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Entropy and Its Application to Number Theory

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Abstract: In this paper, we propose an extension of the Planck distribution function derived from the Boltzmann principle. That is, we consider extending Planck's law with new distribution functions. In addition, using ideas applied to the expansion of the Planck distribution function, Von Koch's inequality is derived without using the Riemann hypothesis, showing that the abc conjecture is negated. We also describe some challenges for the future. Namely, we will discuss that Entropy is related to dynamical systems described by logistic function models, such as the bacterial and the population growth.

Keywords: entropy; Boltzmann's principle; Planck's law; Dynamical system; Von Koch's inequality; Riemann Hypothesis; abc conjecture

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1. Introduction.

In this paper, we will explain in the following order.

1.1.

First, we explain the principle of Boltzmann's Entropy S and the Planck distribution function to aid understanding. The Planck distribution function partitions particles P into resonators N and applies this partitioning method to Entropy S. Furthermore, this Entropy S is made to correspond to the average energy of resonators U and an energy element ε. In addition, The Planck distribution function is derived by differentiating with the average energy of resonators U.

1.2.

Second, we describe that the expansion of the Planck distribution function which is main contents of this article. We consider Entropy $S_{\pi_f}(x)$ which is the Boltzmann principle divided by function $x/f(x)$, where set the function $f(x)$ to $\log(x)$ and let x be a positive real number. The function $x/\log(x)$ is an approximation of the number of prime numbers $\pi(x)$. The function $R^{\pm}_{\alpha}(x)$ is defined and describe the relation between $S_{\pi_f}(x)$ and $R^{\pm}_{\alpha}(x)$. Furthermore, we attempt to compare the possibility of expanding the Planck distribution function by using the function $R^{\pm}_{\alpha}(x)$. Besides, the relation between the constant α of the function $R^{\pm}_{\alpha}(x)$ and fine-structure constant will be considered.

1.3.

Third, we consider applying the constant α of the function $R^{\pm}_{\alpha}(x)$ to number theory. Von Koch's inequality is derived without using the Riemann hypothesis. Namely, it proves the Riemann Hypothesis is correct. Additionally, we verify that the negation of the abc conjecture is true.

1.4.

Finally, we will describe some considerations and issues for the future. We generalize Entropy $S_{\pi_f}(x)$ to Entropy $S_D(x)$, where $D(x)$ is function on x. Using Entropy $S_D(x)$, we will discuss that Entropy is related to dynamical systems described by logistic function models, such as bacterial and population growth.

2. The Boltzmann principle and the Planck distribution function.

2.1. Introduction for Entropy S and the Planck distribution function.

To make it easier the understanding, we would first let us introduce the Boltzmann principle and the Planck distribution function as follows.

Definition 2.1. *We define symbols using on this article as follows :*

$$\begin{aligned} P &: The\ number\ of\ particles, \\ N &: The\ number\ of\ resonators, \\ U &: The\ average\ energy\ per\ a\ resonator, \\ U_N &: Total\ energy, \\ \varepsilon &: An\ element\ of\ energy, \\ \nu &: Frequency, \\ T &: Temperature, \\ k_B &: The\ Boltzmann\ constant, \\ h &: The\ Planck\ constant, \\ \beta &: Inverse\ temperature. \end{aligned} \tag{2.1}$$

□

Using the definitions above, the following equations are satisfied :

$$U_N = NU = P\varepsilon, \tag{2.2}$$

$$\frac{P}{N} = \frac{U}{\varepsilon}, \tag{2.3}$$

$$\beta = \frac{1}{k_B T}, \tag{2.4}$$

where the inequality $P > N$ is satisfied.

The concept of the Planck distribution is that the number of particles P is partitioned by the number of resonators N. Namely, the number of particles P is partitioned by the number of partitions $N - 1$. The number of particles P and resonators N can be regarded positive integer numbers. Therefore, we define the number of states $W_{N,P}$ and Entropy (the Boltzmann Principle) S as follows :

Definition 2.2. *Let the number of particles P and the number of resonators N be positive integers.*

$$W_{N,P} := \frac{(N+P-1)!}{(N-1)!P!}, \quad (the\ number\ of\ states,\ combination), \tag{2.5}$$

$$S_{N,P} := k_B \log W_{N,P}, \quad (the\ Boltzmann\ Principle), \tag{2.6}$$

$$S := \frac{S_{N,P}}{N}, \quad (the\ average\ of\ S_{N,P}). \tag{2.7}$$

□

Using the Stirling's formula, for sufficiently large natural numbers P and N, the following conditions are satisfied :

$$W_{N,P} = \frac{(N+P-1)!}{(N-1)!P!} \approx \frac{(N+P)^{N+P}}{N^N P^P}. \tag{2.8}$$

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Using the Boltzmann principle above, for sufficiently large the number of particles P and resonators N, we can obtain the following equations :

$$\begin{aligned} S_{N,P} &= k_B \log W_{N,P} \\ &= k_B\{(N+P)\log(N+P) - \log N^N - \log P^P\} \\ &= k_B\{(N+P)\log(N+P) - N\log N - P\log P\} \\ &= k_B N\{(\frac{P}{N})\log N + (1+\frac{P}{N})\log(1+\frac{P}{N}) - \frac{P}{N}\log P\} \\ &= k_B N\{(1+\frac{P}{N})\log(1+\frac{P}{N}) - \frac{P}{N}\log\frac{P}{N}\}. \end{aligned} \tag{2.9}$$

Using the definition above, the equality(2.3) $P/N = U/\varepsilon$ and (2.7) $S = S_{N,P}/N$, the equality(2.9) is satisfied as follows :

$$S = k_B\Big\{(1+\frac{U}{\varepsilon})\log(1+\frac{U}{\varepsilon}) - \frac{U}{\varepsilon}\log\frac{U}{\varepsilon}\Big\}. \tag{2.10}$$

Differentiate both sides of the equation(2.10) above with respect to average energy per resonator U. Hence, the following equation is satisfied :

$$\frac{dS}{dU} = \frac{k_B}{\varepsilon}\Big\{\log(1+\frac{U}{\varepsilon}) - \log\frac{U}{\varepsilon}\Big\}. \tag{2.11}$$

Furthermore, differentiate both sides of the equation(2.11) with respect to average energy per resonator U, the following an equation is satisfied :

$$\frac{d^2S}{dU^2} = \frac{-k_B}{U(\varepsilon+U)}. \tag{2.12}$$

The rate of change of Entropy dS is the multiplication of the rate of change of Energy U and the reciprocal of temperature T. Namely, the following relation between Entropy S, average energy per resonator U and Temperature T are satisfied :

$$\frac{dS}{dU} = \frac{1}{T}. \tag{2.13}$$

Thus, using the equation(2.12) and (2.13), the following relation is satisfied :

$$\frac{d}{dU}\left(\frac{1}{T}\right) = \frac{-k_B}{U(\varepsilon+U)}. \tag{2.14}$$

Integrating both sides of the equation(2.14) with respect to average energy per resonator U, the following relation is satisfied :

$$U = \frac{\varepsilon}{\exp(\frac{\varepsilon}{k_B T}) - 1}. \tag{2.15}$$

Here, put ε as follows :

$$\varepsilon = h\nu. \tag{2.16}$$

Therefore, the following equations are satisfied :

$$U = \frac{h\nu}{\exp(\frac{h\nu}{k_B T}) - 1} = \frac{h\nu}{\exp(h\nu\beta) - 1}, \quad (Planck's\ law). \tag{2.17}$$

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The equation above(2.17) is determined the expression for the average energy of particles in a single mode of frequency ν in thermal equilibrium T, that is, called Planck's law. Besides, we define the distribution function $\bar{n}(\nu,\beta)$ as follows:

$$\bar{n}(\nu,\beta) = \frac{1}{\exp(h\nu\beta)-1}, \quad (the\ Planck\ distribution\ function). \tag{2.18}$$

This is expressed the mean particle occupation number in thermal equilibrium T . This is called the Planck distribution function on this paper. Moreover, the equation(2.18) is transformed as follows :

$$\frac{\bar{n}(\nu,\beta)}{\bar{n}(\nu,\beta)+1} = \exp(-h\nu\beta), \quad (the\ Boltzmann\ factor). \tag{2.19}$$

The function $\exp(-h\nu\beta)$ is called the Boltzmann factor. Besides, let N_g and N_e be the mean number of atoms in the ground state and in the excited state. The following equation is satisfied :

$$\frac{N_e}{N_g} = \exp(-h\nu\beta). \tag{2.20}$$

3. Expansion of the Planck distribution function.

3.1. Entropy S_{π_f}.

We will continue the discussion with reference to ideas in subsection2.1. The number of particles P is replaced to the positive real number x. The number of resonator N is replaced to the number of prime number $\pi(x)$. However the function $\pi(x)$ is not differentiable. Therefore, we consider to partition the function a positive real number x by logarithm $\log(x)$, that is, the function $x/\log(x)$. This function $x/\log(x)$ is an approximation to the prime number $\pi(x)$. We show the function $R^{\pm}_{\alpha}(x)$ is derived as follows. First, we start with some definitions.

Definition 3.1. *Let $x > 1$ be a positive real number, and $f(x)$ be a positive real valued function on x.*

$$\pi(x) := \sum_{\substack{p \le x \\ p:prime}} 1\,, \tag{3.1}$$

$$\pi_f(x) := \frac{x}{f(x)}\,, \tag{3.2}$$

$$Q_f(x) := \frac{x}{\pi_f(x)}\,. \tag{3.3}$$

The function $\pi(x)$ is expressed that the number of prime numbers less than or equal to x. By the definition above, it is satisfied that $Q_f(x) = f(x)$.

□

We define the number of states $W_{\pi_f,x}$. Therefore, Entropy $S_{\pi_f,x}$ under $W_{\pi_f,x}$ is defined by the number of states $W_{\pi_f,x}$. Moreover, Entropy $S_{\pi_f}(x)$ under $\pi_f(x)$ is defined to devide by Entropy $S_{\pi_f,x}$ by $\pi_f(x)$ as follows:

Definition 3.2. *Entropy $S_{\pi_f}(x)$ divided by $\pi_f(x)$.*

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Let $x > 1$ be a positive real number.

$$W_{\pi_f,x} := \frac{(\pi_f(x)+x)^{\pi_f(x)+x}}{\pi_f(x)^{\pi_f(x)} x^x}, \tag{3.4}$$

$$S_{\pi_f,x} := \log W_{\pi_f,x}, \tag{3.5}$$

$$S_{\pi_f}(x) := \frac{S_{\pi_f,x}}{\pi_f(x)}. \tag{3.6}$$

□

Note: Since the definition of Combination below formula (3.7) cannot define real values well, therefore, we adopted the definition of formula (3.4) using Stirling's approximation.

$$\frac{(\pi_f(x)+x-1)!}{(\pi_f(x)-1)!x!}. \tag{3.7}$$

In discussion below, unless otherwise specified, let the function $f(x)$ set to $\log(x)$. Namely, the following is satisfied :

$$f(x) = \log(x). \tag{3.8}$$

Therefore, using definitions above and the prime number theorem (Refer to Narkiewicz [1]), the following conditions are satisfied :

$$Q_f(x) = Q_{\log}(x) = \frac{x}{\pi_{\log}(x)} = \log(x) \sim \frac{x}{\pi(x)}. \tag{3.9}$$

By the definition 3.2, for sufficiently large $x > 1$, the following equations are satisfied :

$$\begin{aligned} S_{\pi_f,x} &= (\pi_f(x)+x)\log(\pi_f(x)+x) - \pi_f(x)\log(\pi_f(x)) - x\log(x) \\ &= \pi_f(x)\Big((1+\frac{x}{\pi_f(x)})\log(1+\frac{x}{\pi_f(x)}) - \frac{x}{\pi_f(x)}\log(\frac{x}{\pi_f(x)})\Big), \end{aligned} \tag{3.10}$$

$$S_{\pi_f}(x) = (1+\frac{x}{\pi_f(x)})\log(1+\frac{x}{\pi_f(x)}) - \frac{x}{\pi_f(x)}\log(\frac{x}{\pi_f(x)}). \tag{3.11}$$

Using the function $Q_f(x)$ above, the function $S_{\pi_f}(x)$ under π_f is expressed as follows :

$$S_{\pi_f}(x) = (1+Q_f(x))\log(1+Q_f(x)) - Q_f(x)\log Q_f(x). \tag{3.12}$$

Differentiating Entropy $S_{\pi_f}(x)$ under π_f as follows :

$$\begin{aligned} S'_{\pi_f}(x) &= (\frac{x}{\pi_f(x)})'\log(1+\frac{x}{\pi_f(x)}) + (\frac{x}{\pi_f(x)})' \\ &\quad - \Big((\frac{x}{\pi_f(x)})'\log(\frac{x}{\pi_f(x)}) + (\frac{x}{\pi_f(x)})'\Big) \\ &= (\frac{x}{\pi_f(x)})'\Big(\log(1+\frac{x}{\pi_f(x)}) - \log(\frac{x}{\pi_f(x)})\Big). \end{aligned} \tag{3.13}$$

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Furthermore, differentiating $S'_{\pi_f}(x)$ as follows :

$$S''_{\pi_f}(x) = (\frac{x}{\pi_f(x)})''\Big(\log(1+\frac{x}{\pi_f(x)}) - \log(\frac{x}{\pi_f(x)})\Big) + (\frac{x}{\pi_f(x)})'\Big(\log(1+\frac{x}{\pi_f(x)}) - \log(\frac{x}{\pi_f(x)})\Big)'. \tag{3.14}$$

Therefore, the equations above is expressed by using $Q_f(x)$ as follows :

$$S'_{\pi_f}(x) = Q'_f(x)\Big(\log(1+Q_f(x)) - \log Q_f(x)\Big), \tag{3.15}$$

$$S''_{\pi_f}(x) = Q''_f(x)\Big(\log(1+Q_f(x)) - \log Q_f(x)\Big) + Q'_f(x)Q'_f(x)\Big(\frac{-1}{Q_f(x)(1+Q_f(x))}\Big). \tag{3.16}$$

Repeating differential of the part of $Q''_f(x)\Big(\log(1+Q_f(x)) - \log Q_f(x)\Big)$ on (3.16), the following conditions are satisfied :

$$\Big(Q''_f(x)\Big(\log(1+Q_f(x)) - \log Q_f(x)\Big)\Big)' = Q^{(3)}_f(x)\Big(\log(1+Q_f(x)) - \log Q_f(x)\Big) + Q^{(2)}_f(x)Q'_f(x)\Big(\frac{-1}{Q_f(x)(1+Q_f(x))}\Big), \tag{3.17}$$

$$\Big(Q^{(n)}_f(x)\Big(\log(1+Q_f(x)) - \log Q_f(x)\Big)\Big)' = Q^{(n+1)}_f(x)\Big(\log(1+Q_f(x)) - \log Q_f(x)\Big) + Q^{(n)}_f(x)Q'(x)_f\Big(\frac{-1}{Q_f(x)(1+Q_f(x))}\Big). \tag{3.18}$$

Therefore, for all sufficiently large $x > 1$, the following conditions are satisfied :

$$\textit{Case1) } n > 1 \textit{ is even number}: \quad Q^{(n+1)}_f(x) = \frac{(-1)^n(n)!}{x^{n+1}} > \frac{(-1)^{n-1}(n-1)!}{x^n} = Q^{(n)}_f(x), \tag{3.19}$$

$$\textit{Case2) } n > 1 \textit{ is odd number}: \quad Q^{(n+1)}_f(x) = \frac{(-1)^n(n)!}{x^{n+1}} < \frac{(-1)^{n-1}(n-1)!}{x^n} = Q^{(n)}_f(x). \tag{3.20}$$

Furthermore, for all sufficiently large $x > 1$, the following are satisfied :

$$Q^{(n)}_f(x) > Q^{(2)}_f(x), \tag{3.21}$$

$$|Q^{(n+1)}_f(x)| > |Q^{(n)}_f(x)|. \tag{3.22}$$

Next, we define some functions $k_f(x)$, $R^+_m(x)$ and $R^-_m(x)$ as follows :

Definition 3.3. *The definition of the function $k_f(x)$.*

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Let $x > 1$ be a positive real number, and $f(x)$ be a real valued function. The function $k_f(x)$ is defined as follows :

$$k_f(x) = S''_{\pi_f}(x)\left(\frac{-Q_f(x)(1+Q_f(x))}{Q'_f(x)}\right). \tag{3.23}$$

Namely, the following equation is satisfied :

$$S''_{\pi_f}(x) = k_f(x)\left(\frac{-Q'_f(x)}{Q_f(x)(1+Q_f(x))}\right). \tag{3.24}$$

□

Let us call this function $k_f(x)$ the Boltzmann variable function in the function $f(x)$.

Definition 3.4. *The function $R^+_m(x)$ and $R^-_m(x)$ are defined as follows :*

$$R^+_m(x) := \sum_{n=1}^{m}\left|\frac{(-1)^{n-1}(n-1)!}{x^n}\right|. \tag{3.25}$$

Therefore, the following equations are satisfied :

$$R^+_m(x) = \sum_{n=1}^{m}\left|\frac{(-1)^{n-1}(n-1)!}{x^n}\right| = \sum_{n=1}^{m}\left|(\log(x))^{(n)}\right|. \tag{3.26}$$

Same as discussion, the following inequality are satisfied :

$$R^-_m(x) := \sum_{n=1}^{m}\frac{(-1)^{n-1}(n-1)!}{x^n}. \tag{3.27}$$

Therefore, the following equations are satisfied :

$$R^-_m(x) = \sum_{n=1}^{m}\frac{(-1)^{n-1}(n-1)!}{x^n} = \sum_{n=1}^{m}(\log(x))^{(n)}. \tag{3.28}$$

□

The function $R^+_m(x)$ is called an m-th absolute lower bound approximation of the Boltzmann variable function $k_f(x)$ in the function $f(x)$. Similarly, the function $R^-_m(x)$ is called an m-th lower bound approximation of the Boltzmann variable function $k_f(x)$ in the function $f(x)$. Using the defintion above, the following inequality is satisdied :

$$R^+_m(x) \geq R^-_m(x), \tag{3.29}$$

where the function $(\log(x))^{(n)}$ represents the n-th derivation of $\log(x)$. Moreover, the function $(\log(x))^n$ and $\log^n(x)$ to the n-th power represents $\log(x)$.

Using equivalent(3.24), for all sufficiently large $x > 1$, the following conditions are satisfied :

$$k_f(x) = -S''_{\pi_f}(x)\frac{Q_f(x)(1+Q_f(x))}{Q'_f(x)} \leq \frac{1}{x}(2+\log(x)). \tag{3.30}$$

where the function $Q_f(x)$ is $\log(x)$. Because, by the equation(3.16),

$$\begin{aligned} S''_{\pi_f}(x) &= Q''_f(x)\Big(\log(1+Q_f(x)) - \log Q_f(x)\Big) \\ &\quad + Q'_f(x)Q'_f(x)\left(\frac{-1}{Q_f(x)(1+Q_f(x))}\right). \end{aligned} \tag{3.31}$$

Therefore, for sufficiently large $x > 1$, the following are satisfied :

$$\begin{aligned} k_f(x) &= \frac{1}{x^2}\log\left(1+\frac{1}{\log(x)}\right)x\log(x)(1+\log(x)) + \frac{1}{x} \\ &= \frac{1}{x}\log\left(1+\frac{1}{\log(x)}\right)^{\log(x)}(1+\log(x)) + \frac{1}{x} \\ &\leq \frac{1}{x}\log(e)\Big(1+\log(x)\Big) + \frac{1}{x} \qquad \because) \lim_{x\to\infty}\left(1+\frac{1}{\log(x)}\right)^{\log(x)} \to e \\ &= \frac{1}{x}\Big(1+\log(x)\Big) + \frac{1}{x} \\ &= \frac{1}{x}(2+\log(x)). \end{aligned} \tag{3.32}$$

Furthermore, there is a positive integer $m \geq 1$ such that the following conditions are satisfied:

$$\begin{aligned} S''_{\pi_f}(x) &= k_f(x)\left(\frac{-Q'_f(x)}{Q_f(x)(1+Q_f(x))}\right) \\ &\geq \Big(|Q'_f(x)| + |Q''_f(x)| + \cdots + |Q^{(m)}_f(x)|\Big)\frac{-Q'_f(x)}{Q_f(x)(1+Q_f(x))} \\ &\geq R^+_m(x)\left(\frac{-Q'_f(x)}{Q_f(x)(1+Q_f(x))}\right). \end{aligned} \tag{3.33}$$

Using the same discussion above, there is a positive integer $m \geq 1$ such that the following conditions are satisfied:

$$\begin{aligned} S''_{\pi_f}(x) &= k_f(x)\left(\frac{-Q'_f(x)}{Q_f(x)(1+Q_f(x))}\right) \\ &\geq \Big(Q'_f(x) + Q''_f(x) + \cdots + Q^{(m)}_f(x)\Big)\frac{-Q'_f(x)}{Q_f(x)(1+Q_f(x))} \\ &\geq R^-_m(x)\left(\frac{-Q'_f(x)}{Q_f(x)(1+Q_f(x))}\right). \end{aligned} \tag{3.34}$$

First order differentiation of Entropy $S_{\pi_f}(x)$ is always positive values, that is $S'_{\pi_f}(x) > 0$. Moreover, second order differentiation of Entropy $S_{\pi_f}(x)$ has always negative values, so that $S''_{\pi_f}(x) < 0$. Therefore, Entropy $S_{\pi_f}(x)$ has no inflection points.

3.2. Derivation of the functions $R^{\pm}_{\alpha}(x)$.

Next, the function $R^+_{\alpha}(x)$ and $R^-_{\alpha}(x)$ are derived as follows :

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Definition 3.5. *$R^+_\alpha(x)$, $R^-_\alpha(x)$ and $R^\pm_\alpha(x)$*
Let the constant $\alpha > 0$ be a positive real number . For all positive real number $x > 1$, the function $R^+_\alpha(x)$ and $R^-_\alpha(x)$ are defined as follows :

$$R^+_\alpha(x) = \frac{\sqrt{2\pi}\alpha}{ex+1}, \tag{3.35}$$

$$R^-_\alpha(x) = \frac{\sqrt{2\pi}\alpha}{ex-1}. \tag{3.36}$$

The function $R^\pm_\alpha(x)$ are combined $R^+_\alpha(x)$ and $R^-_\alpha(x)$ as follows :

$$R^\pm_\alpha(x) := \frac{\sqrt{2\pi}\alpha}{ex \pm 1}. \tag{3.37}$$

Therefore, the following conditions are satisfied :

$$xR^\pm_\alpha(x) = \frac{\sqrt{2\pi}\alpha x}{ex \pm 1}, \tag{3.38}$$

$$\frac{1}{xR^\pm_\alpha(x)} = \frac{e}{\sqrt{2\pi}\alpha}\left(1 \pm \frac{1}{ex}\right). \tag{3.39}$$

□

This function $R^\pm_\alpha(x)$ is called an $\pm$lower bound approximation of the Boltzmann variable function $k_f(x)$ in the function $f(x)$ and the constant α.

The relations of functions $R^\pm_\alpha(x)$, $R^+_m(x)$ and $R^-_m(x)$ are satisfied as follows :

Lemma 3.6. *The relation $R^+_m(x) \geq R^+_\alpha(x)$.*
Let $\alpha > 0$ be a positve real number. There is an integer $m \geq 1$ such that for all sufficiently large $x > 1$, the following inequality is satisfied :

$$R^+_m(x) \geq R^+_\alpha(x) = \frac{\sqrt{2\pi}\alpha}{ex+1} \tag{3.40}$$

where a positive real number α is satisfied as follows :

$$x \geq \frac{-1}{e - \sqrt{2\pi}\alpha}, \tag{3.41}$$

that is, satisfied as follows:

$$\frac{e}{\sqrt{2\pi}}(1 + \frac{1}{x}) \geq \alpha. \tag{3.42}$$

Using same as discussion, the following conditions are satisfied :

Lemma 3.7. *The relation $R^-_m(x) \geq R^-_\alpha(x)$.*
Let $\alpha > 0$ be a positve real number. There is an integer $m \geq 1$ such that for all sufficiently large $x > 1$, the following inequality is satisfied :

$$R^-_m(x) \geq R^-_\alpha(x) = \frac{\sqrt{2\pi}\alpha}{ex-1} \tag{3.43}$$

where a positive real number α is satisfied as follows :

$$x \geq \frac{1}{e - \sqrt{2\pi}\alpha}, \tag{3.44}$$

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that is, satisfied as follows:

$$\frac{e}{\sqrt{2\pi}}(1-\frac{1}{x}) \geq \alpha. \tag{3.45}$$

Proof. The proof of Lemma 3.6 and Lemma 3.7 are described the following the section 6.1. □

Consequently, for sufficiently large real number $x > 1$, a real valued function $f(x)$ and a positive integer $m > 1$, the following inequalities are satisfied :

$$S''_{\pi_f}(x) \geq R^{+}_{m}(x)\frac{-Q'_f(x)}{Q_f(x)(1+Q_f(x))} \geq R^{+}_{\alpha}(x)\frac{-Q'_f(x)}{Q_f(x)(1+Q_f(x))}, \tag{3.46}$$

$$S''_{\pi_f}(x) \geq R^{-}_{m}(x)\frac{-Q'_f(x)}{Q_f(x)(1+Q_f(x))} \geq R^{-}_{\alpha}(x)\frac{-Q'_f(x)}{Q_f(x)(1+Q_f(x))}. \tag{3.47}$$

Namely, the following inequality is satisfied :

$$S''_{\pi_f}(x) \geq R^{\pm}_{\alpha}(x)\frac{-Q'_f(x)}{Q_f(x)(1+Q_f(x))}. \tag{3.48}$$

The second derivative of Entropy $S''_{\pi_f}(x)$ is suppressed from the bottom side by formula. Besides, the second derivative of Entropy $S''_{\pi_f}(x)$ is suppressed from the upper side by formula as follows.

Lemma 3.8. $R^{+}_{m}(x) \geq R^{+}_{\alpha}(x)$, *and* $R^{-}_{m}(x) \geq R^{-}_{\alpha}(x)$.
For all sufficiently large $x > 1$ and a positive integer $m > 1$, the following inequalities are satisfied :

$$\frac{1}{(x-1)^{1/2}}\frac{e}{e+1} \geq R^{+}_{m}(x) \geq R^{+}_{\alpha}(x) = \frac{\sqrt{2\pi}\alpha}{ex+1}, \tag{3.49}$$

$$\frac{1}{(x-1)^{1/2}}\frac{e}{e+1} \geq R^{-}_{m}(x) \geq R^{-}_{\alpha}(x) = \frac{\sqrt{2\pi}\alpha}{ex-1}. \tag{3.50}$$

Proof. The proof of Lemma 3.8 are described the following the section 6.2. □

On the next subsection, we discuss the meaning of inequalities above Lemma 3.6 and Lemma 3.7.

3.3. The Expansion of the Planck distribution $n^{\pm}(x,\alpha)$ by using $R^{\pm}_{\alpha}(x)$.

Next, we examine to define the distribution functions $n^{\pm}(x,\alpha)$ by using $R^{\pm}_{\alpha}(x)$. Beside, integrate the inequality(3.46) and (3.47) as a variable x, Therefore, the following inequality is satisfied :

$$S''_{\pi_f}(x)dx \geq R^{\pm}_{\alpha}(x)\frac{-Q'_f(x)}{Q_f(x)(1+Q_f(x))}dx. \tag{3.51}$$

As the following equation is satisfied :

$$S'_{\pi_f}(x) = \int S''_{\pi_f}(x)dx. \tag{3.52}$$

Hence, the following formulas are satisfied :

$$\begin{aligned} S'_{\pi_f}(x) &= Q'_f(x)(\log(1+Q_f(x)) - \log Q_f(x)) + C \\ &\geq R^{\pm}_{\alpha}(x)(\log(1+Q_f(x)) - \log Q_f(x)) + C \\ &= R^{\pm}_{\alpha}(x)\log(1+\frac{1}{Q_f(x)}) + C. \end{aligned} \tag{3.53}$$

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where the constant C is a positive real number.

Here, for all sufficiently large $Q_f(x) > 0$, the following equation is satisfied :

$$\log(1 + \frac{1}{Q_f(x)}) = 0. \tag{3.54}$$

Hence, the first order differentiation $S'_{\pi_f}(x)$ is satisfied as follows :

$$S'_{\pi_f}(x) = Q'_f(x)\log(1 + \frac{1}{Q_f(x)}) = 0. \tag{3.55}$$

Thus, using inequality(3.53), the constant C is satisfied as follows :

$$C = 0. \tag{3.56}$$

Therefore, the inequality(3.53) is satisfied as follows :

$$S'_{\pi_f}(x) \geq R^{\pm}_{\alpha}(x)\log(1 + \frac{1}{Q_f(x)}). \tag{3.57}$$

For sufficiently large positive real number $x > 1$, the function $S'_{\pi_f}(x)$ is satisfied as follows :

$$\frac{1}{x} \geq S'_{\pi_f}(x) = \frac{1}{x}\log(1 + \frac{1}{\log(x)}). \tag{3.58}$$

According to inequalities(3.57) and (3.58),

$$\frac{1}{xR^{\pm}_{\alpha}(x)} \geq \log(1 + \frac{1}{Q_f(x)}). \tag{3.59}$$

Therefore, by (3.59) the following inequality is derived :

$$Q_f(x) \geq \frac{1}{\exp(\frac{1}{xR^{\pm}_{\alpha}(x)}) - 1}. \tag{3.60}$$

Focusing on equality of the inequality (3.60), we define the distribution function $n^{\pm}(x,\alpha)$ as follows:

Definition 3.9. *The distribution functions $n^{\pm}(x,\alpha)$ are defined as follows:*

$$n^{\pm}(x,\alpha) = \frac{1}{\exp(\frac{1}{xR^{\pm}_{\alpha}(x)}) - 1}, \tag{3.61}$$

where $\alpha > 0$. □

The definition above is transformed as follows :

$$\frac{n^{\pm}(x,\alpha)}{n^{\pm}(x,\alpha) + 1} = \exp(\frac{-1}{xR^{\pm}_{\alpha}(x)}), \quad (\alpha > 0). \tag{3.62}$$

Thus, this distribution function $n^{\pm}(x,\alpha)$ is regards as the approximate density of prime numbers $x/\pi(x)$ until the number x. Besides, this function $n^{\pm}(x,\alpha)$ is regarded as one of the distribution

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functions. Furthermore, this function $n^{\pm}(x,\alpha)$ is seems to expand the Planck distribution function $\bar{n}(\nu,\beta)$. According to imitate the Boltzmann factor, the following function

$$\exp(\frac{-1}{xR_{\alpha}^{\pm}(x)}) \tag{3.63}$$

is called the expansion of Boltzmann factor or $R_{\alpha}^{\pm}$ factor . We will consider the further relationship in the next subsection.

3.4. Corresponding the Planck distribution function and the distribution function $n^{\pm}(x,\alpha)$.

We examine to correspond the Planck distribution function $\bar{n}(\nu,\beta)$ and the distribution function $n^{\pm}(x,\alpha)$ as follows :

$$\bar{n}(\nu,\beta) = \frac{1}{\exp(h\nu\beta)-1}, \tag{3.64}$$

where

$$\begin{aligned} &h: \text{ the Planck constant},\\ &\nu: \text{Frequency},\\ &\beta: \text{Inverse temperature}. \end{aligned} \tag{3.65}$$

Here, we consider to correspond the internal parameter of the Boltzmann factor $\exp(-h\nu\beta)$

$$h\nu\beta \tag{3.66}$$

and the internal function of $R_{\alpha}^{\pm}$ factor $\exp(\frac{-1}{xR_{\alpha}^{\pm}(x)})$

$$\frac{1}{xR_{\alpha}^{\pm}(x)} \quad \left(= \frac{e}{\sqrt{2\pi\alpha}}(1 \pm \frac{1}{ex})\right). \tag{3.67}$$

Namely, we suppose the correspondence as follows:

$$h\nu\beta = \frac{e}{\sqrt{2\pi\alpha}}(1 \pm \frac{1}{ex}). \tag{3.68}$$

Furthermore, we can consider by separating the correspondence between and the variable parts and the constant parts as follows :

$$\nu\beta = (1 \pm \frac{1}{ex}), \quad (\text{variable parts}) \tag{3.69}$$

$$h = \frac{e}{\sqrt{2\pi\alpha}}. \quad (\text{constant parts}) \tag{3.70}$$

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The relationship diagram between $S_{N,P}$ and $S_{\pi_f,x}$ is shown below :

$$
\begin{array}{ccc}
W_{N,P} = \frac{(N+P-1)!}{(N-1)!P!}, & & S_{N,P} = k_B \log W_{N,P}, \\
\exp(-h\nu\beta) & \xrightarrow{\log} & -h\nu\beta \\
{\scriptstyle N:=\pi_f(x) \atop \scriptstyle P:=x} \downarrow \uparrow {\scriptstyle \alpha:=\frac{e}{\sqrt{2\pi h}} \atop \scriptstyle x:=\frac{\mp 1}{e(1-\nu\beta)}} & & {\scriptstyle N:=\pi_f(x) \atop \scriptstyle P:=x} \downarrow \uparrow {\scriptstyle \alpha:=\frac{e}{\sqrt{2\pi h}} \atop \scriptstyle x:=\frac{\mp 1}{e(1-\nu\beta)}} \\
W_{\pi_f,x} = \frac{(\pi_f(x)+x)^{\pi_f(x)+x}}{\pi_f(x)^{\pi_f(x)} x^x}, & & S_{\pi_f,x} = \log W_{\pi_f,x}, \\
\exp(\frac{-1}{xR^{\pm}_{\alpha}(x)}) & \xrightarrow{\log} & \frac{-1}{xR^{\pm}_{\alpha}(x)}
\end{array}
\tag{3.71}
$$

Corresponding the above, the distribution function $n^{\pm}(x,\alpha)$ becomes to expand the Planck distribution function. Namely, the following conditions are satisfied :

Suggestion 3.10. *The expansion of the Planck distribution $\bar{n}(\nu,\beta)$.*
Let $\alpha > 0$ be a real number constant. For all real number $x > 1$ the following equation is satisfied :

$$\bar{n}(\nu,\beta) = n^{\pm}(x,\alpha), \tag{3.72}$$

where

$$
\begin{aligned}
x &= \frac{\mp 1}{e(1-\nu\beta)}, \\
\alpha &= \frac{e}{\sqrt{2\pi h}}.
\end{aligned}
\tag{3.73}
$$

Namely, the distribution function $n^{\pm}(x,\alpha)$ can be regarded as representing an expansion of the Planck distribution function $\bar{n}(\nu,\beta)$. □

For sufficiently large $x > 1$, the correspondence of equation(3.69) is satisfied as follows:

$$\nu\beta = \lim_{x\to\infty}(1 \pm \frac{1}{ex}) = 1. \tag{3.74}$$

Moreover, according to the method to divide each S and $S_{\pi_f}(x)$, we remember that the following corresponds :

1. The number of particles P is replaced to the positive real number x.
2. The number of resonators N is replaced to approximate number of prime numbers $\pi(x) \sim \frac{x}{\log(x)}$.

Therefore, we consider the correspondence the between

$$\frac{U}{\varepsilon} = \frac{P}{N} \tag{3.75}$$

and

$$Q_f(x) = \frac{x}{\frac{x}{\log(x)}} \sim \frac{x}{\pi(x)}. \tag{3.76}$$

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Namely, we suppose the following correspondence is considered :

$$U \longleftrightarrow x, \\ \varepsilon \longleftrightarrow \frac{x}{\log(x)}. \tag{3.77}$$

Thereby, We consider the correspondence the between Planck's law U and the following function $U^{\pm}_{x,\alpha}$.

Definition 3.11. *The real valued function $U^{\pm}_{x,\alpha}$ as the expansion of Planck's law U.*
Let $\alpha > 0$ be a real number constant. For sufficiently large real number $x > 1$, the real valued function $U^{\pm}_{x,\alpha}$ is defined as follows :

$$\begin{aligned} U^{\pm}_{x,\alpha} &= n^{\pm}(x,\alpha)\frac{x}{\log(x)} \\ &= \frac{x}{\log(x)\left(\exp(\frac{1}{xR^{\pm}_{\alpha}(x)}) - 1\right)} \\ &\sim \frac{\pi(x)}{\exp(\frac{1}{xR^{\pm}_{\alpha}(x)}) - 1} \quad (\alpha > 0). \end{aligned} \tag{3.78}$$

□

Using the suggestion3.10 and the definition3.11, the following suggestion is given:

Suggestion 3.12. *The expansion of Planck's law U.*
Let $h > 0$, $\nu > 0$ and $\beta > 0$ be real numbers. Each values h, ν and β means the Planck constant, frequency and inverse temperature.
There exists real numbers $x > 1$ and $\alpha > 0$ such that the following equality is satisfied :

$$U = U^{\pm}_{x,\alpha}, \tag{3.79}$$

where the following conditions are satisfied :

$$\begin{aligned} h\nu &= \frac{x}{\log(x)}, \\ \beta &= \frac{h\log(x)}{x}(1 \pm \frac{1}{ex}), \\ \nu &= \frac{x}{h\log(x)}, \\ h &= \frac{e}{\sqrt{2\pi\alpha}}. \end{aligned} \tag{3.80}$$

Namely, the real valued function $U^{\pm}_{x,\alpha}$ can be regarded as representing the expansion of Planck's law U. □

According to the suggestion 3.12 above, under the condition that the product of the Planck constant h and the frequency ν, that is, $h\nu$ is an approximation of the number of prime numbers $\pi(x)$. Planck's law U is seems to take discrete values and has an approximate spectrum of prime numbers. It is possible that the discrete values of an element of energy are related to the distribution of prime numbers.

3.5. A kind of fine-structure constant.

The distribution function $n^{\pm}(x,\alpha)$ and the Planck distribution $\bar{n}(\nu,\beta)$ is associated by a constant α. The constant $\alpha = e/\sqrt{2\pi}h$ is thought like the *fine-structure constant* that associated with the Planck constant h.

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Let a positive real number α set as follows :

$$\alpha = \frac{e}{\sqrt{2\pi}h}. \tag{3.81}$$

For all sufficiently large $x > 1$, the following inequalities are satisfied :

$$\frac{e}{\sqrt{2\pi}}(2 + \log(x)) \geq \alpha. \tag{3.82}$$

According to the Prime numbers theorem, the following relation is satisfied :

$$\pi(x) \sim \frac{x}{\log(x)} \sim \frac{x}{\log(x) + 2}. \tag{3.83}$$

Thus, the positive real number α is satisfied such that

$$\frac{e}{\sqrt{2\pi}} \frac{x}{\pi(x)} \geq \alpha, \quad (\alpha > 0). \tag{3.84}$$

Hence, the positive real number α can be regard as fine-structure constant by $\sqrt{2}$, π, e and $\pi(x)$. Furthermore, the following inequality is satisfied :

Suggestion 3.13. *The ratio of the Boltzmann constant and the Planck constant.*
Let $\alpha > 0$ be a positive real number (constant). For sufficiently large $x > 1$, the following formulas are satisfied :

$$\frac{h}{k_B} = \frac{e}{\sqrt{2\pi}\alpha} \geq \frac{\pi(x)}{x}. \tag{3.85}$$

Namely, the ratio of the Boltzmann constant k_B and the Planck constant h is bigger than the ratio of a positive real number x and the number of prime $\pi(x)$ until x. □

Using discussions above, a constant α can be associated between the Planck distribution function

$$\bar{n}(\nu, \beta) = \frac{1}{\exp(h\nu\beta) - 1}, \tag{3.86}$$

and the expansion of the Planck distribution function

$$n^{\pm}(x, \alpha) = \frac{1}{\exp(\frac{1}{xR_\alpha^{\pm}(x)}) - 1}. \tag{3.87}$$

Namely, suppose that a constant $\alpha > 0$ is decided. Specially, the constant h_α and α_h are decided by α, e and π as follows :

$$h_\alpha = \frac{e}{\sqrt{2\pi}\alpha}, \tag{3.88}$$

$$\alpha_h = \frac{e}{\sqrt{2\pi}h}. \tag{3.89}$$

Namely, The function $R_\alpha^{\pm}(x)$ is changed and depended by a constant $\alpha > 0$. Therefore, the constant h_α can defined for each constant α.

By lemma3.10, Modern physics may be a special case that satisfy the following condition:

$$\alpha_h = \alpha = \frac{e}{\sqrt{2\pi}h}, \tag{3.90}$$

$$h_\alpha = h = \frac{e}{\sqrt{2\pi}\alpha}. \tag{3.91}$$

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Therefore, the following suggestion is stated :

Suggestion 3.14. *Let $\alpha > 0$ be a positive real number (constant). The constant h_α can be selected as follows :*

$$h_\alpha = \frac{e}{\sqrt{2\pi}\alpha}, \tag{3.92}$$

where the following inequality is satisfied :

$$\frac{e}{\sqrt{2\pi}} \frac{x}{\pi(x)} \geq \alpha, \quad (\alpha > 0). \tag{3.93}$$

Namely, the condition of the equality is satisfied as follows :

$$\alpha = \frac{e}{\sqrt{2\pi}h}. \tag{3.94}$$

Therefore, the constant h_α becomes the Planck constant h. □

Note:
Let me mention here for your attention as follows: The fine-structure constant is a *physical constant* α and is originally expressed using the Planck constant as follows.

In this paper, we describe it as original the fine-structure constant α_p to distinguish it from the real number α. Besides, describe it as the elementary charge e_p to distinguish it from the Napier's number e.

$$\alpha_p = \frac{e_p^2}{2h\varepsilon_0 c} = \frac{\mu_0 e_p^2 c}{2h}. \tag{3.95}$$

where

$$\begin{aligned} h &: \text{the Planck constant}, \\ \varepsilon_0 &: \text{the electric constant}, \\ \mu_0 &: \text{the magnetic constant}, \\ e_p &: \text{the elementary charge}, \\ c &: \text{the speed of light}. \end{aligned} \tag{3.96}$$

Therefore, the relation the fine-structure constant α_p and the real number α in this paper is satisfied as follows :

$$\frac{\alpha_p}{\alpha} = \sqrt{\frac{\pi}{2e^2}} \frac{e_p^2}{\varepsilon_0 c} = \sqrt{\frac{\pi}{2e^2}} e_p^2 \mu_0 c. \tag{3.97}$$

On the following section, using the function $R_\alpha^\pm(x)$, we show that some examples such that the constant $\alpha \neq \frac{e}{\sqrt{2\pi}h}$ as follows :

$$\alpha = \frac{1}{\sqrt{2\pi}}, \tag{3.98}$$

$$\alpha = \frac{2}{\sqrt{2\pi}}, \tag{3.99}$$

$$\alpha = \frac{e}{\sqrt{2\pi}}, \tag{3.100}$$

$$\alpha = \frac{e}{\sqrt{2\pi}\log(\frac{\epsilon+2}{\epsilon+1})}. \tag{3.101}$$

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4. Application the function $R^{\pm}_{\alpha}(x)$ to Number Theory.

4.1. Examples using the function $R^{\pm}_{\alpha}(x)$ for deriving Von Koch's inequality.

We derive Von Koch's inequality using the above the constant α and the function $R^{\pm}_{\alpha}(x)$. (Refer to Fujino [20])

Note : In this paper, we consider an element of energy ε and an arbitrary real number ϵ separately.

Theorem 4.1. *Inequalities for evaluating the number of prime numbers (1). Let $\alpha > 0$ be a positive real number (constant). There exist a positive real number $C > 1$ such that for all sufficiently large real number $x \geq 2$, the following conditions are satisfied :*

$$|\pi(x) - \mathrm{li}(x)| \leq C(\frac{\sqrt{2\pi\alpha}}{48})^{\frac{1}{4}} \exp\left(\frac{e}{\sqrt{2\pi\alpha}}\right) x^{\frac{1}{\sqrt{2\pi\alpha}}} \left(\frac{1}{\log(x)}\right)^{\frac{3}{4}} \exp\left(\frac{1}{\log(x)}\right), \tag{4.1}$$

where the positive real number $\alpha > 0$ are satisfied as follows :

$$1 \leq \frac{1}{\sqrt{2\pi\alpha}} \exp\left(\frac{e}{\sqrt{2\pi\alpha}}\right), \tag{4.2}$$

$$\frac{1}{\sqrt{2\pi}} \leq \alpha \leq C\frac{e}{\sqrt{2\pi}}, \tag{4.3}$$

$$\exp\left(\frac{e}{\sqrt{2\pi\alpha}}\right) = \lim_{x\to\infty} \exp\left(\frac{1}{xR^{\pm}(x)}\right). \tag{4.4}$$

□

Corollary 4.2. *Inequalities for evaluating the number of prime numbers (2).*
There exist a positive real number $C > 1$ such that for all $\epsilon > 0$ and for all sufficiently large $x \geq 2$, the following conditions is satisfied:

$$|\pi(x) - \mathrm{li}(x)| \leq C(\frac{1}{48})^{\frac{1}{4}} \exp(e) x \left(\frac{1}{\log(x)}\right)^{\frac{3}{4}} \exp\left(\frac{1}{\log(x)}\right). \tag{4.5}$$

Proof. Using Theorem(4.1), put a positive real number $\alpha > 0$ as follows:

$$\alpha = \frac{1}{\sqrt{2\pi}}. \tag{4.6}$$

Therefore, the inequality(4.5) is satisfied.
□

The result of Corollary(4.1) is similar to the following result : (Refer to Wladyslaw [1])

$$(\exists C > 0)|\pi(x) - \mathrm{li}(x)| \leq O(x \exp(-C\sqrt{log(x)})). \tag{4.7}$$

Comparing inequalities(4.5) and (4.7), the following condition is satisfied :

$$O\left(x\left(\frac{1}{\log(x)}\right)^{\frac{3}{4}} \exp\left(\frac{1}{\log(x)}\right)\right) \leq O(x \exp(-C\sqrt{log(x)})). \tag{4.8}$$

Namely, the asymptotic of (4.5) gives better than that of (4.7).

Therefore, let be $\alpha = 2/\sqrt{2\pi}$, $(h_\alpha = e/2)$. the theorem above are satisfied as follows :

Corollary 4.3. *Inequalities for evaluating the number of prime numbers (3).*
There exist a positive real number $C > 1$ such that for all $\epsilon > 0$ and for all sufficiently large $x \geq 2$, the following condition is satisfied:

$$|\pi(x) - \mathrm{li}(x)| \leq C(\frac{1}{24})^{\frac{1}{4}} \exp{(\frac{e}{2})} x^{\frac{1}{2}} (\frac{1}{\log(x)})^{\frac{3}{4}} \exp(\frac{1}{\log(x)}). \tag{4.9}$$

Proof. Using Theorem(4.1) and the following conditions is satisfied:

$$1 \leq \frac{1}{\sqrt{2\pi\alpha}} \exp{(\frac{e}{\sqrt{2\pi\alpha}})}. \tag{4.10}$$

Put a positive real number $\alpha > 0$ as follows:

$$\alpha = \frac{2}{\sqrt{2\pi}} (\geq \frac{1}{\sqrt{2\pi}}). \tag{4.11}$$

Hence, the following inequalities is satisfied :

$$\begin{aligned} 1 &\leq \frac{1}{\sqrt{2\pi\alpha}} \exp{(\frac{e}{\sqrt{2\pi\alpha}})} \\ &= \frac{1}{\sqrt{2\pi\frac{2}{\sqrt{2\pi}}}} \exp{(\frac{e}{\sqrt{2\pi\frac{2}{\sqrt{2\pi}}}})} \quad (\because \alpha = \frac{2}{\sqrt{2\pi}}) \\ &= \frac{1}{2} \exp{(\frac{e}{2})} \quad (= 1.946424\cdots). \end{aligned} \tag{4.12}$$

Thus, the positive real number $\alpha > 0$ is satisfied conditions of (4.10) and (4.11). Therefore, there exist a positive real number $C > 1$ such that for all sufficiently large $x \geq 2$ the following condition is satisfied :

$$|\pi(x) - \mathrm{li}(x)| \leq C(\frac{1}{24})^{\frac{1}{4}} \exp{(\frac{e}{2})} x^{\frac{1}{2}} (\frac{1}{\log(x)})^{\frac{3}{4}} \exp(\frac{1}{\log(x)}). \tag{4.13}$$

□

Corollary 4.4. *Von Koch's inequality.*

$$(\exists C > 1)(\forall \epsilon > 0)(\forall x >> 2) \ |\pi(x) - \mathrm{li}(x)| \leq Cx^{\frac{1}{2}} \log(x), \tag{4.14}$$

where C, ϵ and x are real numbers. Namely,

$$|\pi(x) - \mathrm{li}(x)| \leq O(x^{\frac{1}{2}} \log(x)). \tag{4.15}$$

Proof. Fix $\epsilon > 0$. For all sufficient large $x \geq 2$, the following conditions are satisfied :

$$\left(\frac{1}{\log(x)}\right)^{\frac{3}{4}} \exp(\frac{1}{\log(x)}) < \log(x) < x^{\epsilon}. \tag{4.16}$$

Therefore, there exist a positive real number $C > 0$ such that for all sufficiently large $x \geq 2$, the following inequalities are satisfied :

$$\begin{aligned} &|\pi(x) - \mathrm{li}(x)| \\ &\leq C(\frac{1}{24})^{\frac{1}{4}} \exp{(\frac{e}{2})} x^{\frac{1}{2}} (\frac{1}{\log(x)})^{\frac{3}{4}} \exp(\frac{1}{\log(x)}) \\ &\leq Cx^{\frac{1}{2}} \log(x). \end{aligned} \tag{4.17}$$

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□

As is well known, Corollary(4.4) is equivalent to the Riemann Hypothesis. (Refer to Wladyslaw [1]) Therefore, the Riemann Hypothesis is considered true.

Furthermore, let $\alpha = e/\sqrt{2\pi}$, $(h_\alpha = 1)$. The following inequality is satisfied :

Corollary 4.5. *The reduction of the upper limit of the inequality that evaluate the number of prime number.*

$$(\exists C > 1)(\forall \epsilon > 0)(\forall x >> 2)\ |\pi(x) - \mathrm{li}(x)| \leq Cx^{\frac{1}{e}} \log(x), \tag{4.18}$$

where C, ϵ and x are real numbers. Namely,

$$|\pi(x) - \mathrm{li}(x)| \leq O(x^{\frac{1}{e}} \log(x)). \tag{4.19}$$

Proof. Using theorem(4.1), put a positive real number $\alpha > 0$ as follows:

$$\alpha = \frac{e}{\sqrt{2\pi}} (\geq \frac{e}{\sqrt{2\pi}}). \tag{4.20}$$

Thus, the following conditions are satisfied :

$$\begin{aligned} 1 &\leq \frac{1}{\sqrt{2\pi\alpha}} \exp\left(\frac{e}{\sqrt{2\pi\alpha}}\right) \\ &= \frac{1}{\sqrt{2\pi\frac{e}{\sqrt{2\pi}}}} \exp\left(\frac{e}{\sqrt{2\pi\frac{e}{\sqrt{2\pi}}}}\right) \quad (\because \alpha = \frac{e}{\sqrt{2\pi}}) \\ &= \frac{1}{e} \exp(1) \quad (= 1). \end{aligned} \tag{4.21}$$

Therefore the following condition is satisfied :

$$\begin{aligned} &|\pi(x) - \mathrm{li}(x)| \\ &\leq C(\frac{e}{48})^{\frac{1}{4}} \exp(1) x^{\frac{1}{e}} (\frac{1}{\log(x)})^{\frac{3}{4}} \exp(\frac{1}{\log(x)}) \\ &\leq Cx^{\frac{1}{e}} \log(x). \end{aligned} \tag{4.22}$$

□

We have a question that the upper limit of the real number d that satisfies the formula $|\pi(x) - \mathrm{li}(x)| \leq O(x^{1/d} \log(x))$ is Napier's constant e correct or not. Namely, the following problem can be considered.

Problem 4.6. *The upper limit of the inequality that evaluate the number of prime number.*

$$e = \sup\{d > 1 \mid (\exists C > 1)(\forall \epsilon > 0)(\forall x >> 2)\ |\pi(x) - \mathrm{li}(x)| \leq Cx^{\frac{1}{d}} \log(x)\}, \tag{4.23}$$

where C, ϵ and x are real numbers. Namely,

$$e = \sup\{d > 1 \mid |\pi(x) - \mathrm{li}(x)| \leq O(x^{\frac{1}{d}} \log(x))\}. \tag{4.24}$$

□

We expect problem 4.6 to be correct. In future, we attempt to solve this problem.

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4.2. Example using the function $R^{\pm}_{\alpha}(x)$ for the abc conjecture.

We derive the weak abc conjecture and the strong abc conjecture using the constant α. Namely, using the constant α, the following theorems are satisfied : (Refer to Fujino [21])

Theorem 4.7. *Let $\alpha > 0$ be a positive real number. For all real number $\epsilon > 0$ and the constant $K_{\epsilon} \geq 1$, there exists countable infinite triples (a, b, c) of coprime positive integers with $a + b = c$ such that the following inequality is satisfied :*

$$K_{\epsilon}\mathrm{rad}(abc) < c^{\exp(\frac{e}{\sqrt{2\pi}\alpha})-1} \tag{4.25}$$

where the following equation is satisfied :

$$\exp\left(\frac{e}{\sqrt{2\pi}\alpha}\right) = \lim_{x\to\infty}\exp\left(\frac{1}{xR^{\pm}(x)}\right). \tag{4.26}$$

□

Set the constant α as follows:

$$\alpha = \frac{e}{\sqrt{2\pi}\log(\frac{\epsilon+2}{\epsilon+1})}. \tag{4.27}$$

Therefore, the following is satisfied :

$$h_{\alpha} = \log\left(\frac{\epsilon+2}{\epsilon+1}\right). \tag{4.28}$$

Therefore, the negation of the weak abc conjecture is satisfied as follows :

Theorem 4.8. *The negation of the weak abc conjecture.*
For all real number $\epsilon > 0$ and constant $\bar{K}_{\epsilon} \geq 1$, there exists countable infinite triples (a, b, c) of coprime positive integers with $a + b = c$ such that the following inequality is satisfied :

$$\bar{K}_{\epsilon}\mathrm{rad}(abc)^{1+\epsilon} < c. \tag{4.29}$$

Namely, There is a counter-example in the weak abc conjecture. Therefore, the weak abc conjecture is not true. □

Furthermore, let $\epsilon = 1$ and $\bar{K}_{\epsilon} = 1$. The negative of the strong abc conjecture is satisfied as follows :

Theorem 4.9. *The negation of the strong abc conjecture.*
There exists countable infinite triples (a, b, c) of coprime positive integers with $a + b = c$ such that the following inequality is satisfied :

$$\mathrm{rad}(abc)^2 < c. \tag{4.30}$$

Namely, the strong abc conjecture is not true. □

The function $R^{\pm}_{\alpha}(x)$ was used the above discussions of Von Koch's inequality and the abc conjecture. We will investigate the relation Entropy and Number Theorem further.

4.3. Conclusion and Application to Number Theory.

The above discussion, we attempted to proceed through applying the Boltzmann Principle and the Planck distribution function to the prime number theory. We considered dividing natural number

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x by an approximation of the number of prime number $\pi(x)$, that is, the function $x/\log(x)$. Thereby, we obtained that the function $R_\alpha^\pm(x)$. Furthermore, using the function $R_\alpha^\pm(x)$, we derived and define new distribution function $n^\pm(x,\alpha)$.

As mentioned above, modern physics is considered to be the special condition that the real number α be satisfied as follows :

$$\alpha = \frac{e}{\sqrt{2\pi h}}, \text{ that is, } h_\alpha = h, \tag{4.31}$$

where the constant h is Planck constant.

Furthermore, using new distribution function $R_\alpha^\pm$, we evaluated Von Koch's inequality that equivalent to the Riemann Hypothesis and the abc conjecture.

Namely, we consider the different system that Von Koch's inequality is satisfied as follows :

$$\alpha = \frac{2}{\sqrt{2\pi}}, \text{ that is, } h_\alpha = \frac{e}{2}. \tag{4.32}$$

Moreover, the abc conjecture is satisfied as follows :

$$\alpha = \frac{e}{\sqrt{2\pi}\log(\frac{\epsilon+2}{\epsilon+2})}, \text{ that is, } h_\alpha = \log(\frac{\epsilon+2}{\epsilon+1}). \tag{4.33}$$

Namely, we considered that there exist the relation between new distribution function $n^\pm(x,\alpha)$ and the Boltzmann principle, furthermore the Planck distribution function. Furthermore, it is meaningful that there exist the relation between statistical mechanics and Number theory.

In the future, based on the above measures, It may be possible that there exists new constant h_α and non-constant $k_f(x)$ that are different from the constants of modern physics, such as Planck's constant and Boltzmann's constant.

5. Generalization and application to dynamical systems.

We would describe some future issues on this section. Namely, we consider that that Entropy is related to dynamical systems described by logistic function models, such as bacterial and population growth.

5.1. *What are $S'_{\pi_f}(x)$ and $S''_{\pi_f}(x)$?*

We would like to consider what $S'_{\pi_f}(x)$ and $S''_{\pi_f}(x)$ are (Refer to Nicolis, Prigogine [18], [19]). Let $x > 1$ be a real number. These functions $S_{\pi_f}(x)$, $S'_{\pi_f}(x)$ and $S''_{\pi_f}(x)$ above are regarded as follows:

$$\begin{aligned} &S_{\pi_f}(x) : \textit{Entropy partitioned by } \pi_f(x), \\ &S'_{\pi_f}(x) : \textit{Entropy velocity } S_{\pi_f}(x), \ (\textit{Entropy generation}) \\ &S''_{\pi_f}(x) : \textit{Entropy acceleration } S_{\pi_f}(x). \end{aligned} \tag{5.1}$$

where these functions are satisfied as follows:

$$\begin{aligned} f(x) &:= \log(x), \\ Q_f(x) &:= \log(x), \\ \pi_f(x) &:= \frac{x}{f(x)} = \frac{x}{\log(x)}. \end{aligned} \tag{5.2}$$

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The first derivative of function $S_{\pi_f}(x)$, that is, $S'_{\pi_f}(x)$ and the second derivative of function $S_{\pi_f}(x)$, that is, $S''_{\pi_f}(x)$ can be describe as follows :

$$S'_{\pi_f}(x) = Q'_f(x)\log\left(1+\frac{1}{Q_f(x)}\right),$$
$$S''_{\pi_f}(x) = k_f(x)\left(\frac{-Q'_f(x)}{Q_f(x)(1+Q_f(x))}\right), \tag{5.3}$$

where the function $k_f(x)$ is regard as a function decided by a real number x and the function $\pi_f(x)$. The function $Q_f(x)$ can be regard as the position divided a real number x by $Q_f(x)$. The first derivative of $Q_f(x)$, that is, $Q'_f(x)$ can be regard as the slope of the function $Q_f(x)$ (the change $Q'_f(x)$ of the position $f(x)$, the charge or potential $Q_f(x)$ of the position $f(x)$). Entropy velocity can be regraded as Entropy generation (Refer to Nicolis, Prigogine [18], [19]). We consider the generalization below subsection.

5.2. Generalize of the function $S''_D(x)$.

We generalize the equation above (3.51) as follows. Let $x > 1$ be real number and $\xi > 0$ be a constant. The function $D(x)$ be a positive real valued function such that $D(x) \leq x$. The function $D(x)$ can be thought of as a division of x. Therefore, the above $S_D(x)$, $S'_D(x)$ and $S''_D(x)$ are regarded and defined as follows:

$$S_D(x) = (1+\frac{Q_D(x)}{\xi})\log(1+\frac{Q_D(x)}{\xi}) - \frac{Q_D(x)}{\xi}\log\frac{Q_D(x)}{\xi}, \tag{5.4}$$

$$S'_D(x) = \frac{Q'_D(x)}{\xi}\log\left(1+\frac{\xi}{Q_D(x)}\right), \tag{5.5}$$

$$S''_D(x) = k_D(x)\left(\frac{-Q'_D(x)}{Q_D(x)(\xi+Q_D(x))}\right), \tag{5.6}$$

where the relation between the functions $D(x)$ and $Q_D(x)$ are satisfied as follows:

$$D(x) = \frac{\xi x}{Q_D(x)}. \tag{5.7}$$

Moreover, the function $k_D(x)$ can be regard as the function decided by x, ξ and $D(x)$. The function $Q_D(x)$ can be regard as the position divided a real value ξx by $Q_D(x)$. The first derivative of $Q_D(x)$, that is, $Q'_D(x)$ can be regard as the change of the position by x and ξ. Each functions above are real valued functions.

5.3. New distribution function $R^{\pm}_{\alpha}(x)$.

We examine that the correspondence between generalized the equation(5.6) of $S''_D(x)$ and the equation(3.51) of $S''_{\pi_f}(x)$. We put as follows:

$$\begin{aligned} S''_D(x) &:= S''_{\pi_f}(x)\ (<0),\\ Q_D(x) &:= Q_f(x),\\ Q'_D(x) &:= Q'_f(x).\\ k_D(x) &:= R^{\pm}_{\alpha}(x),\\ \xi &:= 1. \end{aligned} \tag{5.8}$$

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Therefore, we can obtain the following equation :

$$S''_{\pi_f}(x)dx = R^{\pm}_{\alpha}(x)\frac{-Q'_f(x)}{Q_f(x)(1+Q_f(x))}dx. \tag{5.9}$$

This equation (5.9), that is (3.51), can be regarded as an application of the equation (5.6). In the subsections below, we examine some laws can be regarded as the accelerations of $S_D(x)$, that is (5.6).

5.4. Logistic function of the bacterial and the population growth.

We examine the relationship between generalized the function $S''_D(x)$ of equation (5.6) and the logistic function such that the bacterial and the population growth.

Let r and K be positive integer constants. For positive real number $t > 0$, let $N(t)$ be a positive real valued function. We put to set as follows:

$$\begin{aligned} S''_D(x) &:= -r(<0), \\ Q_D(x) &:= -\frac{N(t)}{K}, \\ Q'_D(x) &:= -\frac{1}{K}\frac{dN(t)}{dt}, \\ k_D(x) &:= 1, \\ x &:= t, \\ \xi &:= 1, \end{aligned} \tag{5.10}$$

where parameters r, K, t and the function $N(x)$ mean as follows :

$$\begin{aligned} r &: \textit{ the growth rate}, \\ N(t) &: \textit{ the bacterial or the population growth} \\ K &: \textit{ carrying capacity}, \\ t &: \textit{ time or step}. \end{aligned} \tag{5.11}$$

Thus, the equation(5.6) becomes the equation of the dynamical system as follows:

$$\int -rdt \geq \int \frac{-(\frac{-1}{K})}{-\frac{N(t)}{K}(1-\frac{N(t)}{K})}\frac{dN(t)}{dt}dt. \tag{5.12}$$

Thus, transforming the formula above as follows :

$$\int rdt \geq \int \frac{K}{N(t)(K-N(t))}dN(t). \tag{5.13}$$

Therefore, the following equation is obtained :

$$\frac{dN(t)}{dt} = rN(t)\frac{(K-N(t))}{K}. \tag{5.14}$$

The equation(5.14) above is the logistic function of dynamics. In other words, the equation (5.14) derived from the equation(5.6) can be regarded as an application of the dynamical system. Therefore, we consider that Entropy and dynamical system are closely related and are studying these applications.

5.5. Conclusion.

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We considered the following possibilities :
for sufficiently large $x > 1$ and a constant $\xi > 0$, the equation (5.6)

$$S''_D(x) = k_D(x)\left(\frac{-Q'_D(x)}{Q_D(x)(\xi + Q_D(x))}\right).$$

is regarded as a generalized expression and approximate representation of the equation (5.14)

$$\frac{dN(t)}{dt} = rN(t)\frac{(K - N(t))}{K},$$

where the function $D(x)$ needs to be chosen appropriately.

According to partition the Boltzmann principle by the number of prime numbers, Entropy $S_{\pi_f}(x)$ (the second law of π_f) can be related to Quantum mechanics(statistical mechanics) and Number theory.
Besides, Entropy acceleration $S''_D(x)$ (the second derivative of Entropy $S_D(x)$) can be related to dynamical systems described by logistic function models such as the bacterial and the population growth.
Furthermore, according to the theory of quantum mechanics, an atom can only take discrete spectral values. Similarly, the placement of planetary systems such as the solar system, galaxies, and clusters of galaxies can be considered to only take discrete spectral values. This discrete spectral arrangements seems to be related to Entropy.
Entropy was related to quantum mechanics by Boltzmann and Planck. Prigogine and Nicolis associated Entropy with complex systems. We further developed these concept of Entropy, proposed an extension of the Planck distribution, and attempted to relate Entropy to dynamical systems and classical mechanics. In addition we applied the expansion of Entropy to Number theory.
In the future, We would investigate why these relationships appear when the partitions number of the Boltzmann principle is partition by the number of prime numbers. We hope that this paper will serve as a bridge to further research and that Entropy will be further studied and many things will develop in the future. Increasing Entropy (the Second Law) does not mean becoming disordered.
On the contrary, the second law has the potential to cause movement and order in phenomena. (Refer to Fujino [22])

6. The proof of Lemma 3.6 , Lemma 3.7 and Lemma 3.8

6.1. The proof of Lemma 3.6 and Lemma 3.7

Proof. Let $n \geq 1$ a positive integer. For all sufficiently large positive real number $x > 0$, the following conditions are satisfied :

$$\begin{aligned} |(Q_f(x))^{(n)}| &= |\frac{(-1)^{n-1}(n-1)!}{x^n}| \\ &= \frac{(n-1)!}{x^n} \\ &\geq \frac{\sqrt{2\pi}(n-1)^{(n-1+\frac{1}{2})}e^{-(n-1)}}{x^n} \\ &(\because \text{Stirling's formula}: \ n! \geq \sqrt{2\pi}e^{-n}n^{n+\frac{1}{2}}) \\ &= \left(\frac{\sqrt{2\pi}(n-1)^{n-(\frac{1}{2})}}{e^{(n-1)}x^n}\right) \\ &= \sqrt{2\pi}(n-1)^{n-(\frac{1}{2})}\frac{1}{e^{(n-1)}x^n} \quad (*2). \end{aligned} \tag{6.1}$$

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Therefore, dividing the end of the formula(6.1), that is (*2), by the number $n^{n-(\frac{1}{2})}$, for sufficiently large $x > 1$, the following condition are satisfied :
Case 1) $x > n$: Because $x \geq x - (\frac{1}{2})$, therefore

$$\begin{aligned}(*2) &\geq \sqrt{2\pi}(\frac{n-1}{n})^{n-(\frac{1}{2})}\frac{1}{e^{(n-1)}x^n} \\ &\geq \sqrt{2\pi}(\frac{x-1}{x})^{x-(\frac{1}{2})}\frac{1}{e^{(n-1)}x^n} \quad (\because x > n) \\ &\geq \sqrt{2\pi}(\frac{x-1}{x})^{x}\frac{1}{e^{(n-1)}x^n} \\ &\quad (\because x \geq x-(\frac{1}{2}) \text{ and } (\frac{x-1}{x})^x \geq \lim_{x\to\infty}(\frac{x-1}{x})^x = e^{-1}) \\ &\geq \sqrt{2\pi}e^{-1}\frac{1}{e^{(n-1)}x^n}.\end{aligned} \tag{6.2}$$

Case 2) $n \geq x$:

$$\begin{aligned}(*2) &\geq \sqrt{2\pi}(\frac{n-1}{n})^{n-(\frac{1}{2})}\frac{1}{e^{(n-1)}x^n} \\ &\geq \sqrt{2\pi}(\frac{n-1}{n})^{n}\frac{1}{e^{(n-1)}x^n} \\ &\quad (\because n \geq n-(\frac{1}{2}) \text{ and } (\frac{n-1}{n})^n \geq \lim_{n\to\infty}(\frac{n-1}{n})^n = e^{-1}) \\ &\geq \sqrt{2\pi}e^{-1}\frac{1}{e^{(n-1)}x^n}.\end{aligned} \tag{6.3}$$

Therefore, using Case 1) and Case 2) above, for all sufficiently large $x > 0$, the following inequality is satisfied :

$$|(Q_f(x))^{(n)}| \geq \sqrt{2\pi}e^{-1}\frac{1}{e^{(n-1)}x^n}. \tag{6.4}$$

Therefore, the following conditions are satisfied :

$$\begin{aligned}R_m^+(x) &\geq \lim_{N\to\infty}\sum_{n=1}^{N}\left|(Q_f(x))^{(n)}\right| \\ &\geq \lim_{N\to\infty}\sqrt{2\pi}e^{-1}\sum_{n=1}^{N}\left|\frac{(-1)^{n-1}}{e^{(n-1)}x^n}\right| \\ &\geq \lim_{N\to\infty}\sqrt{2\pi}e^{-1}\sum_{n=1}^{N}\frac{(-1)^{n-1}}{e^{(n-1)}x^n}.\end{aligned} \tag{6.5}$$

Put the function $A(x)$ as follows:

$$A(x) := \sum_{n=1}^{N}\frac{(-1)^{n-1}}{e^{(n-1)}x^n}. \tag{6.6}$$

Therefore,

$$A(x) = \frac{1}{e^0x^1} - \frac{1}{e^1x^2} + \frac{1}{e^2x^3} - \cdots, \tag{6.7}$$

$$\frac{1}{ex}A(x) = \frac{1}{e^1x^2} - \frac{1}{e^2x^3} + \frac{1}{e^3x^4} - \cdots + \frac{(-1)^N}{e^Nx^{(N+1)}}. \tag{6.8}$$

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Add the equation(6.7) and (6.8). Hence, the following conditions are satisfied :

$$\left(1+\frac{1}{ex}\right)A(x)=\lim_{N\to\infty}\left(\frac{1}{x}+\frac{(-1)^N}{e^N x^{(N+1)}}\right)=\frac{1}{x}. \tag{6.9}$$

Therefore,

$$\lim_{N\to\infty}A(x)=\frac{1}{x}\left(\frac{1}{1+\frac{1}{ex}}\right)=\frac{e}{ex+1}. \tag{6.10}$$

Therefore, for integer $m>1$, the function R_m^+ is approximated as follows:

$$R_m^+(x)\geq\sqrt{2\pi}e^{-1}\lim_{N\to\infty}A(x)=\frac{\sqrt{2\pi}}{ex+1}. \tag{6.11}$$

Here, the following conditions are satisfied :

$$R_m^+(x)>\frac{1}{x}>\frac{\sqrt{2\pi}\alpha}{ex+1}=R_\alpha^+(x). \tag{6.12}$$

Put the positive real number $\alpha>0$ such that as follows:

$$x\geq\frac{-1}{e-\sqrt{2\pi}\alpha}. \tag{6.13}$$

Consequently, the following conditions are satisfied :

$$R_m^+(x)\geq\sqrt{2\pi}e^{-1}\alpha\lim_{N\to\infty}A(x)=\frac{\sqrt{2\pi}\alpha}{ex+1}=R_\alpha^+(x), \tag{6.14}$$

$$x\geq\frac{-1}{e-\sqrt{2\pi}\alpha},\quad\text{that is,}\quad\frac{ex+1}{\sqrt{2\pi}x}\geq\alpha. \tag{6.15}$$

Furthermore, for all sufficiently large $x>1$, the following conditions are satisfied:

$$\frac{1}{x}(2+\log(x))\geq k_f(x)\geq R_\alpha^+(x)=\frac{\sqrt{2\pi}\alpha}{ex+1}, \tag{6.16}$$

$$\frac{e}{\sqrt{2\pi}}(2+\log(x))\geq\frac{ex+1}{\sqrt{2\pi}x}\geq\alpha>0, \tag{6.17}$$

where $\alpha>0$ is a real number. (The end of the proof of Lemma3.6)

By the same method, for integer $m>1$, the function R_m^- is approximated as follows:

$$\sqrt{2\pi}e^{-1}\alpha\lim_{N\to\infty}A(x)=\frac{\sqrt{2\pi}\alpha}{ex-1}=R_\alpha^-(x), \tag{6.18}$$

$$x\geq\frac{1}{e-\sqrt{2\pi}\alpha}. \tag{6.19}$$

For all $x>1$, the following conditions are satisfied :

$$\frac{1}{x}(2+\log(x))\geq k_f(x)\geq R_\alpha^-(x)=\frac{\sqrt{2\pi}\alpha}{ex-1}, \tag{6.20}$$

$$\frac{e}{\sqrt{2\pi}}(2+\log(x))\geq\frac{ex-1}{\sqrt{2\pi}x}\geq\alpha>0. \tag{6.21}$$

(The end of the proof of Lemma3.7). □

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6.2. The proof of Lemma 3.8

Proof. Let $n \geq 1$, and $x > 0$ is sufficiently large.

$$\begin{aligned}
|(Q_f(x))^{(n)}| &= |\frac{(-1)^{n-1}(n-1)!}{x^n}| \\
&= \frac{(n-1)!}{x^n} \\
&\leq \frac{e(n-1)^{(n-1+\frac{1}{2})}e^{-(n-1)}}{x^n} \\
&(\because \text{Stirling's formula: } ee^{-n}n^{n+\frac{1}{2}} \geq n! \geq \sqrt{2\pi}e^{-n}n^{n+\frac{1}{2}}) \\
&= \left(\frac{e(n-1)^{n-(1/2)}}{e^{(n-1)}x^n}\right) \\
&= e(n-1)^{n-(1/2)}\frac{1}{e^{(n-1)}x^n} \\
&= e\left(\frac{n-1}{x}\right)^n\frac{(n-1)^{-1/2}}{e^{(n-1)}} \\
&= e\left(\frac{x-1}{x}\right)^x\frac{1}{e^{(n-1)}(x-1)^{1/2}} \quad (x \gg n) \\
&\leq e\left(\frac{x-1}{x}\right)^x\frac{1}{e^{(n-1)}(x-1)^{1/2}} \\
&\leq ee^{-1}\frac{1}{e^{(n-1)}(x-1)^{1/2}} \quad (\because \lim_{x\to\infty}(\frac{x-1}{x})^x = \frac{1}{e}) \\
&\leq \frac{1}{e^{(n-1)}(x-1)^{1/2}}.
\end{aligned} \tag{6.22}$$

Therefore, the following inequality are satisfied :

$$|(Q_f(x))^{(n)}| \leq \frac{1}{e^{(n-1)}(x-1)^{1/2}}. \tag{6.23}$$

Furthermore, the inequality

$$|(Q_f(x))^{(n)}| > |(Q_f(x))^{(n+1)}|. \tag{6.24}$$

is satisfied. Besides, the following relation are satisfied :

$$if\ n\ is\ even \quad : \quad 0 < (Q_f(x))^{(n)} \leq \frac{1}{e^{(n-1)}(x-1)^{1/2}}, \tag{6.25}$$

$$if\ n\ is\ odd \quad : \quad 0 > (Q_f(x))^{(n)} \geq \frac{-1}{e^{(n-1)}(x-1)^{1/2}}. \tag{6.26}$$

The discussion above, the following are satisfied :

$$\begin{aligned}
R_m^+(x) &= \lim_{N\to\infty}\sum_{n=1}^{N}\left|(Q_f(x))^{(n)}\right| \\
&\leq \lim_{N\to\infty}\sum_{n=1}^{N}\frac{1}{e^{(n-1)}(x-1)^{1/2}}.
\end{aligned} \tag{6.27}$$

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Put the function $A_2(x)$ as follows :

$$A_2(x) = \sum_{n=1}^{N} \frac{(-1)^{n-1}}{e^{(n-1)}(x-1)^{1/2}}. \tag{6.28}$$

Hence, we can describe as follows :

$$A_2(x) = \frac{1}{e^0(x-1)^{1/2}} - \frac{1}{e^1(x-1)^{1/2}} + \frac{1}{e^2(x-1)^{1/2}} - \cdots, \tag{6.29}$$

$$\frac{1}{e}A_2(x) = \qquad \frac{1}{e^1(x-1)^{1/2}} - \frac{1}{e^2(x-1)^{1/2}} + \cdots + \frac{(-1)^N}{e^N(x-1)^{1/2}} \tag{6.30}$$

Add the equivalent(6.29) and (6.30), the following equivalent are satisfied :

$$\begin{aligned}\left(1+\frac{1}{e}\right)A_2(x) &= \lim_{N\to\infty}\left(\frac{1}{(x-1)^{1/2}} + \frac{(-1)^N}{e^N(x-1)^{1/2}}\right) \\ &= \frac{1}{(x-1)^{1/2}}.\end{aligned} \tag{6.31}$$

Therefore,

$$\begin{aligned}\lim_{N\to\infty} A_2(x) &= \frac{1}{(x-1)^{1/2}}\left(\frac{1}{1+\frac{1}{e}}\right) \\ &= \frac{1}{(x-1)^{1/2}}\frac{e}{e+1}.\end{aligned} \tag{6.32}$$

Therefore, the function $R_m^+(x)$ is satisfied as follows :

$$R_m^+(x) \le \lim_{N\to\infty} A_2(x) = \frac{1}{(x-1)^{1/2}}\frac{e}{e+1}. \tag{6.33}$$

where $m > 1$.

For all sufficiently large $x > 1$, the following conditions are satisfied :

$$k_f(x) \le \frac{1}{x}(2+\log(x)) \le \frac{1}{(x-1)^{1/2}}\frac{e}{e+1}. \tag{6.34}$$

By the same method, R_m^+ is approximated as follows :

$$R_m^+(x) \le \frac{1}{(x-1)^{1/2}}\frac{e}{e-1}. \tag{6.35}$$

where $m > 1$.

For all sufficiently large $x > 1$, the following conditions are satisfied :

$$k_f(x) \le \frac{1}{x}(2+\log(x)) \le \frac{1}{(x-1)^{1/2}}\frac{e}{e-1}. \tag{6.36}$$

The proof of $R_m^-(x)$ is similar. □

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