.000	-2.627	-3.566	-3.445	-2.640	-1.419	0.000	1.419	2.640	3.445	3.566	2.627	0.000

The warping displacements are zero in the symmetry axes and in the diagonals, in agreement with the exact results presented in Sadd [14] whereby a displacement pattern with eight zones of symmetry is noted. Some values obtained using FDM are satisfactorily compared with the exact results in Table 2b below.

Table 2b $100 \times$ Coefficient U of the warping displacement: Exact results Sadd [14] and results using FDM

		y =						
		-a/2	-5a/12	-a/3	-a/4	-a/6	-a/12	0.00
Exact	Z = b/2	0.000	2.678	3.604	3.474	2.659	1.429	0.00
FDM		0.000	2.627	3.566	3.445	2.640	1.419	0.000
Exact	Z = b/3	-3.604	-1.336	0.000	0.602	0.678	0.420	0.000
FDM		-3.566	-1.317	0.000	0.593	0.669	0.414	0.000

Cross-sectional Analysis of Beams Subjected to Saint-Venant Torsion

3.2 Torsion of a rectangular beam with multiply connected cross-sections

The torsional analysis of a rectangular beam with multiply connected cross-section, as represented in Figure 14, was conducted. The dimensions of the cross-section in y- and z-directions are denoted by a and b, respectively. The hole was centered to the cross-section with dimensions a/2 and b/2 in y- and z-directions, respectively.

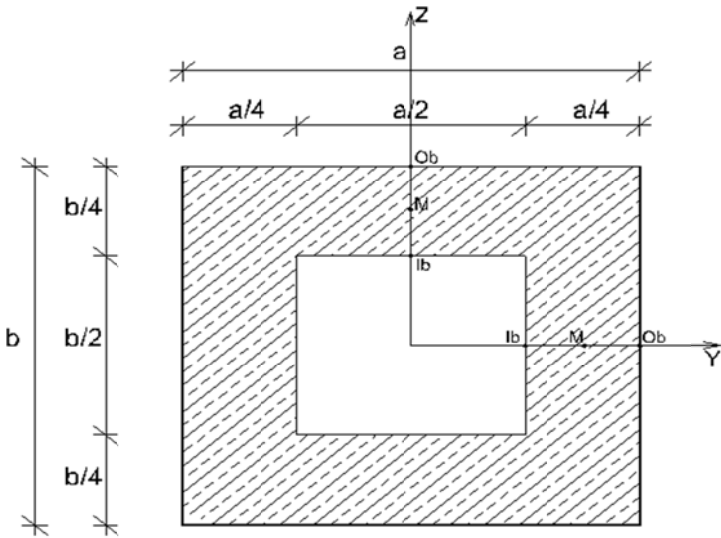


Figure 14 Beam with multiply connected cross-section

The shear stresses were calculated at the middle of the sides depending on the ratio b/a and the FDM discretization, whereby $a = 1.0\text{m}$; the torsion constant and the value of the stress function on the inner boundary were also determined. Details of the results are presented in the supplementary material “Torsion of a beam with rectangular multiply connected cross-section.” Table 3 lists the results obtained in this study for a torque $M_t = 1.0 \text{ kNm}$.

Table 3 Shear stresses, torsion constant, and stress function in the inner boundary

		b/a = 1.00			b/a = 2.00			b/a = 3.00		
		FDM 8 × 8	FDM 12×12	FDM 16×16	FDM 8 × 8	FDM 12×12	FDM 16×16	FDM 8 × 8	FDM 12×12	FDM 16×16
τ_{xy}	outer boundary	-4.904	-4.925	-4.955	-1.674	-1.658	-1.661	-1.023	-0.986	-0.978
	Midline	-3.289	-3.273	-3.275	-0.809	-0.783	-0.787	-0.350	-0.314	-0.319
	inner boundary	-1.673	-1.633	-1.604	0.057	0.029	0.026	0.323	0.261	0.238
τ_{xz}	inner boundary	1.673	1.633	1.604	1.092	1.069	1.056	0.742	0.722	0.712
	Midline	3.289	3.273	3.275	1.617	1.608	1.604	1.051	1.039	1.034
	Outer boundary	4.904	4.925	4.955	2.143	2.146	2.152	1.361	1.356	1.356
Torsion constant		0.1480	0.1445	0.1418	0.4745	0.4630	0.4553	0.8073	0.7886	0.7769
Stress function in the inner boundary		0.1217	0.1183	0.1161	0.1919	0.1861	0.1826	0.2122	0.2049	0.2008

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These results are consistent with a proper torsional behavior of the beam. However, some inaccuracies regarding the torsion constant are noted (values greater than that of the solid section for $b/a = 1.0$ i.e); they are due to the integration of the stress function over the cross section.

The warping displacement $u = d\theta/dx \times U$ was calculated for a beam with dimensions $a = b = 1.0\text{m}$, whereby a 12×12 mesh with spacings $\Delta y = \Delta z = h$ was considered. Recalling that $y_T = z_T = 0$ for the rectangular cross-section, the boundary conditions were applied using Equations (38c-d). As kinematic boundary condition the warping displacement was set to zero at a point in the symmetry axis y . Details of the results are presented in the above mentioned supplementary material. The coefficient U of the warping (for clarity $100 \times U$ is displayed) is given in Table 4.

Table 4 $100 \times$ Coefficient U of the warping for a beam with dimensions $a = b = 1.0\text{m}$

	y =												
	-a/2	-5a/12	-a/3	-a/4	-a/6	-a/12	0.00	a/12	a/6	a/4	a/3	5a/12	a/2
b/2	0,000	2,615	3,514	3,335	2,487	1,307	0,000	-1,307	-2,487	-3,335	-3,514	-2,615	0,000
5b/12	-2,615	0,000	1,275	1,587	1,263	0,676	0,000	-0,676	-1,263	-1,587	-1,275	0,000	2,615
b/3	-3,514	-1,275	0,000	0,473	0,304	0,134	0,000	-0,134	-0,304	-0,473	0,000	1,275	3,514
b/4	-3,335	-1,587	-0,473	0,000	-0,653	-0,443	0,000	0,443	0,653	0,000	0,473	1,587	3,335
b/6	-2,487	-1,263	-0,304	0,653						-0,653	0,304	1,263	2,487
b/12	-1,307	-0,676	-0,134	0,443						-0,443	0,134	0,676	1,307
0.00	0,000	0,000	0,000	0,000						0,000	0,000	0,000	0,000
-b/12	1,307	0,676	0,134	-0,443						0,443	-0,134	-0,676	-1,307
-b/6	2,487	1,263	0,304	-0,653						0,653	-0,304	-1,263	-2,487
-b/4	3,335	1,587	0,473	0,000	0,653	0,443	0,000	-0,443	-0,653	0,000	-0,473	-1,587	-3,335
-b/3	3,514	1,275	0,000	-0,473	-0,304	-0,134	0,000	0,134	0,304	0,473	0,000	-1,275	-3,514
-5b/12	2,615	0,000	-1,275	-1,587	-1,263	-0,676	0,000	0,676	1,263	1,587	1,275	0,000	-2,615
-b/2	0,000	-2,615	-3,514	-3,335	-2,487	-1,307	0,000	1,307	2,487	3,335	3,514	2,615	0,000

For the present square section case, the warping displacements are zero as expected in the symmetry axes and in the diagonals. Similarly to the solid square section, a displacement pattern with eight zones of symmetry is noted.

Cross-sectional Analysis of Beams Subjected to Saint-Venant Torsion

3.3 Torsion of beams with thin-walled sections

The elastic torsional behavior of beams with thin-walled sections, as represented in Figure 15, was analyzed. The dimensions of the cross-section in y- and z-directions are denoted by $a = 1.0\text{m}$ and $b = 1.0\text{m}$, respectively.

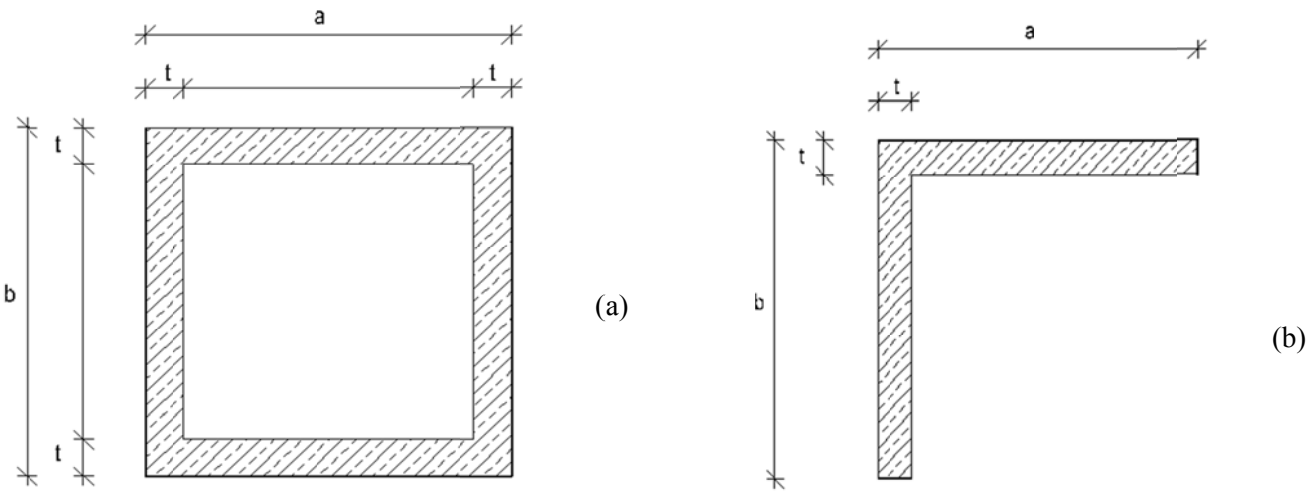


Figure 15 Beams with (a) thin-walled closed section and (b) thin-walled open section

Thin-walled closed section: The shear stresses were determined for a torque $M_t = 1.0 \text{ kNm}$ for values of thicknesses $a/6$, $a/8$, $a/10$, and $a/12$. Detailed results were presented in the supplementary material “Torsion of beam with thin-walled sections.” The results obtained according to the closed-form solutions (CFS, Equations (33b-c)) and FDM (for $t = a/8$, $a/12$) were compared to those using Bredt’s formulas (Equations (23a-b)) and were listed in Table 5. The localized shear stresses at reentrant corners were also determined using FDM

Table 5 Shear stresses τ_t , stresses at reentrant corner, torsion constant I_t , stress function on inner boundary

		a/6		a/8		a/10		a/12	
Stress τ_t (Bredt)		4.32		5.22		6.17		7.14	
I_t (Bredt)		0.0965		0.0837		0.0729		0.0642	
		CFS	CFS	FDM 16×16	FDM 32×32	CFS	CFS	FDM 24×24	FDM 36×36
Shear stresses τ_t	Outer boundary	5.82	6.58	6.42	6.45	7.45	8.37	8.26	8.27
	Midline	4.15	5.12	5.14	5.13	6.10	7.08	7.09	7.09
	Inner boundary	2.49	3.66	3.86	3.82	4.74	5.79	5.91	5.91
Stresses at the reentrant corners		τ_{xy}		2.65				3.80	
		τ_{xz}		2.65				3.80	
I_t (Present study)		0.1003	0.0854	0.0975	0.0952	0.0738	0.0647	0.0708	0.0704
Stress function in the inner boundary			0.0547	0.0626	0.0611		0.0382	0.0418	0.0416

Cross-sectional Analysis of Beams Subjected to Saint-Venant Torsion

The shear stresses in this study vary linearly over the thickness. As Table 5 shows, this paper presents values of shear stresses at midline close to those using Bredt's analysis but the maximal values are higher. Furthermore, the results obtained with the derived closed-form solutions are in good agreement with those using FDM. However, FDM analysis is likely to deliver more accurate results since it can model discontinuity regions such as angles and corners.

Thin-walled open section: The beam with a thin-walled open section and thickness $t = a/8$, as represented in Figure 15 (b), was analyzed and the shear stresses were determined for a torque $M_t = 1.0$ kNm. The results obtained using FDM (mesh 16×16 and 32×32) were compared to those using the commonly used formulas (CUF, Equations (23c-d)) and were listed in Table 6, whereby detailed results were presented in the above mentioned supplementary file. The localized shear stresses at reentrant corners were also determined using FDM.

Table 6 Shear stresses and torsion constant for the thin-walled open section with $t = a/8$

		CUF	FDM 16 × 16	FDM 32 × 32
Shear stresses τ_t	Outer boundary	-102.40	-107.94	-106.21
	Midline	0.00	0.00	0.00
	Inner boundary	102.40	107.94	106.21
Stresses at the reentrant corners		τ_{xy}	26.98	46.68
		τ_{xz}	26.98	46.68
1000 × Torsion Constant I_t		1.207	1.1581	1.1770

As Table 6 shows, this paper presents values of shear stresses and torsion constant in good agreement with those of the common analysis. It is recalled that FDM is likely to yield more accurate results since it can model discontinuity regions such as angles and corners. As the grid got refined, a high increase of shear stresses at reentrant corners was noted.

Cross-sectional Analysis of Beams Subjected to Saint-Venant Torsion

3.4 Torsion of a symmetrical beam with double-cell thin-walled section

The torsional behavior of a beam with symmetrical double-cell thin-walled section, as represented in Figure 16, was analyzed. The thicknesses are t_k , $k = 1$ to 4.

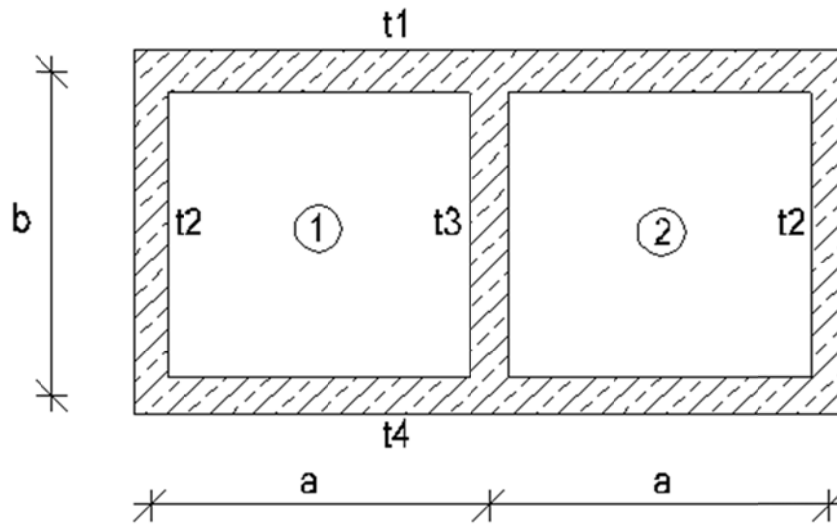


Figure 16 Beam with double-cell thin-walled section

Let ψ_1 and ψ_2 be the values of the modified stress function at the inner boundaries of the cells 1 and 2, respectively; they are calculated using Equation (34c) and the torsion constant is determined using Equation (35d)

$$\psi_1 = \psi_2 = \frac{2A_{M1}}{\frac{a}{t_1} + \frac{b}{t_2} + \frac{a}{t_4}}, \quad I_t = 2\psi_1 (A_{i1} + A_{o1}) \quad (39)$$

$$A_{M1} = A_{M2} = ab, \quad A_{i1} = \left(b - \frac{t_1}{2} - \frac{t_4}{2}\right) \left(a - \frac{t_2}{2} - \frac{t_3}{2}\right), \quad A_{o1} = \left(b + \frac{t_1}{2} + \frac{t_4}{2}\right) \left(a + \frac{t_2}{2} + \frac{t_3}{2}\right)$$

The results show that contrarily to Bredt's analysis, the central web (thickness t_3) contributes to the torsional stiffness and carries shear stresses; the stress distribution is linear and odd with respect to the midline with maximal and minimal value

$$\tau_{t,\max,\min} = \pm \frac{M_t}{I_t} t_3 \quad (40a)$$

The maximal and minimal values in other segments are obtained using Equation (36b), with $k = 1, 2$, and 4

$$\tau_{t,\max,\min} = \frac{M_t}{I_t} \times \left(\pm t_k + \frac{\psi_1}{t_k} \right) \quad (40b)$$

Cross-sectional Analysis of Beams Subjected to Saint-Venant Torsion

4 Conclusion

The FDM-based model developed in this paper enables, with certain easiness, the elastic analysis of beams subjected to Saint-Venant torsion. Beams with solid sections and multiply connected cross-sections were analyzed. In addition, beams with thin-walled closed sections were also analyzed and closed form solutions were derived; these solutions which consider a linear distribution of the shear stresses over the thickness can be regarded as an improvement of the Bredt's analysis. The results showed that the calculations, as described in this paper, yield accurate results. The FDM analysis presented in this study is likely to yield more accurate results than analytical methods since it can model discontinuity regions such as angles and corners. Moreover, the FDM analysis utilizes strong formulation of equations while the finite element method (FEM) is based on weak formulation of equations; therefore, the results using FDM are likely to be more accurate than those using FEM.

Supplementary Materials: The following files were uploaded during submission:

- “Torsion of beam with rectangular cross-section”
- “Torsion of a beam with rectangular multiply connected cross-section.”
- “Torsion of beam with thin-walled sections.”

Funding:

Acknowledgments:

Conflicts of Interest: The author declares no conflict of interest.

Appendix A Beam with rectangular cross-section

Given a beam with rectangular cross-section and known values of ψ . The integral $\int \psi dy dz$ needed for the torsion constant I_t is determined here. The grid spacings in y- and z-direction are denoted by Δy and Δz , respectively. The cross-section is divided in rectangles around grid points, as represented in Figure 17.

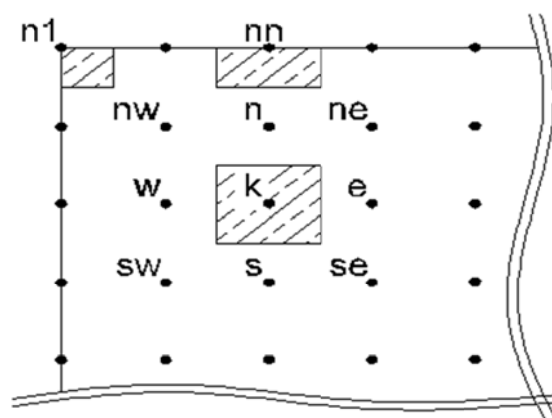


Figure 17 Rectangles for the calculation of the torsion constant

Cross-sectional Analysis of Beams Subjected to Saint-Venant Torsion

Given the rectangle bounded by nodes k, e, s, and se in Figure 17. The stress function $\psi(y, z)$ is approximated as follows

$$\psi(y, z) = \psi_k f_k(y, z) + \psi_e f_e(y, z) + \psi_s f_s(y, z) + \psi_{se} f_{se}(y, z) \quad (A1)$$

The shape function $f_e(y, z)$ i.e. can be expressed as follows

$$f_e(y, z) = \frac{yz}{\Delta y \times \Delta z} \quad (A2)$$

Using Equations (A1) and (A2), the integral for the rectangle of interest is

$$\iint \psi dy dz = \frac{1}{4} (\psi_k + \psi_e + \psi_s + \psi_{se}) \times \Delta y \times \Delta z \quad (A3)$$

Using Equation (A3) for a uniform grid, $\int \psi dy dz$ on a rectangle around an interior node (node k) is $\psi_k \Delta y \Delta z$.

Similarly, using Equation (A3) and recalling that the values of ψ are zero along the boundary, the integral for a rectangle around a boundary node (node nn) is $1/8 \times \psi_n \Delta y \Delta z$, and for a rectangle around an angle node (node n1) $1/64 \times \psi_{nw} \Delta y \Delta z$.

Appendix B Three-cell thin-walled section

Given the three-cell thin-walled section represented in Figure 18, with thicknesses t_k , $k = 1$ to 8. Let ψ_1 , ψ_2 , and ψ_3 be the values of the modified stress function at the inner boundaries of the cells 1, 2, and 3, respectively.

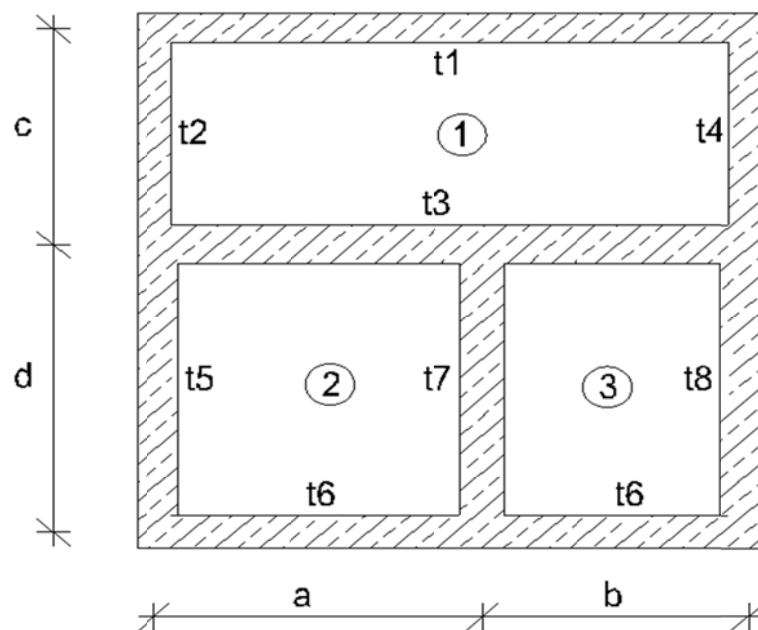


Figure 18 Three-cell thin-walled section

Cross-sectional Analysis of Beams Subjected to Saint-Venant Torsion

The areas A_{Mk} , A_{ik} , and A_{ok} with $k = 1, 2$, and 3 are as follows

$$\begin{aligned} A_{M1} &= (a+b)c, A_{I1} = \left(a+b - \frac{t_2}{2} - \frac{t_4}{2}\right) \left(c - \frac{t_1}{2} - \frac{t_3}{2}\right), A_{O1} = \left(a+b + \frac{t_2}{2} + \frac{t_4}{2}\right) \left(c + \frac{t_1}{2} + \frac{t_3}{2}\right) \\ A_{M2} &= ad, A_{I2} = \left(a - \frac{t_5}{2} - \frac{t_7}{2}\right) \left(d - \frac{t_3}{2} - \frac{t_6}{2}\right), A_{O2} = \left(a + \frac{t_5}{2} + \frac{t_7}{2}\right) \left(d + \frac{t_3}{2} + \frac{t_6}{2}\right) \\ A_{M3} &= bd, A_{I3} = \left(b - \frac{t_7}{2} - \frac{t_8}{2}\right) \left(d - \frac{t_3}{2} - \frac{t_6}{2}\right), A_{O3} = \left(b + \frac{t_7}{2} + \frac{t_8}{2}\right) \left(d + \frac{t_3}{2} + \frac{t_6}{2}\right) \end{aligned} \quad (B1)$$

Equation (34c) applied at each cell as follows yields the values ψ_k .

$$\begin{aligned} \psi_1 \left[\frac{a+b}{t_1} + \frac{c}{t_2} + \frac{a+b}{t_3} + \frac{c}{t_4} \right] - \psi_2 \frac{a}{t_3} - \psi_3 \frac{b}{t_3} &= 2A_{M1} \\ \psi_2 \left[\frac{a}{t_3} + \frac{d}{t_5} + \frac{a}{t_6} + \frac{d}{t_7} \right] - \psi_1 \frac{a}{t_3} - \psi_3 \frac{d}{t_7} &= 2A_{M2} \\ \psi_3 \left[\frac{b}{t_3} + \frac{d}{t_7} + \frac{b}{t_6} + \frac{d}{t_8} \right] - \psi_1 \frac{b}{t_3} - \psi_2 \frac{d}{t_7} &= 2A_{M3} \end{aligned} \quad (B1)$$

Then the torsion constant and the shear stresses are calculated using Equations (35d) and (36c), respectively.

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