A Generalization of the Sum of Divisors Function

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Abstract

A generalization of the sum of divisors function involves a recursive definition. This leads to variants of superabundant numbers, colossally abundant numbers, and Gronwall's theorem (relevant to the Riemann hypothesis).

1   Introduction

For real or complex α and any integer $n \geq 1$ we define $\sigma_\alpha = \sum_{d|n} d^\alpha$, the sum of the αth powers of the divisors of n. The functions σ_α are called divisor functions. The sum of divisors function is commonly denoted by $\sigma(n)$. The divisor functions are multiplicative and a Dirichlet product of $\sigma(n)$ with the constant function $g(n) = 1$ gives another multiplicative function. Similarly, the Dirichet product of this multiplicative function with the constant function gives yet another multiplicative function.

2   Multiplicative Arithmetical Functions

An arithmetical function f is called multiplicative if f is not identically zero and if $f(mn) = f(m)f(n)$ whenever $(m, n) = 1$. Let $f_\alpha(n) = n^\alpha$. This function is completely multiplicative (m and n are not required to be relatively prime). We denote the function f_α by N^α. The divisor functions are multiplicative since $\sigma_\alpha = \mu \cdot N^\alpha$, the Dirichlet product of two multiplicative functions. The Dirichlet product of arithmetical functions f and g has the form $\sum_{d|n} f(d)g(\frac{n}{d})$. The

Möbius function μ is defined as follows: $\mu(1) = 1$. If $n > 1$, write $n = p_1^{a_1} \cdots p_k^{a_k}$. Then $\mu(n) = (-1)^k$ if $a_1 = a2 = \cdots = a_k = 1$ or $\mu(n) = 0$ otherwise. That the Möbius function is multiplicative is easily seen from its definition.

(1) If f and g are multiplicative, so is their Dirichlet product $f \cdot g$.

See Apostol's [1] Theorem 2.14.

3 Recursive Definition of Divisor Functions

Let $r_{n,1} = \sigma(n)$, $r_{n,2} = \sum_{d|n} r_{d,1}$, $r_{n,3} = \sum_{d|n} r_{d,2}$, etc. Note that these are simplified Dirichlet products so that the functions are multiplicative.

(2) If n is squarefree, $\prod_{p|n}(p+1) = r_{n,1}$. More generally, if p^m divides n and p^{m+1} does not divide n, then $p^m + p^{m-1} + p^{m-2} + \ldots + 1$ divides $r_{n,1}$, that is, $\frac{p^{m+1}-1}{p-1}$ divides $r_{n,1}$.

(3) If n is squarefree, $\prod_{p|n}(p+2) = r_{n,2}$. More generally, if p^m divides n and p^{m+1} does not divide n, then $p^m + 2p^{m-1} + 3p^{m-2} + \ldots + (m+1)$ divides $r_{n,2}$, that is, $\frac{\frac{p^{m+2}-1}{p-1}-(m+2)}{p-1}$ divides $r_{n,2}$.

(4) If n is squarefree, $\prod_{p|n}(p+3) = r_{n,3}$. More generally, if p^m divides n and p^{m+1} does not divide n, then $p^m + 3p^{m-1} + 6p^{m-2} + \ldots + (m+1)(m+2)/2$ divides $r_{n,3}$. An expression similar to the above can be derived using the binomial theorem.

4 Superabundant Numbers

A natural number is called a superabundant number if $\sigma(n)/n > \sigma(k)/k$ for $1 \le k \le n-1$. The first few superabundant numbers are 2, 4, 6, 12, 24, 36, 48, 60, 120, 180, 240, 360, 720, 840, 1260, 1680, 2520, 5040,.... Gronwall's theorem (Theorem 323 in Hardy and Wright's [2] book) is

(5) $\lim_{n \to \infty} \sup \frac{\sigma(n)}{n \log \log n} = e^\lambda$

λ denotes Euler's constant. A plot of $\frac{\sigma(n)}{n \log \log n}$ for $n = 2$, 4, 6, 12, 24, 36, 48, 60, 120, 180, 240, 360, 720, 840, 1260, 1680, 2520, 5040, 10080, 15120, 25200, 27720, 55440, 110880, 166320, 277200, 332640, 554400, 665280, 720720, and 1441440 (the superabundant numbers less than 2 million) is
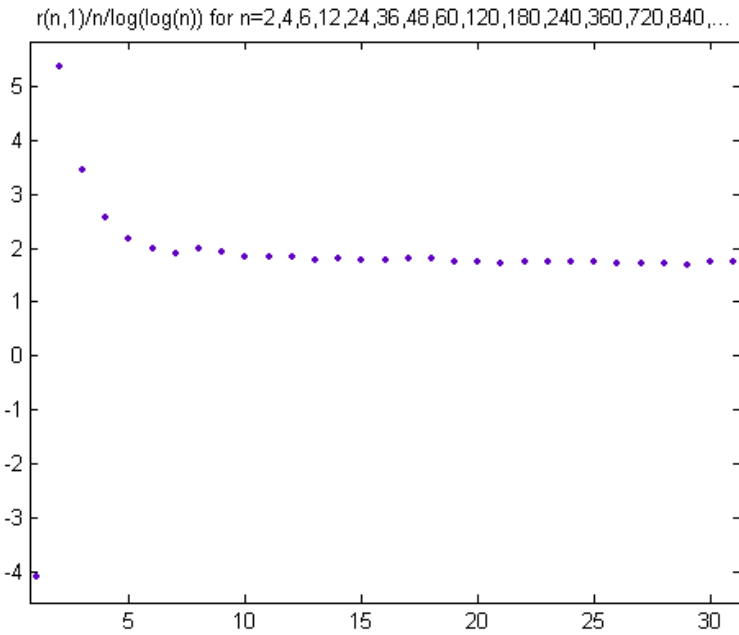
Figure 1: Plot of r(n,1) ratios

e^λ is approximately equal to 1.7811. A quadratic least-squares fit of $r_{n,1}/n$ versus $\log \log n$ for these n values is
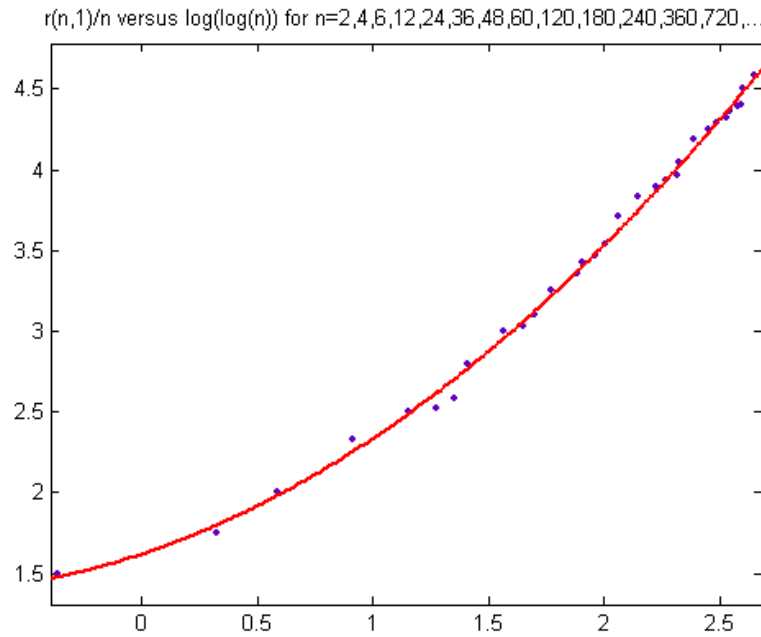
Figure 2: Quadratic least-squares fit

$p_1 = 0.2386$ with a 95% confidence interval of $(0.21, 0.2672)$, $p_2 = 0.4811$ with a 95% confidence interval of $(0.3966, 0.5657)$, $p_3 = 1.62$ with a 95% confidence interval of $(1.556, 1.683)$, SSE=0.07629, R-squared=0.9966, and RMSE=0.0522.

Superabundant numbers consist of a product of all the small primes up to some bound with exponents which are non-increasing as the prime increases. Alaoglu and Erdös' [3] theorems on superabundant numbers are

Theorem 1 *If $n = 2^{k_2} \cdots p^{k_p}$, then $k_2 \geq k_3 \geq \ldots \geq k_p$.*

Theorem 2 *Let $q < r$, and set $\beta = [k_q \log q / \log r]$. Then k_r has one of the three values: $\beta - 1$, $\beta + 1$, β.*

Theorem 3 *If p is the largest prime factor of n, then $k_p = 1$, except when $n = 4$, 36.*

In the remainder of the theorems, p always denotes the largest prime factor of n.

Theorem 4 *If q is either the greatest prime of exponent k or the least prime of exponent $k-1$, and if $q^{1-\theta} > \log p$, then $q^k = (p \log p / \log q)[1 + O(\log p / q^{1-\theta} \log q)]$.*

If Riemann's hypothesis is true any $\theta > 1/2$ can be used.

Theorem 5 *If $k_q = k$ and $q < (\log p)^\alpha$, where α is a constant, then (i)* $\log \frac{q^{k+1}-1}{q^{k+1}-q} > \frac{\log q}{p \log p}[1 + O((\log \log p)^2 / \log p \log q)]$, *(ii)* $\log \frac{q^{k+2}-1}{q^{k+2}-q} < \frac{\log q}{p \log p}[1 + O((\log \log p)^2 / \log p \log q)]$.

Let $K_q + 1$ be the integral part and θ_q the fractional part of $\log[(q^{1+\epsilon} - 1)/(q^\epsilon - 1)]/\log q$, where $\epsilon = \log(1 + 1/p) \log p$.

Theorem 6 *(i) If $\epsilon_q < \theta_q \le 1 - \epsilon_q$, then $k_q = K_q$. (ii) If $\theta_q \le \epsilon_q$, then $k_q = K_q$ or $k_q = K_q - 1$. (iii) If $1 - \epsilon_q < \theta_q$, then $k_q = K_q$ or $K_q + 1$.*

Theorem 7 $p \sim \log n$.

Theorem 8 *The quotient of two consecutive superabundant numbers tends to 1.*

In the proof of Theorem 8 it was shown that the ratio of two consecutive superabundants n and n' is less than $1 + c(\log \log n)^2 / \log n$. The order of $(\log p)^c$ is used.

Theorem 9 *The number of superabundant numbers less than x exceeds* $c \log x \log \log x / (\log \log \log x)^2$.

Similar superabundant numbers can be defined for $r_{n,2}$, $r_{n,3}$, $r_{n,4}$, etc. and usually occur at the same n values. For example, the superabundant numbers for $r_{n,2}$ satisfy $r_{n,2}/n > r_{k,2}/k$ for $1 \le k \le n - 1$. As for the usual superabundant numbers, these superabundant numbers consist of a product of all the small primes up to some bound with exponents which are non-increasing as the prime increases. Exceptions to Theorem 2 for $r_{n,2}$, $r_{n,3}$, $r_{n,4}$, $r_{n,5}$, and $r_{n,6}$ and $n < 2000000$ occur at $n = 96$, 480, 5760, 241920, and 1209600. In these cases, k_r has the value $\beta - 2$ for $q = 2$ and $r = 3$. This conclusion is based on the assumption that the brackets in Theorem 2 denote the floor function. There are also exceptions to Theorem 3 when $q = 2$ and $r = 3$ (such as when $n = 6$, 12, 24,...). The number of superabundant numbers less than 2000000 for $r_{n,2}$, $r_{n,3}$, $r_{n,4}$,...,$r_{n,12}$ is 38, 39, 44, 43, 41, 41, 42, 48, 48, 49, and 51 respectively. The number of ordinary superabundant numbers less than 2000000 is 31, so the counts for the new superabundant numbers are consistent with Theorem 9. All of these theorems should be applicable to the new superabundant numbers. As Alaoglu and Erdös state in the introduction of their article - "In comparing the magnitude of $\sigma(n)$ and $\sigma(n')$ it is clear from the multiplicative property of $\sigma(n)$ that one need only consider the behavior of those primes which divide the two numbers to different powers. The same is true for $d(n)$ and $\sigma(n)/n$." This appears to be the only property of the sum of divisors function that they use in proving their theorems.

A plot of $\sqrt{\frac{r_{n,2}}{n \log \log n}}$ for $n = 2$, 4, 6, 12, 24, 36, 48, 60, 72, 120, 180, 240, 360, 720, 1440, 1680, 2160, 2520, 5040, 10080, 15120, 20160, 25200, 30240,

50400, 55440, 110880, 166320, 221760, 277200, 332640, 554400, 665280, 831600, 1108800, 1330560, 1441440, and 1663200 is
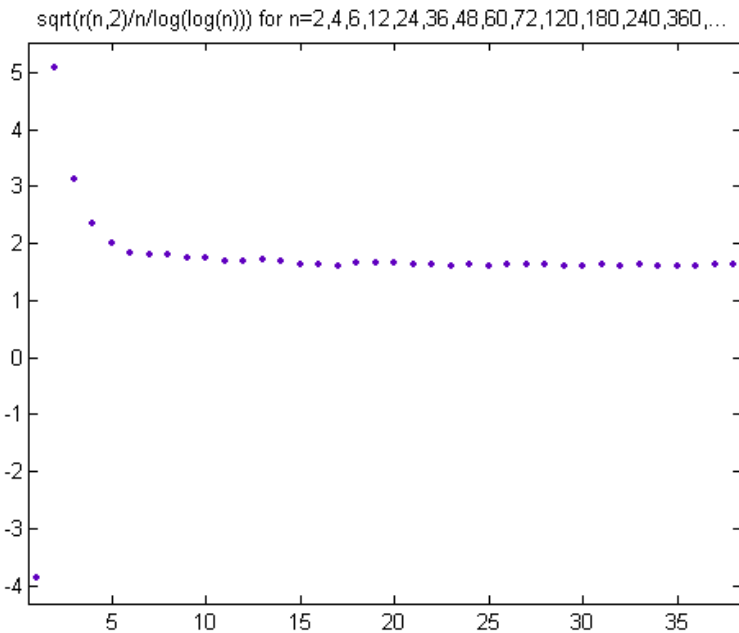
Figure 3: Square root of r(n,2) ratios

In this variant of Gronwall's formula, the supremum limit is unknown but is slightly less than e^λ.

A plot of $\sqrt[3]{\frac{r_{n,3}}{n \log \log n}}$ for $n = 2$, 4, 6, 8, 12, 24, 36, 48, 60, 72, 120, 180, 240, 360, 720, 1440, 2160, 2520, 4320, 5040, 10080, 15120, 20160, 30240, 50400, 55440, 60480, 100800, 110880, 151200, 166320, 221760, 302400, 332640, 554400, 665280, 1108800, 1330560, and 1663200 is
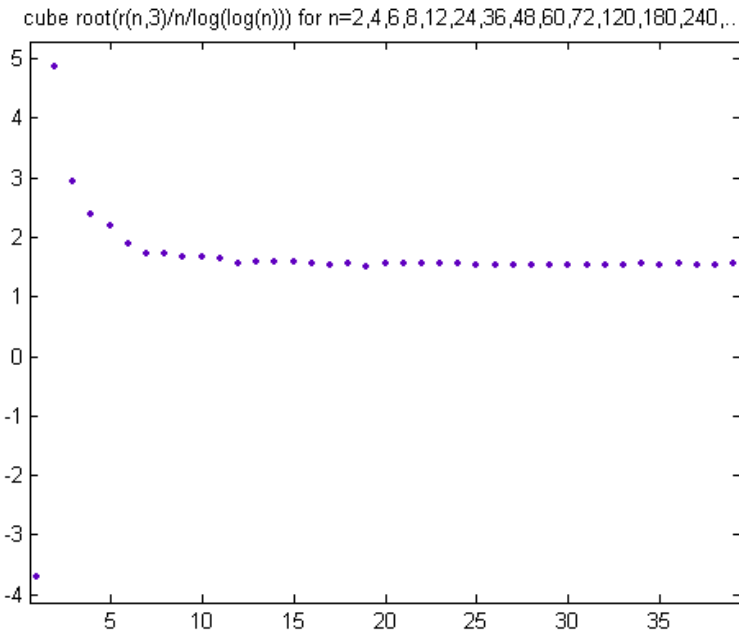
Figure 4: Cube root of r(n,3) ratios

In this variant of Gronwall's formula, the supremum limit is smaller than above.

A plot of $\sqrt[4]{\frac{r_{n,4}}{n \log\log n}}$ for $n = 2, 4, 6, 8, 12, 24, 36, 48, 72, 96, 120, 144, 240, 360,$ 480, 720, 1440, 2160, 2520, 2880, 4320, 5040, 8640, 10080, 15120, 20160, 30240, 50400, 60480, 100800, 110880, 120960, 151200, 181440, 221760, 302400, 332640, 55440, 604800, 665280, 1108800, 1330560, 1663200, and 1995840 is
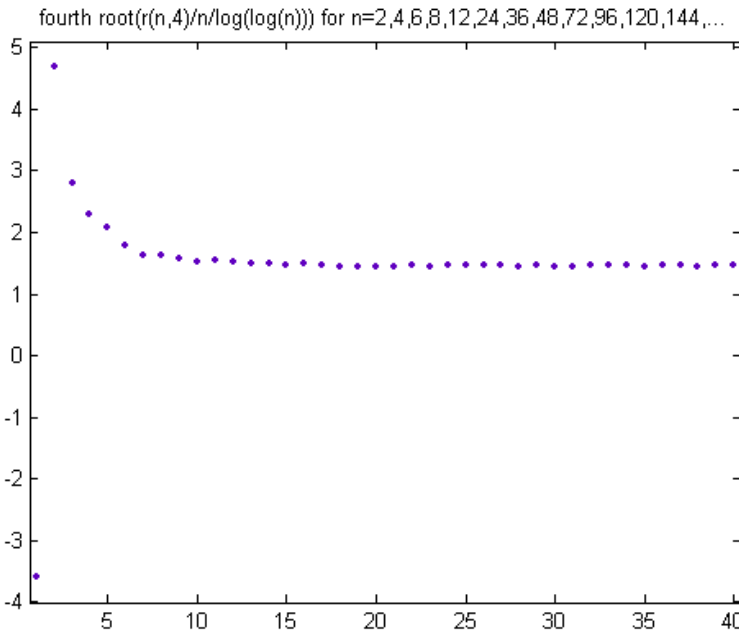
Figure 5: Fourth root of r(n,4) ratios

In this variant of Gronwall's formula, the supremum limit is still smaller.

A quadratic least-squares fit of the smallest positive $\sqrt[i]{\frac{r_{n,i}}{n \log \log n}}$ values for $n <$ 2000000 and $i = 1, 2, 3, \ldots, 12$ is
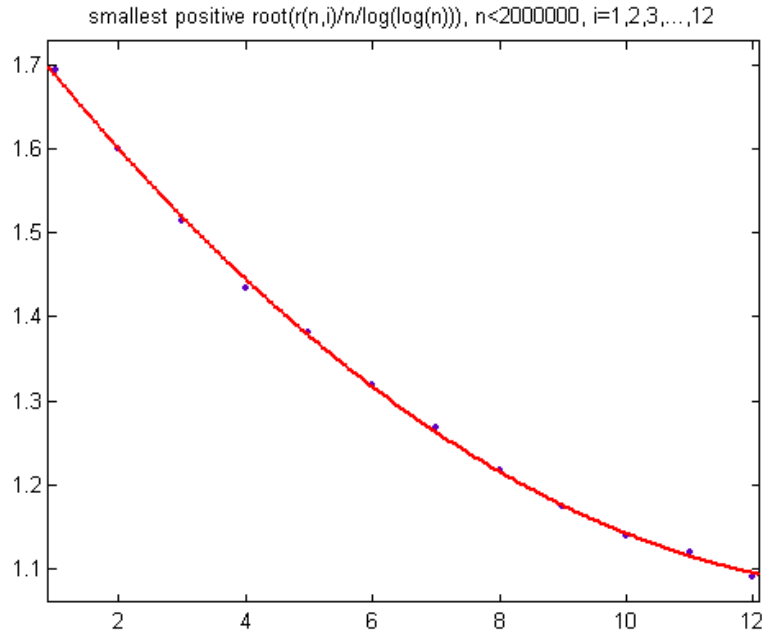
Figure 6: Quadratic least-squares fit

$p_1 = 0.003381$ with a 95% confidence interval of $(0.003023, 0.003739)$, $p_2 = -0.09788$ with a 95% confidence of $(-0.1027, -0.0931)$, $p_3 = 1.782$ with a 95% confidence interval of $(1.769, 1.796)$, SSE=0.0003007, R-squared =0.9993, and RMSE=0.00578. The first value is for $\frac{r_{n,1}}{n \log \log n}$. The values are 1.6943, 1.5998, 1.5135, 1.4339, 1.3809, 1.3190, 1.2683, 1.2181, 1.1736, 1.1395, 1.1199, and 1.0902 respectively. For small sample sizes, quadratic curves give good approximations of some logarithmic functions.

A cubic least-squares fit of the smallest positive $\sqrt[i]{\frac{r_{n,i}}{n \log \log n}}$ values for $n < 2000000$ and $i = 1, 2, 3, \ldots, 16$ is
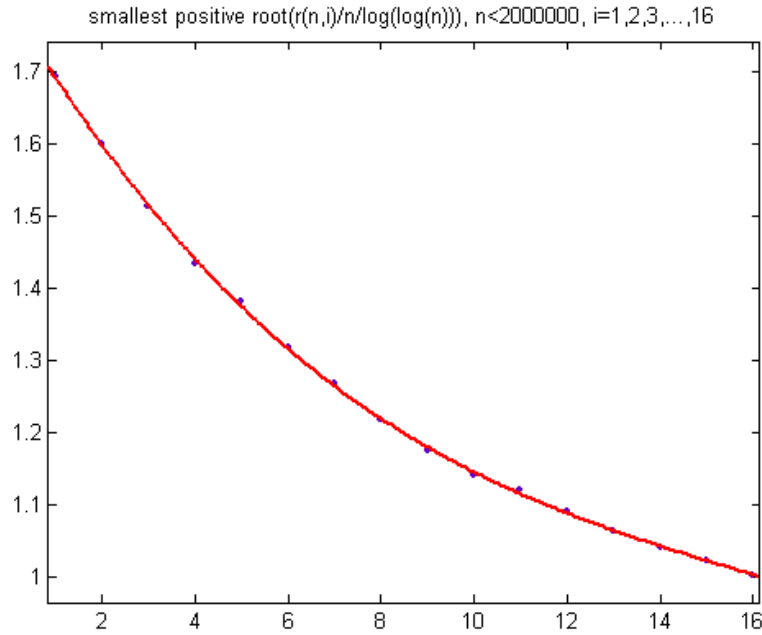
Figure 7: Cubic least-squares fit

RMSE=0.0002309, R-squared=0.9997, and RMSE=0.004386. The values are 1.6943, 1.5998, 1.5135, 1.4339, 1.3809, 1.3190, 1.2683, 1.2181, 1.1736, 1.1395, 1.1199, 1.0902, 1.0623, 1.0392, 1.0278, and 1.0018 respectively. Eventually, higher order curves will be required to approximate the values (which are expected to approach zero).

5 Euler's Phi Function and the Sum of Divisors Function

Hardy and Wright's Theorem 328 is

(6) $\lim_{n \to \infty} \inf \frac{\phi(n) \log \log n}{n} = e^{-\lambda}$

$\phi(n)$ denotes Euler's phi function. Their Theorem 329 is

(7) $A < \frac{\sigma(n)\phi(n)}{n^2} < 1$ for a positive constant A.

This relationship between $\phi(n)$ and $\sigma(n)$ is used to prove Gronwall's theorem.

Superabundant numbers can also be defined for the largest $\frac{r_{n,2}\phi(n)}{n^2}$ values. A plot of $\frac{r_{n,2}\phi(n)}{n^2 \log \log n}$ for $n = 3, 4, 8, 16, 24, 32, 48, 72, 144, 288, 432, 576, 720, 864,$

1440, 2160, 2880, 3600, 4320, 7200, 8640, 10800, 14400, 15120, 20160, 21600, 30240, 43200, 60480, 75600, 100800, 120960, 151200, 302400, 604800, 907200, 1663200, and 1814400 is
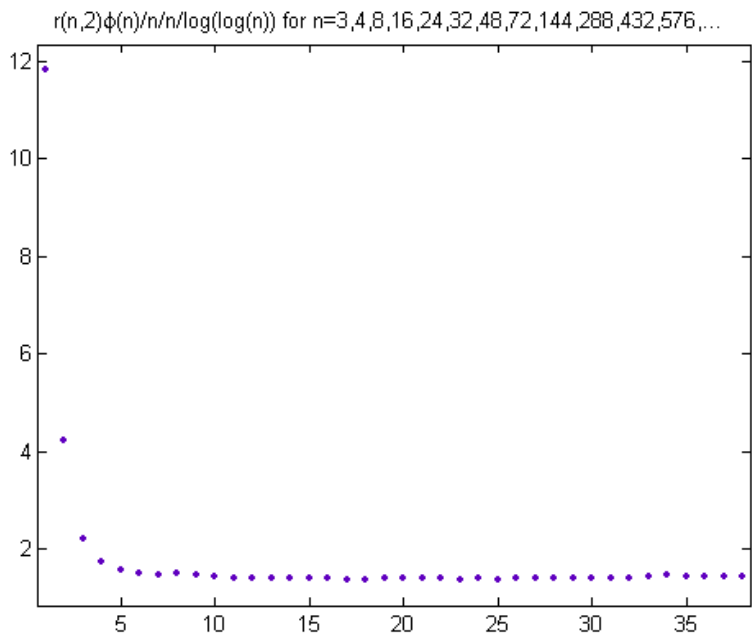
Figure 8: Plot of r(n,2) ratios

In this variant of Gronwall's formula, the supremum limit is unknown. A quadratic least-squares fit of $\frac{r_{n,2}\phi(n)}{n^2}$ versus $\log\log n$ for these n values is
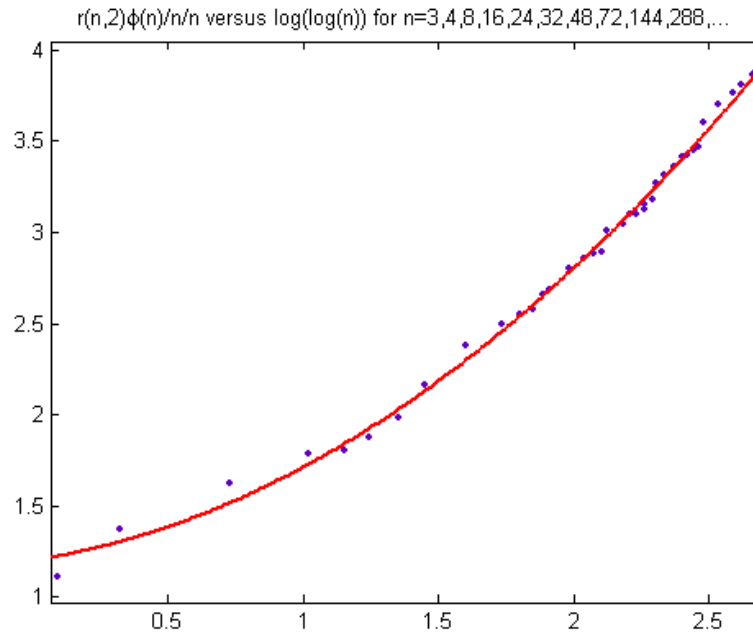
Figure 9: Quadratic least-squares fit of $r_{n,2}$ ratios

$p_1 = 0.2912$ with a 95% confidence interval of $(0.2573, 0.325)$, $p_2 = 0.2204$ with a 95% confidence interval of $(0.1131, 0.3276)$, $p_3 = 1.201$ with a 95% confidence interval of $(1.121, 1.281)$, SSE=0.08065, R-squared=0.9958, and RMSE=0.048. Note the resemblance to the curve in Figure 2.

Superabundant numbers can also be defined for the largest $\frac{r_{n,3}\phi(n)\phi(n)}{n^3}$ values. A plot of $\sqrt{\frac{r_{n,3}\phi(n)\phi(n)}{n^3 \log\log n}}$ for $n = 5$, 7, 8, 16, 32, 64, 128, 256, 288, 432, 576, 864, 1728, 3456, 5184, 6912, 10368, 20736, 31104, 41472, 43200, 86400, 129600, 172800, 259200, 518400, 777600, 1036800, 1296000, and 1555200 is
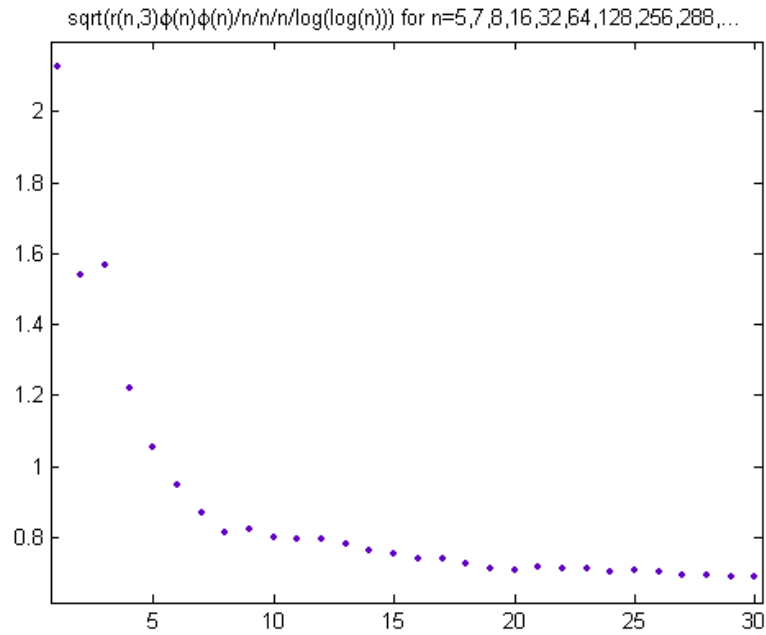
Figure 10: Square root of r(n,3) ratios

Superabundant numbers can also be defined for the largest $\frac{r_{n,4}\phi(n)\phi(n)\phi(n)}{n^4}$ values. A plot of $\sqrt[3]{\frac{r_{n,4}\phi(n)\phi(n)\phi(n)}{n^4 \log\log n}}$ for $n = 9$, 11, 13, 16, 32, 64, 128, 256, 512, 1024, 1728, 3456, 5184, 6912, 10368, 20736, 31104, 41472, 62208, 124416, 248832, 373248, 497664, 518400, 777600, 1036800, and 1555200 is
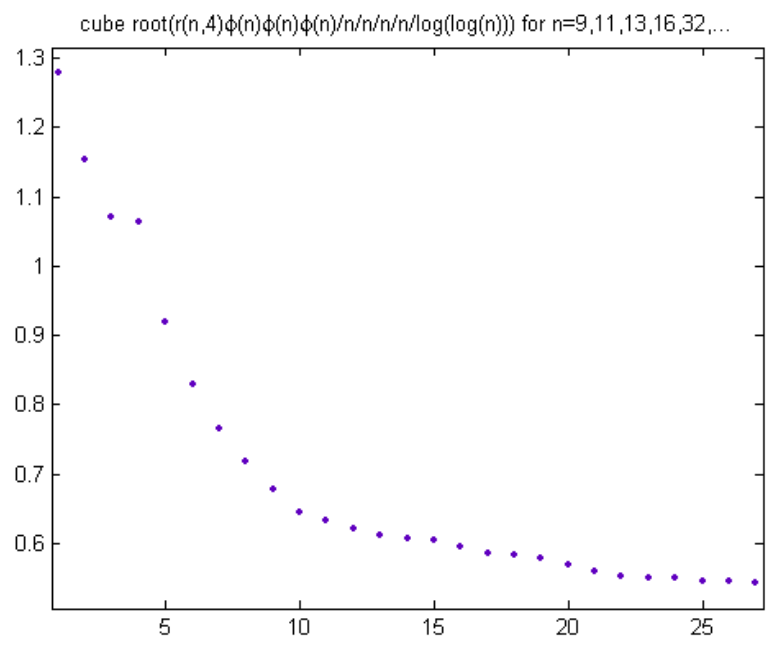
Figure 11: Cube root of r(n,4) ratios

Other superabundant numbers are similarly defined for $r_{n,5}$, $r_{n,6}$, $r_{n,7}$, etc. The smallest of the above values for $r_{n,2}$, $r_{n,3}$, $r_{n,4}$, $r_{n,5}$, $r_{n,6}$, $r_{n,7}$, and $r_{n,8}$ are 1.3734, 0.6888, 0.5427, 0.4862, 0.4561, 0.4350, and 0.4187 respectively. A plot of the normalized reciprocals of these minima (decremented by 1/1.3734) and $\log(i)$ for $i = 1, 2, 3, 4, 5, 6$, and 7 is
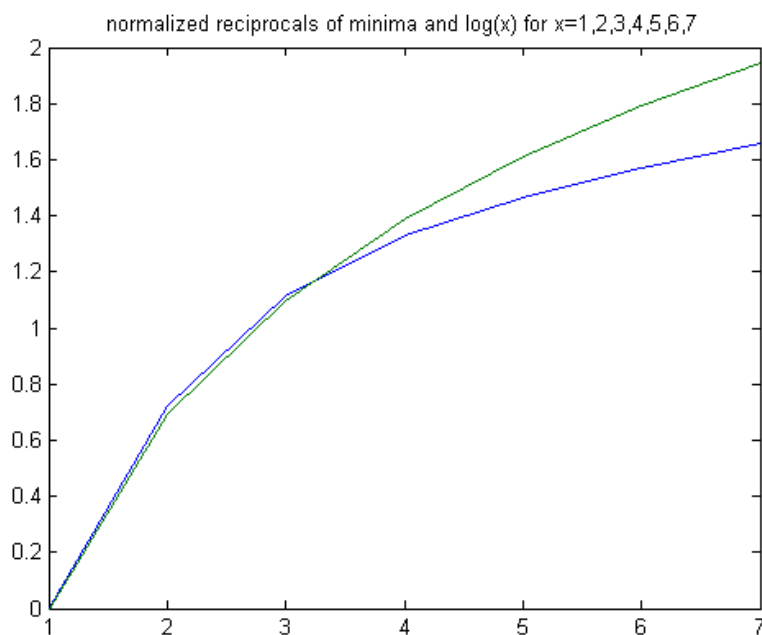
Figure 12: Normalized reciprocals of minimum and $\log(x)$

The logarithm increases more rapidly.

6 Colossally Abundant Numbers

In the above, the n values start with a few odd values followed by powers of 2. Afterwards, the n values are the product of powers of 2 and 3, the product of powers of 2, 3, and 5, the product of powers of 2, 3, 5, and 7, etc. For $r_{n,2}$, the odd numbers and powers of 2 are 3, 4, 8, and 16. For $r_{n,3}$, the odd numbers and powers of 2 are 5, 7, 8, 16, 32, 64, 128, 256. For $r_{n,4}$, the odd numbers and powers of 2 are 9, 11, 13, 16, 32, 64, 128, 256, 512, and 1024. For $r_{n,5}$, the odd numbers and powers of 2 are 13, 17, 19, 23, 25, 27, 32, 64, 128, 256, 512, 1024, 2048, 4096, and 8192. For $r_{n,6}$, the odd numbers and powers of 2 are 19, 23, 29, 31, 37, 49, 64, 128, 256, 512, 1024, 2048, 4096, 8192, 16384, and 32768. For $r_{n,7}$, the odd numbers and powers of 2 are 29, 31, 37, 41, 43, 47, 49, 81, 125, 128, 243, 256, 512, 1024, 2048, 4096, 8192, 16384, 32768, 65536, and 131072. Note that a power of 3 (243) is interspersed between 128 and 256. For $r_{n,8}$, the odd numbers and powers of 2 are 31, 37, 41, 43, 47, 53, 59, 61, 67, 121, 125, 243, 256, 512, 1024, 2048, 4096, 8192, 16384, 32768, 65536, 131072, 262144, 524288, and 1048576. For $r_{n,9}$, the odd numbers and powers of 2 (up to 2 million) are 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 121, 169, 243, 289, 343, 512, 729, 1024, 2048, 4096, 8192, 16384, 32768, 65536, 131072, 262144, 524288, 1048576,.... Note that a power of 3 (729) is interspersed between 512 and 1024.

For $r_{n,10}$, the odd numbers and powers of 2 (up to 2 million) are 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 169, 289, 343, 625, 729, 1024, 2048, 4096, 8192, 16384, 32768, 65536, 131072, 262144, 524288, 1048576,.... For $r_{n,11}$, the odd numbers and powers of 2 (up to 2 million) are 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 169, 289, 361, 529, 625, 1331, 2048, 4096, 8192, 16384, 32768, 65536, 131072, 262144, 524288, 1048576,.... For $r_{n,12}$, the odd numbers and powers of 2 (up to 2 million) are 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 289, 361, 529, 841, 1331, 2187, 2401, 3125, 4096, 6561, 8192, 16384, 32768, 65536, 131072, 262144, 524288, 1048576,.... Note that a power of 3 (6561) is interspersed between 4096 and 8192. For $r_{n,13}$, the odd numbers and powers of 2 (up to 2 million) are 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151, 157, 163, 167, 289, 361, 529, 841, 1331, 2197, 2401, 3125, 6561, 8192, 15625, 16384, 32768, 65536, 131072, 262144, 524288, 1048576,.... Note that a power of 5 (15625) is interspersed between 8192 and 16384. For $r_{n,14}$, the odd numbers and powers of 2 (up to 2 million) are 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 361, 529, 841, 961, 1331, 2197, 2401, 6561, 14641, 15625, 16384, 32768, 65536, 131072, 262144, 524288, 1048576,.... For $r_{n,15}$, the odd numbers and powers of 2 (up to 2 million) are 113, 127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199, 221. 223, 227, 529, 841, 961, 1331, 2197, 4913, 14641, 15625, 19683, 32768, 59049, 65536, 131072, 262144, 524288, 1048576,.... For $r_{n,16}$, the odd numbers and powers of 2 (up to 2 million) are 127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199, 211, 223, 227, 229, 233, 239, 241, 251, 257, 529, 841, 961, 1369, 1681, 2197, 4913, 6859, 14641, 15625, 16807, 59049, 131072, 262144, 524288, 1048576,.... The odd numbers appear to be prime powers. All the n values appear to satisfy Theorem 2 (with the caveat of the $\beta - 2$ case). Other than the odd values, the n values behave like colossally abundant numbers. Colossally abundant numbers are a subset of the superabundant numbers.

Colossally abundant numbers are those numbers n for which there is a positive exponent ϵ such that $\frac{\sigma(n)}{n^{1+\epsilon}} \geq \frac{\sigma(k)}{k^{1+\epsilon}}$ for all $k > 1$. Colossally abundant numbers consist of a product of all the small primes up to some bound, with exponents which are non-increasing as the primes increase. The first few colossally abundant numbers are 2, 6, 12, 60, 120, 360, 2520, 5040, 55440, 720720, 1441440, 4324320, 21621600, 367567200, ... Washington and Yang [4] investigated odd colossally abundant numbers.

7   Relevance to the Riemann Hypothesis

Robin's [5] theorem states that the Riemann hypothesis is true if and only if $\sigma(n) < e^\lambda n \log \log n$ for all $n \geq 5041$. Robin showed that if the Riemann hypothesis is false, there will necessarily exist a counterexample to this inequality which is a colossally abundant number.

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