

# A Generalization of the Sum of Divisors Function

Darrell Cox<sup>1</sup>, Sourangshu Ghosh<sup>2</sup> and Eldar Sultanow<sup>3</sup>

<sup>1</sup> Department of Mathematics, Grayson County College, United States

<sup>2</sup> Department of Civil Engineering, Indian Institute of Technology Kharagpur, India

<sup>3</sup> Potsdam University, Chair of Business Informatics, Processes and Systems, Potsdam, Germany

**E-mail** — Eldar.Sultanow@wi.uni-potsdam.de

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## Abstract

Superabundant and colossally abundant numbers are generated using the sum of divisors function. If the Riemann hypothesis is false, there will necessarily exist a counterexample to an inequality involving the maximal order of the sum of divisors function which is a colossally abundant number. This paper introduces a generalization of the sum of divisors function having a recursive definition. The multiplicative property of the sum of divisors function is preserved so that infinitely many variants of superabundant and colossally abundant numbers can be generated. Besides the usual method of generating superabundant and colossally abundant numbers, we introduce a new method involving both the sum of divisors function and Euler's totient function. This method is applicable to the variant superabundant and colossally abundant numbers. Also, a method for "linearizing" certain quadratic, cubic, biquadratic, etc. curves is introduced. This gives "linear" curves similar to the second Chebyshev function but having different slopes.

## 1 Introduction

The sum of divisors function is commonly denoted by  $\sigma(n)$ . An arithmetical function  $f$  is called multiplicative if  $f$  is not identically zero and if  $f(mn) = f(m)f(n)$  whenever  $(m, n) = 1$ . The Dirichlet product of arithmetical functions  $f$  and  $g$  has the form  $\sum_{d|n} f(d)g\left(\frac{n}{d}\right)$ . If  $f$  and  $g$  are multiplicative, so is their Dirichlet product  $f \cdot g$ . See Theorem 2.14 of Apostol's [1] book. The sum of divisors function is multiplicative. See section 2.13 of Apostol's book. The Dirichlet product of  $\sigma(n)$

with the constant function  $g(n) = 1$  gives another multiplicative function. Similarly, the Dirichet product of this multiplicative function with the constant function gives yet another multiplicative function. The multiplicative property of these functions is essential for generating variants of superabundant numbers.

The connection between the Riemann hypothesis and the sum of divisors function dates back to Ramanujan's [2] work on highly composite numbers. In 1913, Gronwall [3] determined the maximal order of the sum of divisors function. More recently Alaoglu and Erdős [4] improved Ramanujan's work and proved theorems applicable to superabundant and colossally abundant numbers. In 1984, Robin [5] gave a much more refined version of the asymptotic upper bound of the sum of divisors function than Gronwall and proved a statement equivalent to the Riemann hypothesis. Since then, many statements involving the maximal order of the sum of divisors function have been proved to be equivalent to the Riemann hypothesis.

One purpose of this work is to show that there are infinitely many variants of superabundant numbers and thus that there are likely to be infinitely many statements equivalent to the Riemann hypothesis. No variant superabundant number is proven to lead to a statement equivalent to the Riemann hypothesis - empirical evidence is just given. Another purpose is to investigate the alternate method for generating superabundant and colossally abundant numbers.

## 2 Results

Let  $r_{n,1} = \sigma(n)$ ,  $r_{n,2} = \sum_{d|n} r_{d,1}$ ,  $r_{n,3} = \sum_{d|n} r_{d,2}$ , etc.

**Theorem 1** *If  $n$  is squarefree,  $\prod_{p|n}(p+1) = r_{n,1}$ . More generally, if  $p^m$  divides  $n$  and  $p^{m+1}$  does not divide  $n$ , then  $p^m + p^{m-1} + p^{m-2} + \dots + 1$  divides  $r_{n,1}$ , that is,  $\frac{p^{m+1}-1}{p-1}$  divides  $r_{n,1}$ .*

**Proof 1** *See Lemma 2.1 of Carella's [6] article.*

$\phi(n)$  denotes Euler's totient function. The alternate representation  $(\sigma(n) = \prod_{p|n} \frac{p^{m+1}-1}{p-1})$  is of importance due to the similar representation of  $\phi(n)$ .  $\phi(n) = n \prod_{p|n} (1 - p^{-1})$ . See Theorem 2.4 of Apostol's book.

**Theorem 2** *If  $n$  is squarefree,  $\prod_{p|n}(p+2) = r_{n,2}$ . More generally, if  $p^m$  divides  $n$  and  $p^{m+1}$  does not divide  $n$ , then  $p^m + 2p^{m-1} + 3p^{m-2} + \dots + (m+1)$  divides  $r_{n,2}$ , that is,  $\frac{p^{m+2}-1-(m+2)}{p-1}$  divides  $r_{n,2}$ .*

**Proof 2** *If  $p^m||n$ , then  $1 + (p+1) + (p^2 + p + 1) + \dots + (p^m + p^{m-1} + \dots + 1) = p^m + 2p^{m-1} + 3p^{m-2} + \dots + (m+1)$  divides  $r_{n,2}$ . This can easily be verified to equal  $\frac{p^{m+2}-1-(m+2)}{p-1}$ .*

**Theorem 3** *If  $n$  is squarefree,  $\prod_{p|n}(p+3) = r_{n,3}$ . More generally, if  $p^m$  divides  $n$  and  $p^{m+1}$  does not divide  $n$ , then  $p^m + 3p^{m-1} + 6p^{m-2} + \dots + (m+1)(m+2)/2$  divides  $r_{n,3}$ . An expression similar to the above can be derived using the binomial theorem.*

**Proof 3** If  $p|n$ , then  $1 + (p+2) + (p^2+2p+3) + (p^3+2p^2+3p+4) + \dots + (p^m+2p^{m-1}+3p^{m-2}+\dots+(m+1) = p^m + \frac{2(2+1)}{2}p^{m-1} + \frac{3(3+1)}{2}p^{m-2} + \dots + \frac{(m+1)(m+2)}{2}$  divides  $r_{n,3}$ .

## 2.1 Superabundant Numbers

A natural number is called a superabundant number if  $\sigma(n)/n > \sigma(k)/k$  for  $1 \leq k \leq n-1$ . The first few superabundant numbers are 2, 4, 6, 12, 24, 36, 48, 60, 120, 180, 240, 360, 720, 840, 1260, 1680, 2520, 5040,.... Gronwall's theorem is

**Theorem 4**  $\lim_{n \rightarrow \infty} \sup \frac{\sigma(n)}{n \log \log n} = e^\lambda$

**Proof 4** See Theorem 323 in Hardy and Wright's [8] book.

$\lambda$  denotes Euler's constant. A plot of  $\frac{r_{n,1}}{n \log \log n}$  for  $n = 2, 4, 6, 12, 24, 36, 48, 60, 120, 180, 240, 360, 720, 840, \dots$  is

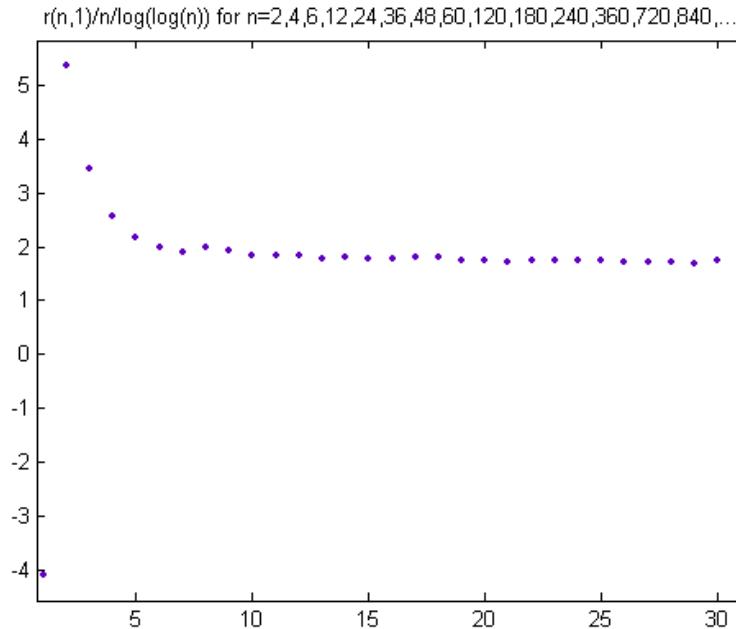


Figure 1: Plot of  $\frac{r_{n,1}}{n \log \log n}$  values

$e^\lambda$  is approximately equal to 1.7811. A quadratic least-squares fit of  $r_{n,1}/n$  versus  $\log \log n$  for these  $n$  values is

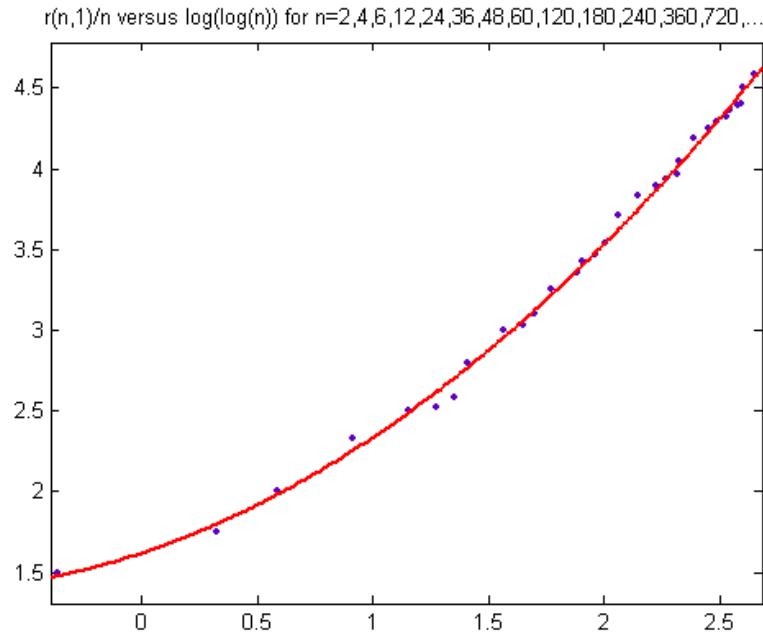


Figure 2: Quadratic least-squares fit of  $r_{n,1}/n$  versus  $\log \log n$

$p_1 = 0.2386$  with a 95% confidence interval of (0.21, 0.2672),  $p_2 = 0.4811$  with a 95% confidence interval of (0.3966, 0.5657),  $p_3 = 1.62$  with a 95% confidence interval of (1.556, 1.683), SSE=0.07629, R-squared=0.9966, and RMSE=0.0522. SSE denotes sum-squared error and RMSE denotes root-mean-squared error.

Let  $z(n)$  denote  $\log \log n$ . A linear least squares fit of  $r_{n,1}/n$  versus  $z(n)(z(n) + 2)$  is

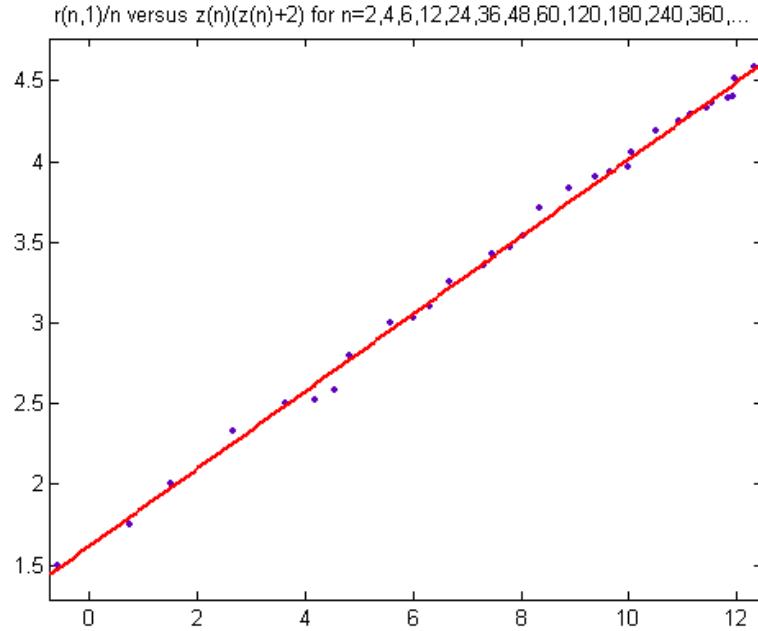


Figure 3: Linear least-squares fit of  $r_{n,1}/n$  versus  $z(n)(z(n) + 2)$

$p_1 = 0.2394$  with a 95% confidence interval of (0.2341, 0.2447),  $p_2 = 1.621$  with a 95% confidence interval of (1.576, 1.666), SSE=0.0763, R-squared=0.9966, and RMSE=0.05129. Let  $\Lambda(i)$  denote the Mangoldt function ( $\Lambda(i)$  equals  $\log(p)$  if  $i = p^m$  for some prime  $p$  and some  $m \geq 1$  or 0 otherwise). Let  $\psi(x)$  denote the second Chebyshev function ( $\psi(x) = \sum_{i \leq x} \Lambda(i)$ ). The Riemann hypothesis is equivalent to the arithmetic statement  $\psi(x) - x = o(x^{1/2+\epsilon})$  for all  $\epsilon > 0$ . A similar statement may be applicable to the above linear relationship. Cox [7], et. al. investigated similar statements involving Farey sequences where the slopes are not 1 (unlike the second Chebyshev function).

Superabundant numbers consist of a product of all the small primes up to some bound with exponents which are non-increasing as the prime increases. Alaoglu and Erdős' theorems on superabundant numbers (given for comparison purposes) are

**Theorem 5** *If  $n = 2^{k_2} \cdots p^{k_p}$ , then  $k_2 \geq k_3 \geq \dots \geq k_p$ .*

**Theorem 6** *Let  $q < r$ , and set  $\beta = [k_q \log q / \log r]$ . Then  $k_r$  has one of the three values:  $\beta - 1, \beta + 1, \beta$ .*

**Theorem 7** *If  $p$  is the largest prime factor of  $n$ , then  $k_p = 1$ , except when  $n = 4, 36$ .*

In the remainder of the theorems,  $p$  always denotes the largest prime factor of  $n$ .

**Theorem 8** If  $q$  is either the greatest prime of exponent  $k$  or the least prime of exponent  $k-1$ , and if  $q^{1-\theta} > \log p$ , then  $q^k = (p \log p / \log q)[1 + O(\log p / q^{1-\theta} \log q)]$ .

If Riemann's hypothesis is true any  $\theta > 1/2$  can be used.

**Theorem 9** If  $k_q = k$  and  $q < (\log p)^\alpha$ , where  $\alpha$  is a constant, then (i)  $\log \frac{q^{k+1}-1}{q^{k+1}-q} > \frac{\log q}{p \log p} [1 + O((\log \log p)^2 / \log p \log q)]$ , (ii)  $\log \frac{q^{k+2}-1}{q^{k+2}-q} < \frac{\log q}{p \log p} [1 + O((\log \log p)^2 / \log p \log q)]$ .

Let  $K_q + 1$  be the integral part and  $\theta_q$  the fractional part of  $\log[(q^{1+\epsilon} - 1)/(q^\epsilon - 1)] / \log q$ , where  $\epsilon = \log(1 + 1/p) \log p$ .

**Theorem 10** (i) If  $\epsilon_q < \theta_q \leq 1 - \epsilon_q$ , then  $k_q = K_q$ . (ii) If  $\theta_q \leq \epsilon_q$ , then  $k_q = K_q$  or  $k_q = K_q - 1$ . (iii) If  $1 - \epsilon_q < \theta_q$ , then  $k_q = K_q$  or  $K_q + 1$ .

**Theorem 11**  $p \sim \log n$ .

**Theorem 12** The quotient of two consecutive superabundant numbers tends to 1.

In the proof of Theorem 12 it was shown that the ratio of two consecutive superabundants  $n$  and  $n'$  is less than  $1 + c(\log \log n)^2 / \log n$ . The order of  $(\log p)^c$  is used.

**Theorem 13** The number of superabundant numbers less than  $x$  exceeds  $c \log x \log \log x / (\log \log \log x)^2$ .

Similar superabundant numbers can be defined for  $r_{n,2}$ ,  $r_{n,3}$ ,  $r_{n,4}$ , etc. and frequently occur at the same  $n$  values. For example, the superabundant numbers for  $r_{n,2}$  satisfy  $r_{n,2}/n > r_{k,2}/k$  for  $1 \leq k \leq n-1$ . Like the usual superabundant numbers, these superabundant numbers consist of a product of all the small primes up to some bound with exponents which are non-increasing as the prime increases. For  $r_{n,2}$ ,  $r_{n,3}$ ,  $r_{n,4}, \dots, r_{n,13}$  and  $n < 2000000$ , there are no exceptions to Theorem 5.

For  $r_{n,2}$ ,  $r_{n,3}$ , and  $n < 2000000$  there are no exceptions to Theorem 6. For  $r_{n,4}$  and  $n < 2000000$  exceptions occur at  $n=96$  and  $480$ . In these cases,  $k_r$  has the value  $\beta - 2$  for  $q = 2$  and  $r = 3$ . There are more such exceptions for  $r_{n,5}$ ,  $r_{n,6}$ ,  $r_7, \dots, r_{n,13}$  and higher order  $r_{n,i}$  values. Typical values are  $n = 96$ ,  $480$ ,  $960$ ,  $1152$ ,  $5760$ ,  $34560$ ,  $40320$ ,  $241920$ ,  $483840$ , and  $1209600$ . For  $r_{n,9}$  and  $r_{n,11}$ , there are exceptions at  $n = 1088640$ . In this case,  $k_r$  has the value  $\beta - 2$  for  $q = 3$  and  $r = 5$ . For  $r_{n,11}$  and  $n < 2000000$ , exceptions occur at  $n = 11520$  and  $80640$ . In these cases,  $k_r$  has the value  $\beta - 3$  for  $q = 2$  and  $r = 3$ . For higher order  $r_{n,i}$  values, typical  $n$  values where such exceptions occur are  $2304$ ,  $11520$ ,  $80640$ , and  $967680$ .

There are also exceptions to Theorem 7 when  $q = 2$  and  $r = 3$ . For  $r_{n,1}$ , exceptions occur at  $4$  and  $36$  (the exceptions for ordinary superabundant numbers). For  $r_{n,2}$ , exceptions occur at  $4$ ,  $36$ , and  $72$ . For  $r_{n,3}$ , exceptions occur at  $4$ ,  $8$ ,  $36$ , and  $72$ . For  $r_{n,4}$ , exceptions occur at  $4$ ,  $8$ ,  $36$ ,  $72$ , and  $144$ . Usually there is one more new exception for the next higher order of variant superabundant numbers. For  $r_{n,13}$ , the exceptions are  $4$ ,  $8$ ,  $16$ ,  $36$ ,  $72$ ,  $144$ ,  $288$ ,  $432$ ,  $576$ ,  $864$ ,  $1152$ ,  $1728$ ,  $2304$ ,  $86400$ , and  $172800$ .

Empirically, it is not practical to determine if Theorems 8, 9, and 10 are applicable to variant superabundant numbers.

A plot of  $p$  versus  $\log n$  for  $n = 2, 4, 6, 12, 24, \dots, 1441440$  (the ordinary superabundant numbers) is

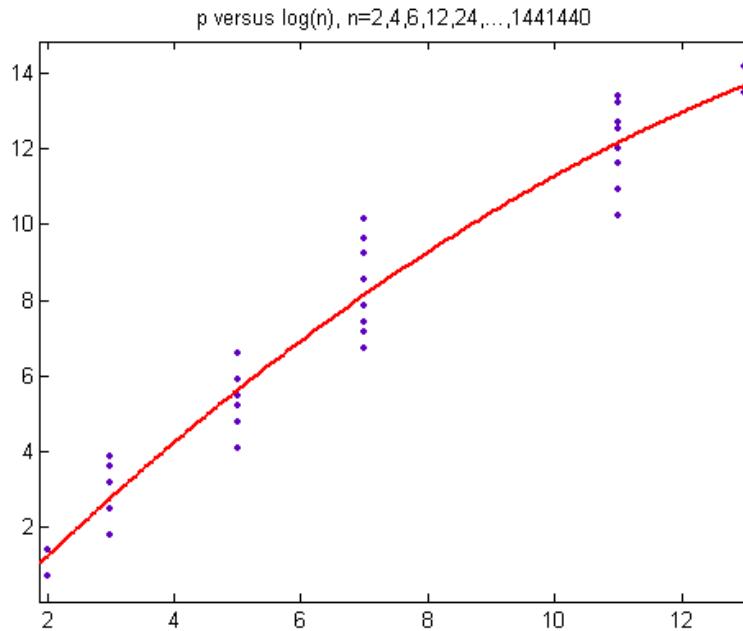


Figure 4:  $p$  versus  $\log n$  for ordinary superabundant numbers

This is pertinent to Theorem 11. Similar plots occur for the variant superabundant numbers. For  $n < 2000000$ , the largest  $p$  values for  $r_{n,1}, r_{n,2}, r_{n,3}, \dots, r_{n,13}$  are 13, 13, 11, 11, 11, 7, 7, 7, 7, 7, 7, and 7 respectively.

For  $r_{n,1}, r_{n,2}, r_{n,3}, \dots, r_{n,13}$  and  $n < 2000000$ , a superabundant number (ordinary or variant) is at most twice as large as the previous superabundant number. This is pertinent to Theorem 12.

The number of superabundant numbers less than 2000000 for  $r_{n,2}, r_{n,3}, r_{n,4}, \dots, r_{n,13}$  is 38, 39, 44, 43, 41, 41, 42, 48, 48, 49, 51, and 53 respectively. The number of ordinary superabundant numbers less than 2000000 is 31, so the counts for the new superabundant numbers are consistent with Theorem 13.

A plot of  $\sqrt{\frac{r_{n,2}}{n \log \log n}}$  for  $n = 2, 4, 6, 12, 24, 36, 48, 60, 72, 120, 180, 240, 360, 720, 1440, 1680, 2160, 2520, 5040, 10080, 15120, 20160, 25200, 30240, 50400, 55440, 110880, 166320, 221760, 277200, 332640, 554400, 665280, 831600, 1108800, 1330560, 1441440, and 1663200 is$

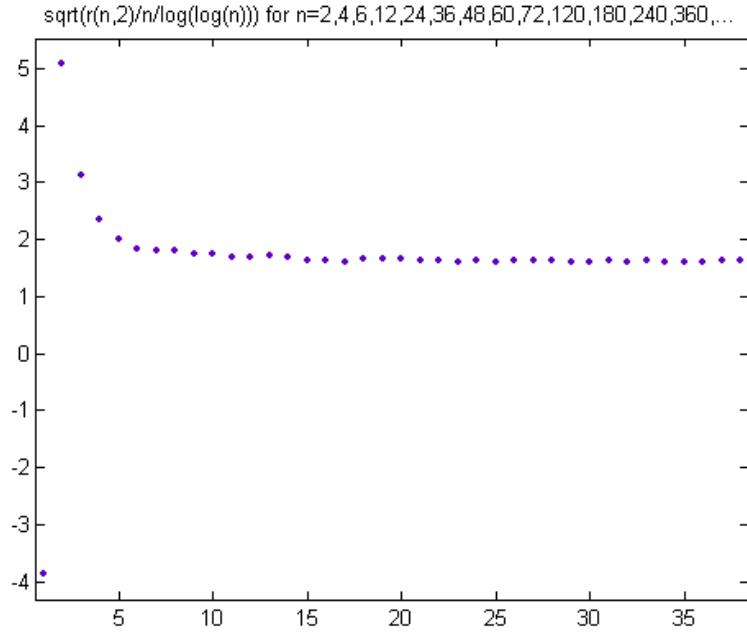


Figure 5: Square root of  $\frac{r_{n,2}}{n \log \log n}$

In this variant of Gronwall's formula, the supremum limit is unknown but is slightly less than  $e^\lambda$ . For a cubic least-squares fit of  $r_{n,2}/n$  versus  $\log \log n$  for these  $n$  values, SSE=2.172, R-squared=0.9977, and RMSE=0.2527.

A linear least squares fit of  $r_{n,2}/n$  versus  $z(n)(z(n) + 1)(z(n) + 2)$  for  $n = 2, 4, 6, 12, 24, 36, 48, 60, 72, 120, 180, 240, 360, 720, 1440, 1680, 2160, 2520, 5040, 10080, 15120, 20160, 25200, 30240, 50400, 55440, 110880, 166320, 221760, 277200, 332640, 554400, 665280, 831600, 1108800, 1330560, 1441440, 1663200, 2162160, 2882880, 3326400, 4324320, 7207200, 8648640, 10810800, 14414400, 17297280, 21621600, 43243200, 73513440, 86486400, 129729600, 147026880, 183783600, 245044800, 259459200, 294053760, 302702400, and 367567200 (the variant superabundant numbers less than 500 million) is$

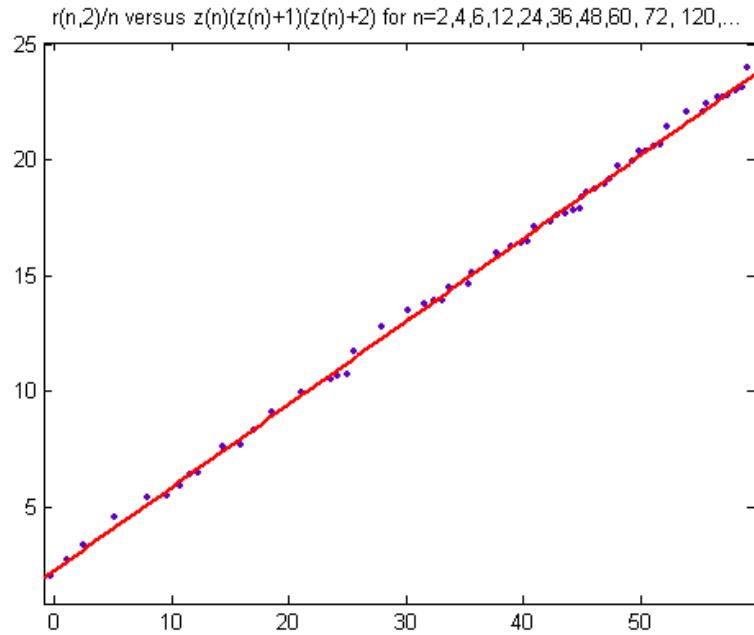


Figure 6: Linear least squares fit of  $r_{n,2}/n$  versus  $z(n)(z(n) + 1)(z(n) + 2)$

$p_1 = 0.3583$  with a 95% confidence interval of (0.3547, 0.362),  $p_2 = 2.297$  with a 95% confidence interval of (2.151, 2.444), SSE=3.369, R-squared=0.9985, and RMSE=0.2431.

A plot of  $\sqrt[3]{\frac{r_{n,3}}{n \log \log n}}$  for  $n = 2, 4, 6, 8, 12, 24, 36, 48, 60, 72, 120, 180, 240, 360, 720, 1440, 2160, 2520, 4320, 5040, 10080, 15120, 20160, 30240, 50400, 55440, 60480, 100800, 110880, 151200, 166320, 221760, 302400, 332640, 554400, 665280, 1108800, 1330560, and 1663200 is$

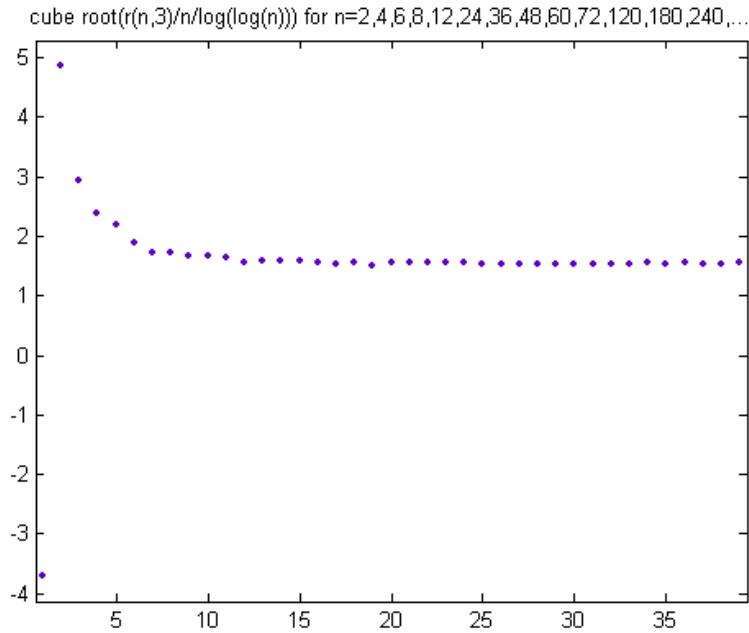


Figure 7: Cube root of  $\frac{r_{n,3}}{n \log \log n}$

In this variant of Gronwall's formula, the supremum limit is smaller than above. For a fourth power least-squares fit of  $r_{n,3}/n$  versus  $\log \log n$  for these  $n$  values, SSE=35.02, R-squared=0.9979, and RMSE=1.015.

A linear least-squares fit of  $r_{n,3}/n$  versus  $z(n)(z(n) + 0.5)(z(n) + 1.0)(z(n) + 1.5)$  for the above  $n$  values is

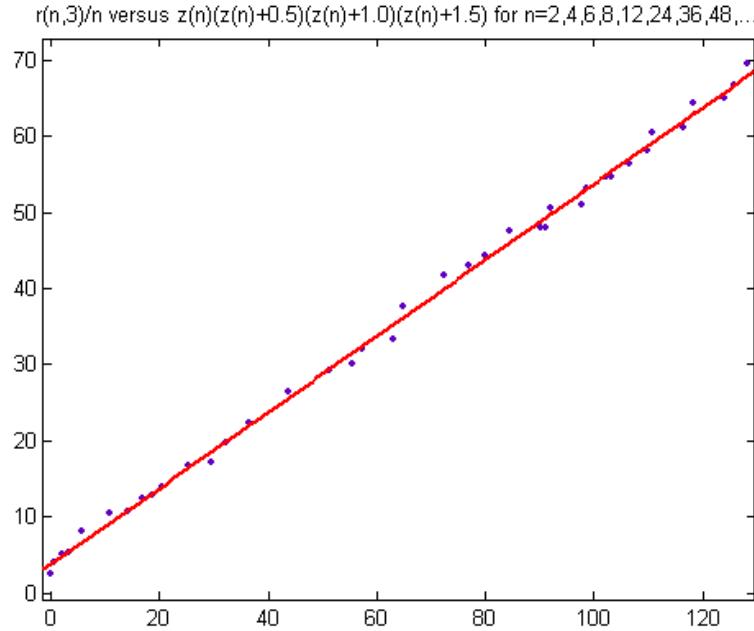


Figure 8: Linear least-squares fit of  $r_{n,3}/n$  versus  $z(n)(z(n) + 0.5)(z(n) + 1.0)(z(n) + 1.5)$

$p_1 = 0.5008$  with a 95% confidence interval of (0.4929, 0.5087),  $p_2 = 3.711$  with a 95% confidence interval of (3.113, 4.309), SSE=37.53, R-squared=0.9978, and RMSE=1.007.

A linear least-squares fit of  $r_{n,4}/n$  versus  $z(n)(z(n) + 0.25)(z(n) + 0.50)(z(n) + 0.75)(z(n) + 1.0)$  for  $n = 2, 4, 6, 8, 12, 24, 36, 48, 72, 96, 120, 144, 240, 360, 480, 720, 1440, 2160, 2520, 2880, 4320, 5040, 8640, 10080, 15120, 20160, 30240, 50400, 60480, 100800, 110880, 120960, 151200, 181440, 221760, 302400, 332640, 554400, 604800, 665280, 1108800, 1330560, 1663200, and 1995840 is$

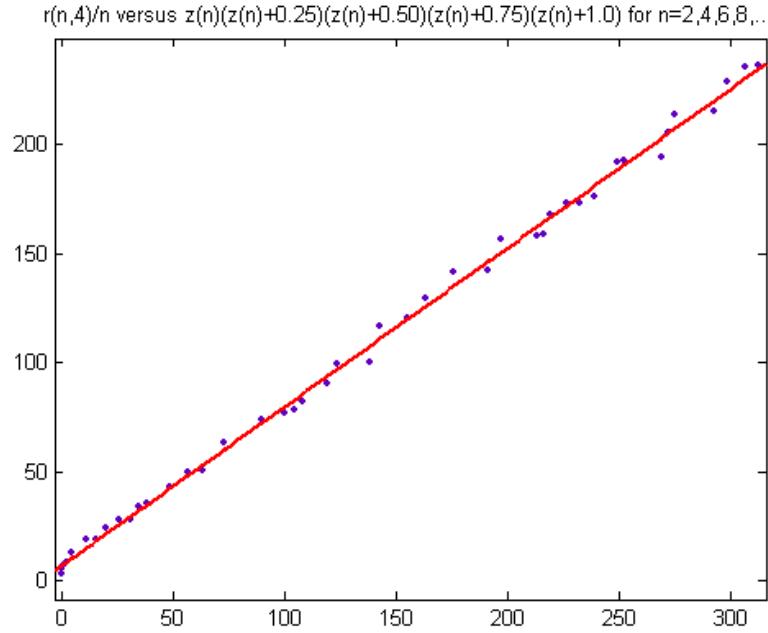


Figure 9: Linear least-squares fit of  $r_{n,4}/n$  versus  $z(n)(z(n) + 0.25)(z(n) + 0.50)(z(n) + 0.75)(z(n) + 1.0)$

$p_1 = 0.7297$  with a 95% confidence interval of (0.7179, 0.7414),  $p_2 = 6.721$  with a 95% confidence interval of (4.695, 8.748), SSE=650.5, R-squared=0.9973, and RMSE=3.936.

For a linear least-squares fit of  $r_{n,5}/n$  versus  $z(n)(z(n) + 0.125)(z(n) + 0.25)(z(n) + 0.375)(z(n) + 0.5)(z(n) + 0.625)$  for  $n = 2, 4, 6, 8, 12, 24, 36, 48, 72, 96, 120, 144, 240, 288, 360, 480, 720, 1440, 2160, 2880, 4320, 5040, 8640, 10080, 15120, 20160, 30240, 50400, 60480, 100800, 120960, 151200, 181440, 241920, 302400, 604800, 665280, 907200, 1209600, 1330560, 1663200, 1814400, and 1995840,  $p_1 = 1.058$  with a 95% confidence interval of (1.046, 1.071),  $p_2 = 13.8$  with a 95% confidence interval of (9.175, 18.42), SSE=3984, R-squared=0.9985, and RMSE=9.857.$

For a linear least-squares fit of  $r_{n,6}/n$  versus  $z(n)(z(n) + 0.0625)(z(n) + 0.125)(z(n) + 0.1875)(z(n) + 0.25)(z(n) + 0.3125)(z(n) + 0.375)$  for  $n = 2, 4, 6, 8, 12, 24, 36, 48, 72, 96, 120, 144, 240, 288, 360, 480, 720, 1440, 2160, 2880, 4320, 5040, 5760, 8640, 10080, 15120, 17280, 20160, 30240, 40320, 60480, 120960, 181440, 241920, 302400, 362880, 604800, 907200, 1209600, 1330560, and 1814400,  $p_1 = 1.514$  with a 95% confidence interval of (1.496, 1.533),  $p_2 = 23.53$  with a 95% confidence interval of (11.35, 35.71), SSE=29340, R-squared=0.9985, and RMSE=27.43.$

A quadratic least-squares fit of the above slopes (0.2394, 0.3583, 0.5008, 0.7297, 1.058, and 1.514) is

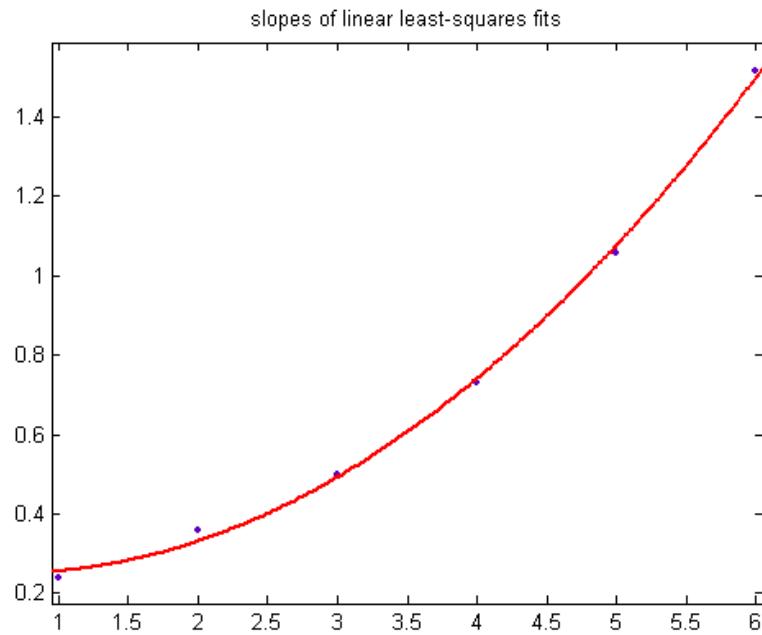


Figure 10: Quadratic least-squares fit of slopes

RMSE=0.001798, R-squared=0.9984, and RMSE=0.02445 A cubic least-squares fit of the y-intercepts (1.621, 2.297, 3.711, 6.721, 13.8, and 27.43) is

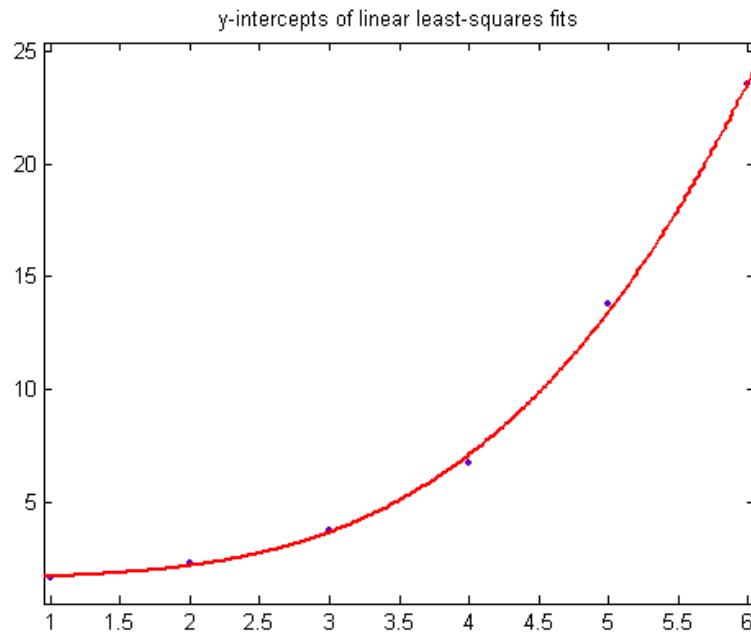


Figure 11: Cubic least-squares fit of y-intercepts

SSE=0.3053, R-squared=0.9992, and RMSE=0.3907.

A cubic least-squares fit of the smallest positive  $\sqrt[i]{\frac{r_{n,i}}{n \log \log n}}$  values for  $n < 2000000$  and  $i = 1, 2, 3, \dots, 16$  is

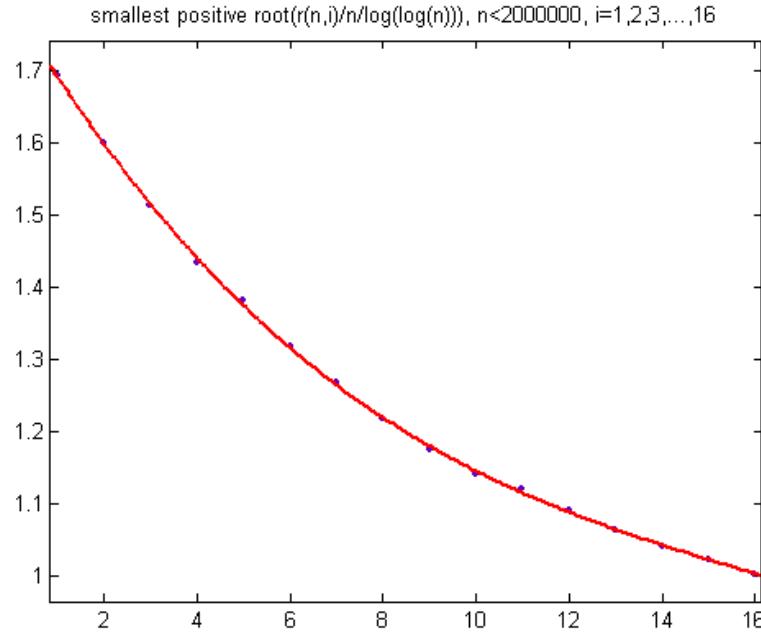


Figure 12: Cubic least-squares fit of smallest positive values of roots of ratios

RMSE=0.0002309, R-squared=0.9997, and RMSE=0.004386. The values are 1.6943, 1.5998, 1.5135, 1.4339, 1.3809, 1.3190, 1.2683, 1.2181, 1.1736, 1.1395, 1.1199, 1.0902, 1.0623, 1.0392, 1.0278, and 1.0018 respectively. For small sample sizes, quadratic and cubic curves give good approximations of some logarithmic functions. Eventually, higher order curves will be required to approximate the values (which are expected to approach zero).

## 2.2 Euler's Phi Function and the Sum of Divisors Function

Hardy and Wright's Theorem 328 is

**Theorem 14**  $\lim_{n \rightarrow \infty} \inf \frac{\phi(n) \log \log n}{n} = e^{-\lambda}$

Their Theorem 329 is

**Theorem 15**  $A < \frac{\sigma(n)\phi(n)}{n^2} < 1$  for a positive constant  $A$ .

The alternate representations of  $\phi(n)$  and  $\sigma(n)$  are used to prove the above theorem. No counterpart of the alternate representation of  $r_{n,2}$   $(\prod_{p|n} \frac{\frac{p^{m+2}-1}{p-1}-(m+2)}{p-1})$  has been found. In general, the relationship between  $\phi(n)$  and  $\sigma(n)$  is used to prove Gronwall's theorem.

### 2.3 New Method for Generating Superabundant Numbers

Superabundant numbers can also be defined for the largest  $\frac{r_{n,2}\phi(n)}{n^2}$  values. A plot of  $\frac{r_{n,2}\phi(n)}{n^2 \log \log n}$  for  $n = 3, 4, 8, 16, 24, 32, 48, 72, 144, 288, 432, 576, 720, 864, 1440, 2160, 2880, 3600, 4320, 7200, 8640, 10800, 14400, 15120, 20160, 21600, 30240, 43200, 60480, 75600, 100800, 120960, 151200, 302400, 604800, 907200, 1663200, 1814400, 2116800, 3326400, 6350400, 6652800, 9979200, 19958400, 23284800, 39916800, 43243200, 69854400, and 86486400 (the variant superabundant numbers less than 100 million) is$

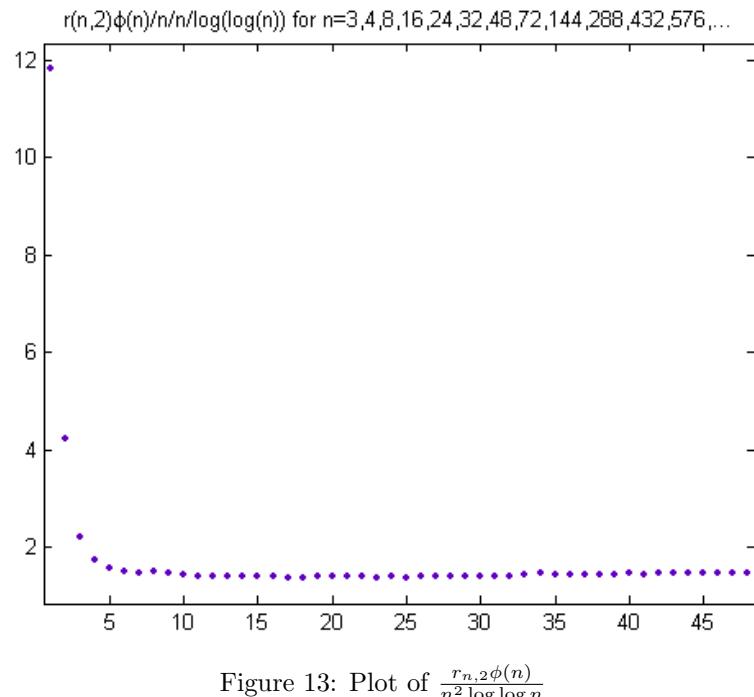


Figure 13: Plot of  $\frac{r_{n,2}\phi(n)}{n^2 \log \log n}$

In this variant of Gronwall's formula, the supremum limit is unknown. A quadratic least-squares fit of  $\frac{r_{n,2}\phi(n)}{n^2}$  versus  $\log \log n$  for these  $n$  values is

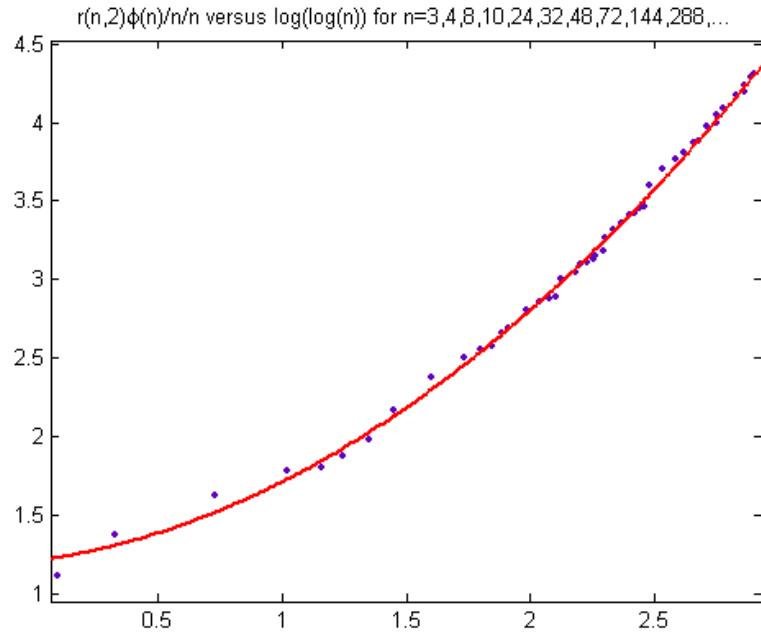


Figure 14: Quadratic least-squares fit of  $\frac{r_{n,2}\phi(n)}{n^2}$  versus  $\log \log n$

$p_1 = 0.2946$  with a 95% confidence interval of  $(0.2719, 0.3174)$ ,  $p_2 = 0.2116$  with a 95% confidence interval of  $(0.1312, 0.292)$ ,  $p_3 = 1.204$  with a 95% confidence interval of  $(1.136, 1.273)$ , SSE=0.08482, R-squared=0.9974, and RMSE=0.04342. Note the resemblance to the curve in Figure 2.

A linear least-squares fit of  $\frac{r_{n,2}\phi(n)}{n^2}$  versus  $z(n)(z(n) + 2)$  for these  $n$  values is

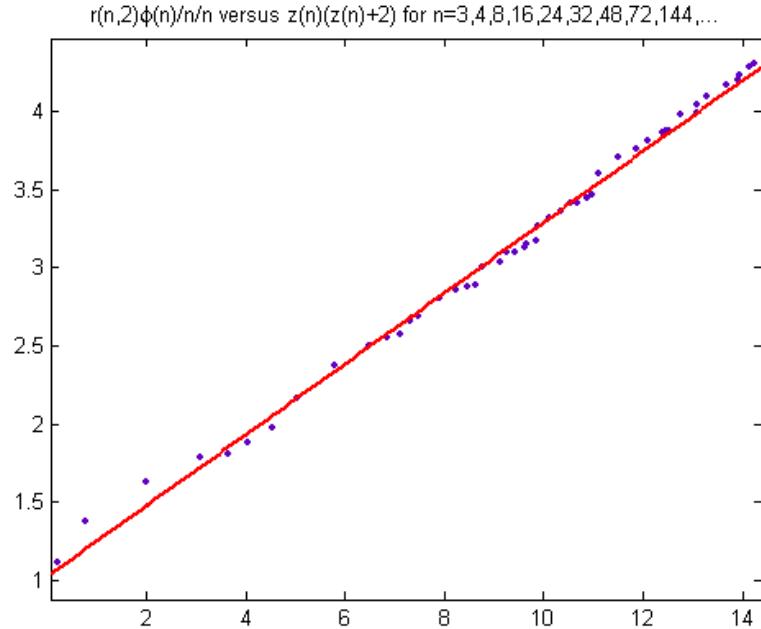


Figure 15: Linear least-squares fit of  $\frac{r_{n,2}\phi(n)}{n^2}$  versus  $z(n)(z(n) + 2)$

$p_1 = 0.2267$  with a 95% confidence interval of (0.222, 0.2314),  $p_2 = 1.027$  with a 95% confidence interval of (0.9804, 1.073), SSE=0.1545, R-squared=0.9952, and RMSE=0.05796.

Superabundant numbers can also be defined for the largest  $\frac{r_{n,3}\phi(n)\phi(n)}{n^3}$  values. A plot of  $\sqrt{\frac{r_{n,3}\phi(n)\phi(n)}{n^3 \log \log n}}$  for  $n = 5, 7, 8, 16, 32, 64, 128, 256, 288, 432, 576, 864, 1728, 3456, 5184, 6912, 10368, 20736, 31104, 41472, 43200, 86400, 129600, 172800, 259200, 518400, 777600, 1036800, 1296000, and 1555200 is$

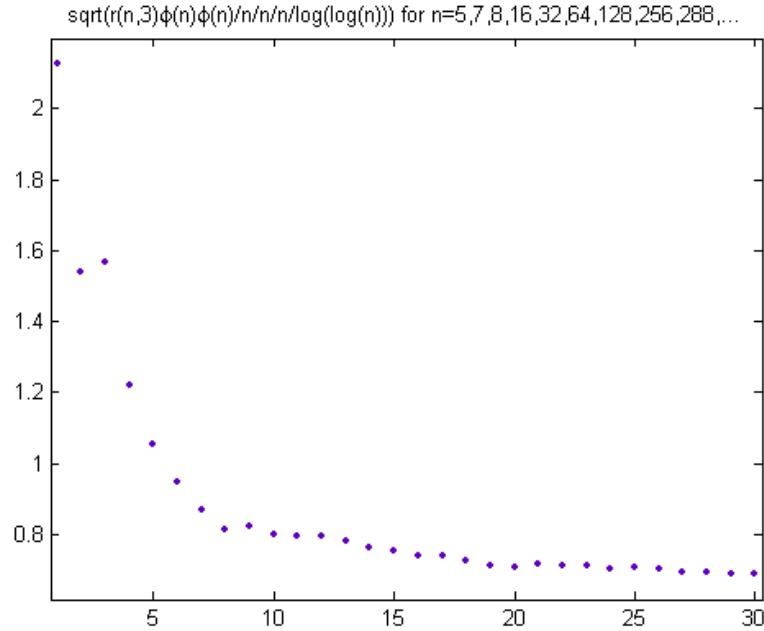


Figure 16: Square root of  $\frac{r_{n,3}\phi(n)\phi(n)}{n^3 \log \log n}$

For a cubic least-squares fit of  $\frac{r_{n,3}\phi(n)\phi(n)}{n^3}$  versus  $\log \log n$  for these  $n$  values, SSE=0.09052, R-squared=0.9935, and RMSE=0.05901. The  $n$  values equal to powers of 2 (at the beginning of the sequence) contribute substantially to the approximation errors.

Superabundant numbers can also be defined for the largest  $\frac{r_{n,4}\phi(n)\phi(n)\phi(n)}{n^4}$  values. The smallest value of  $\sqrt[3]{\frac{r_{n,4}\phi(n)\phi(n)\phi(n)}{n^4 \log \log n}}$  for  $n = 9, 11, 13, 16, 32, 64, 128, 256, 512, 1024, 1728, 3456, 5184, 6912, 10368, 20736, 31104, 41472, 62208, 124416, 248832, 373248, 497664, 518400, 777600, 1036800, and 1555200 is 0.5427.$

Other superabundant numbers are similarly defined for  $r_{n,5}, r_{n,6}, r_{n,7}$ , etc. The smallest of the above values for  $r_{n,2}, r_{n,3}, r_{n,4}, r_{n,5}, r_{n,6}, r_{n,7}$ , and  $r_{n,8}$  are 1.3734, 0.6888, 0.5427, 0.4862, 0.4561, 0.4350, and 0.4187 respectively.

## 2.4 Variants of Colossally Abundant Numbers

Colossally abundant numbers are those numbers  $n$  for which there is a positive exponent  $\epsilon$  such that  $\frac{\sigma(n)}{n^{1+\epsilon}} \geq \frac{\sigma(k)}{k^{1+\epsilon}}$  for all  $k > 1$ . The first few colossally abundant numbers are 2, 6, 12, 60, 120, 360, 2520, 5040, 55440, 720720, 1441440, 4324320, 21621600, 367567200, .... Colossally abundant numbers are a subset of the superabundant numbers. In the above, the  $n$  values start with a few odd values followed by powers of 2. Afterwards, the  $n$  values are the product of powers of 2 and 3, the product of powers of 2, 3, and 5, the product of powers of 2, 3, 5, and 7, etc. This is similar

to colossally abundant numbers. Unlike colossally abundant numbers, the factorization of the  $n$  values does not result in a sequence of relatively large primes with an exponent of 1. For  $r_{n,2}$ , the odd numbers and powers of 2 are 3, 4, 8, and 16. For  $r_{n,3}$ , the odd numbers and powers of 2 are 5, 7, 8, 16, 32, 64, 128, 256. For  $r_{n,4}$ , the odd numbers and powers of 2 are 9, 11, 13, 16, 32, 64, 128, 256, 512, and 1024. For  $r_{n,5}$ , the odd numbers and powers of 2 are 13, 17, 19, 23, 25, 27, 32, 64, 128, 256, 512, 1024, 2048, 4096, and 8192. For  $r_{n,6}$ , the odd numbers and powers of 2 are 19, 23, 29, 31, 37, 49, 64, 128, 256, 512, 1024, 2048, 4096, 8192, 16384, and 32768. For  $r_{n,7}$ , the odd numbers and powers of 2 are 29, 31, 37, 41, 43, 47, 49, 81, 125, 128, 243, 256, 512, 1024, 2048, 4096, 8192, 16384, 32768, 65536, and 131072. Note that a power of 3 (243) is interspersed between 128 and 256. For  $r_{n,8}$ , the odd numbers and powers of 2 are 31, 37, 41, 43, 47, 53, 59, 61, 67, 121, 125, 243, 256, 512, 1024, 2048, 4096, 8192, 16384, 32768, 65536, 131072, 262144, 524288, and 1048576. For  $r_{n,9}$ , the odd numbers and powers of 2 (up to 2 million) are 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 121, 169, 243, 289, 343, 512, 729, 1024, 2048, 4096, 8192, 16384, 32768, 65536, 131072, 262144, 524288, 1048576,.... Note that a power of 3 (729) is interspersed between 512 and 1024. For  $r_{n,10}$ , the odd numbers and powers of 2 (up to 2 million) are 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 169, 289, 343, 625, 729, 1024, 2048, 4096, 8192, 16384, 32768, 65536, 131072, 262144, 524288, 1048576,.... For  $r_{n,11}$ , the odd numbers and powers of 2 (up to 2 million) are 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 169, 289, 361, 529, 625, 1331, 2048, 4096, 8192, 16384, 32768, 65536, 131072, 262144, 524288, 1048576,.... For  $r_{n,12}$ , the odd numbers and powers of 2 (up to 2 million) are 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 289, 361, 529, 841, 1331, 2187, 2401, 3125, 4096, 6561, 8192, 16384, 32768, 65536, 131072, 262144, 524288, 1048576,.... Note that a power of 3 (6561) is interspersed between 4096 and 8192. For  $r_{n,13}$ , the odd numbers and powers of 2 (up to 2 million) are 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151, 157, 163, 167, 289, 361, 529, 841, 1331, 2197, 2401, 3125, 6561, 8192, 15625, 16384, 32768, 65536, 131072, 262144, 524288, 1048576,.... Note that a power of 5 (15625) is interspersed between 8192 and 16384. For  $r_{n,14}$ , the odd numbers and powers of 2 (up to 2 million) are 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 361, 529, 841, 961, 1331, 2197, 2401, 6561, 14641, 15625, 16384, 32768, 65536, 131072, 262144, 524288, 1048576,.... For  $r_{n,15}$ , the odd numbers and powers of 2 (up to 2 million) are 113, 127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199, 221, 223, 227, 529, 841, 961, 1331, 2197, 4913, 14641, 15625, 19683, 32768, 59049, 65536, 131072, 262144, 524288, 1048576,.... For  $r_{n,16}$ , the odd numbers and powers of 2 (up to 2 million) are 127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199, 211, 223, 227, 229, 233, 239, 241, 251, 257, 529, 841, 961, 1369, 1681, 2197, 4913, 6859, 14641, 15625, 16807, 59049, 131072, 262144, 524288, 1048576,.... The odd numbers appear to be prime powers. All the  $n$  values appear to satisfy Theorem 6 (with the caveat of the  $\beta - 2$  case).

A spectrum similar to the Riemann spectrum can be computed for the odd numbers and the powers of 2 by using the function  $F_{\leq C}(t) =$

$-\sum_{p^n \leq C} \frac{\log(p)}{p^{n/2}} \cos(t \log(p^n))$ . See Chapter 30 of Mazur and Stein's [9] book for graphs of the Riemann spectrum. A plot of the positive values of this function for the odd numbers and powers of 2 corresponding to  $r_{n,4}$  and  $t \leq 100$  is

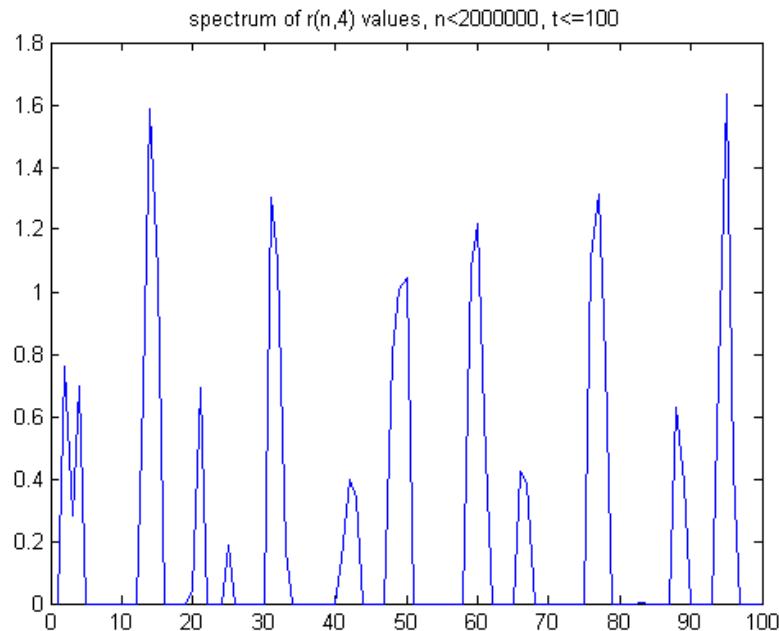


Figure 17: spectrum of odd numbers and powers of 2 corresponding to  $r_{n,4}$

All 27 of the variant colossally abundant numbers for  $r_{n,4}$  are checked (to determine if they are prime powers), so  $C$  is effectively 2 million. The locations of the peaks for  $t \leq 300$  are

2	3	4
13	14	15
20	21	
25		
31	32	33
41	42	43
48	49	50
59	60	61
66	67	
76	77	78
83		
88	89	
94	95	96
104	105	106
111	112	113
122	123	124
130		
134		
139	140	141
150	151	152
157	158	159
167	168	169
174	175	176
185	186	187
192		
196	197	198
202	203	204
213	214	215
220	221	222
231	232	233
248	249	250
259	260	
266	277	
276	277	278
283		
294	295	296

The width of the peaks is usually three. The inverse function for the Riemann spectrum is  $H_{\leq C}(s) = 1 + \sum_{i \leq C} \cos(\log(s)\theta_i)$ . The non-trivial zeta function zeros are denoted by  $\theta_i$ . In the application of this function here, the maxima of the spectrum are taken to be just the positive values. A plot of the inverse function is

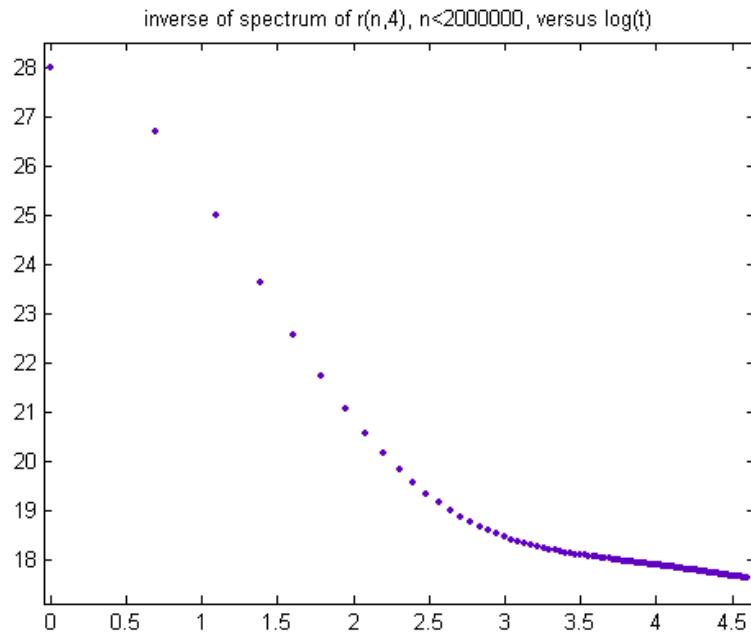


Figure 18: inverse of spectrum of odd numbers and powers of 2 corresponding to  $r_{n,4}$

The inverse of the spectrum of the variant colossally abundant numbers for  $r_{n,2}$  is a constant 39. The locations of the peaks of the spectrum of the variant colossally abundant numbers for  $r_{n,3}$  and  $t \leq 300$  are 14, 76, 95, 186, 276, and 295. (Note that these locations are common to the locations for  $r_{n,4}$ .) The largest value of the inverse is 31.

The locations of the peaks of the spectrum of the variant colossally abundant numbers for  $r_{n,5}$  and  $t \leq 300$  are 2, 15, 21, 25, 43, 48, (60, 61), 88, 106, 111, 124, 134, 151, 157, 170, 185, 197, 202, (220, 221), 232, 248, (260, 261), 266, 283, and 295. (Except for the location 261, these locations are common to the locations for  $r_{n,4}$ .) The largest value of the inverse is 28.

The locations of the peaks of the spectrum of the variant colossally abundant numbers for  $r_{n,6}$  and  $t \leq 300$  are 2, 14, 21, 25, 30, 33, (48, 49), (60, 61), 65, 76, 88, 96, 107, (111, 112), (123, 124), 134, 139, 151, 158 (169, 170), 174, 185, 197, 202 (220, 221), (232, 233), 248, 260, 279, 283, and 295. (Two of the locations are one less than a location for  $r_{n,4}$  and two are one more than a location.) The largest value of the inverse is 27.

The locations of the peaks of the spectrum of the variant colossally abundant numbers for  $r_{n,7}$  and  $t \leq 300$  are 25, 30, 33, 48, 61, 65, 76, (88, 89), 96, 107, 111, 124, 134, 139, 170, 174, 185, 198, 202, 205, (220, 221), (232, 2233), (247, 248), (260, 261), 279, and 283. (Three of the locations are one less than a location for  $r_{n,4}$  and four are one more than a location.) The largest value of the inverse is 28.

The locations of the peaks of the spectrum of the variant colossally abundant numbers for  $r_{n,8}$  and  $t \leq 300$  are 2, 21, 25, 30, 33, 41, 44, 50, 53, 61, 65, 76, 83, 89, (95, 96), 107, 111, 124 (134, 135), 138, 147, 150, 156, 159, 170, 174, 185, 192, 198 (201, 202), 205, 216, 221, 225, 233, 237, 244, 247, 261, 276, 279, (282, 283), and (295, 296). (Seven of the locations are one less than a location for  $r_{n,4}$  and seven are one more than a location. Five more locations do not match a location for  $r_{n,4}$ .) A plot of the inverse function is

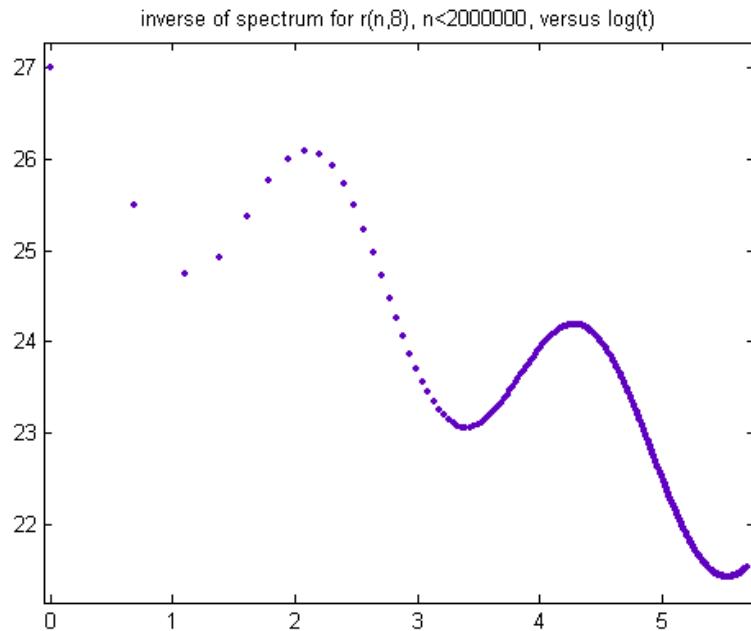


Figure 19: inverse of spectrum variant colossally abundant numbers corresponding to  $r_{n,8}$

These are all the variant colossally abundant numbers for  $r_{n,i}$  values where the largest power of 2 does not exceed two million. The spectra of the variant colossally abundant numbers appear to have some significance. Interpreting the inverses of the spectra is more difficult.

For the variant colossally abundant numbers corresponding to  $r_{n,17}$ , the odd numbers are 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199, 211, 223, 227, 229, 233, 239, 241, 251, 257, 263, 269, 271, 277, 281, 283, 293,  $29^2$ ,  $31^2$ ,  $37^2$ ,  $41^2$ ,  $43^2$ ,  $13^3$ ,  $17^3$ ,  $19^3$ ,  $11^4$ ,  $7^5$ , and  $3^{11}$ . There are 39 of them.

For the variant colossally abundant numbers corresponding to  $r_{n,18}$ , the odd numbers are 163, 167, 173, 179, 181, 191, 193, 197, 199, 211, 223, 227, 229, 233, 239, 241, 251, 257, 263, 269, 271, 277, 281, 283, 293, 307, 311, 313, 317, 331,  $29^2$ ,  $31^2$ ,  $37^2$ ,  $41^2$ ,  $43^2$ ,  $47^2$ ,  $17^3$ ,  $19^3$ ,  $23^3$ ,  $11^4$ ,  $13^4$ ,  $5^7$ ,  $3^{11}$ , and  $5^8$ . There are 44 of them.

For the variant colossally abundant numbers corresponding to  $r_{n,19}$ , the odd numbers are 179, 181, 191, 193, 197, 199, 211, 223, 227, 229, 233, 239, 241, 251, 257, 263, 269, 271, 277, 281, 283, 293, 307, 311, 313, 317, 331, 337, 347, 349, 353, 359,  $31^2$ ,  $37^2$ ,  $41^2$ ,  $43^2$ ,  $47^2$ ,  $53^2$ ,  $17^3$ ,  $19^3$ ,  $23^3$ ,  $11^4$ ,  $13^4$ ,  $5^7$ ,  $7^6$ ,  $5^8$ , and  $3^{12}$ . There are 47 of them.

A quadratic least-squares fit of the number of odd numbers is

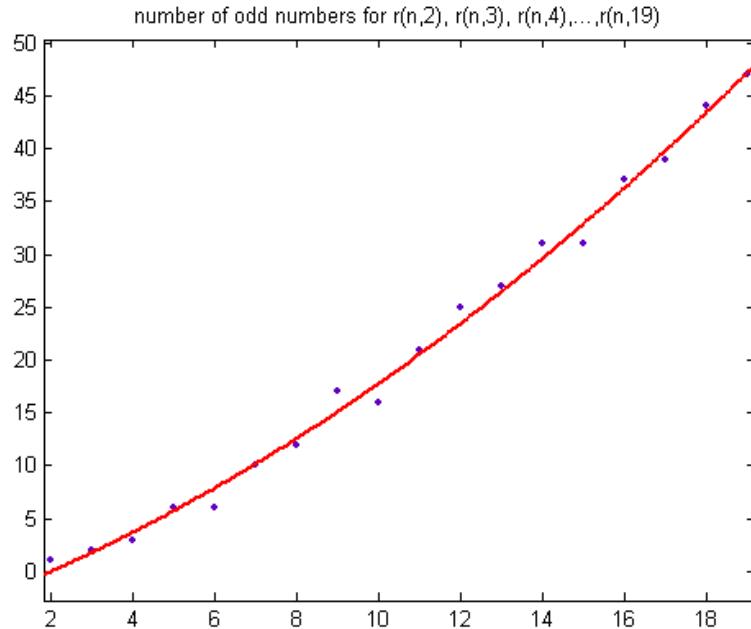


Figure 20: Quadratic least-squares fit of number of odd numbers corresponding to  $r_{n,i}$  values

$p_1 = 0.06153$  with a 95% confidence interval of (0.03608, 0.08699),  $p_2 = 1.491$  with a 95% confidence interval of (0.9438, 2.038),  $p_3 = -3.263$  with a 95% confidence interval of (-5.792, -0.7346), SSE=22.11, R-squared=0.9942, and RMSE=1.214.

Apparently, the number of odd numbers can become arbitrarily large. Perhaps another variant of the Riemann hypothesis can be derived.

## 2.5 Factorization of the Variants of Colossally Abundant Numbers

Neglecting the odd numbers and powers of 2, the variant colossally abundant numbers less than 2 million corresponding to  $r_{n,2}$  are  $2^4 \cdot 3$ ,  $2^3 \cdot 3^2$ ,  $2^4 \cdot 3^2$ ,  $2^5 \cdot 3^2$ ,  $2^4 \cdot 3^3$ ,  $2^6 \cdot 3^2$ ,  $2^4 \cdot 3^2 \cdot 5$ ,  $2^5 \cdot 3^3$ ,  $2^5 \cdot 3^2 \cdot 5$ ,  $2^4 \cdot 3^3 \cdot 5$ ,  $2^6 \cdot 3^2 \cdot 5$ ,  $2^4 \cdot 3^2 \cdot 5^2$ ,  $2^5 \cdot 3^3 \cdot 5$ ,  $2^5 \cdot 3^2 \cdot 5^2$ ,  $2^6 \cdot 3^3 \cdot 5$ ,  $2^4 \cdot 3^3 \cdot 5^2$ ,  $2^6 \cdot 3^2 \cdot 5^2$ ,  $2^4 \cdot 3^3 \cdot 5 \cdot 7$ ,  $2^6 \cdot 3^2 \cdot 5 \cdot 7$ ,  $2^5 \cdot 3^3 \cdot 5^2$ ,  $2^5 \cdot 3^3 \cdot 5 \cdot 7$ ,  $2^6 \cdot 3^3 \cdot 5^2$ ,  $2^6 \cdot 3^3 \cdot 5 \cdot 7$ ,  $2^4 \cdot 3^3 \cdot 5 \cdot 7$ ,  $2^6 \cdot 3^2 \cdot 5^2 \cdot 7$ ,  $2^7 \cdot 3^3 \cdot 5 \cdot 7$ ,  $2^5 \cdot 3^3 \cdot 5^2 \cdot 7$ ,  $2^6 \cdot 3^3 \cdot 5^2 \cdot 7$ ,  $2^7 \cdot 3^3 \cdot 5^2 \cdot 7$ ,  $2^6 \cdot 3^4 \cdot 5^2 \cdot 7$ ,

$2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11$ , and  $2^7 \cdot 3^4 \cdot 5^2 \cdot 7$ . The numbers  $2^5 \cdot 3^3$ ,  $2^5 \cdot 3^3 \cdot 5^2$ ,  $2^6 \cdot 3^3 \cdot 5^2$ , and  $2^7 \cdot 3^4 \cdot 5^2 \cdot 7$  are of interest because they are exceptions to the orderly increase of the last primes of the factorizations. What they have in common is that all the  $\beta$  values of a number equal the exponent of the previous prime. Using  $2^5 \cdot 3^3$  as a “seed”, the sequence of numbers  $2^5 \cdot 3^3$ ,  $2^5 \cdot 3^3 \cdot 5^2$ ,  $2^7 \cdot 3^4 \cdot 5^2 \cdot 7$ ,  $2^8 \cdot 3^5 \cdot 5^3 \cdot 7^2 \cdot 11$ ,  $2^{10} \cdot 3^6 \cdot 5^4 \cdot 7^3 \cdot 11^2 \cdot 13$ ,  $2^{13} \cdot 3^8 \cdot 5^5 \cdot 7^4 \cdot 11^3 \cdot 13^2 \cdot 17$ ,... is obtained. When the next-larger prime is added, the exponents of the previous primes are adjusted from right to left. The least exponent of the previous prime that equals  $\beta$  is selected. There is then a well-defined procedure for generating these numbers.

Neglecting the odd numbers and powers of 2, the variant colossally abundant numbers less than 2 million corresponding to  $r_{n,3}$  are  $2^5 \cdot 3^2$ ,  $2^5 \cdot 3^3$ ,  $2^6 \cdot 3^2$ ,  $2^5 \cdot 3^3$ ,  $2^6 \cdot 3^3$ ,  $2^7 \cdot 3^3$ ,  $2^6 \cdot 3^4$ ,  $2^8 \cdot 3^3$ ,  $2^7 \cdot 3^4$ ,  $2^8 \cdot 3^4$ ,  $2^7 \cdot 3^5$ ,  $2^8 \cdot 3^5$ ,  $2^6 \cdot 3^3 \cdot 5^2$ ,  $2^7 \cdot 3^3 \cdot 5^2$ ,  $2^6 \cdot 3^4 \cdot 5^2$ ,  $2^8 \cdot 3^3 \cdot 5^2$ ,  $2^7 \cdot 3^4 \cdot 5^2$ ,  $2^8 \cdot 3^5 \cdot 5^2$ ,  $2^7 \cdot 3^5 \cdot 5^2$ ,  $2^9 \cdot 3^4 \cdot 5^2$ ,  $2^7 \cdot 3^4 \cdot 5^3$ , and  $2^8 \cdot 3^5 \cdot 5^2$ . Unlike ordinary colossally abundant numbers, the exponents of the primes are strictly decreasing as the primes increase. This appears to be the case for variant colossally abundant numbers corresponding to  $r_{n,4}$ ,  $r_{n,5}$ ,  $r_{n,6}$ ,...

### 3 Methods

A C program for computing superabundant numbers using the new method is given in the Appendix. The files “out7g1.h”, “out7g2.h”, “out7g3.h”, and “out7g4.h” contain  $r_{n,2}$ ,  $r_{n,3}$ ,  $r_{n,4}$ , and  $r_{n,5}$  values up to  $n = 6000$  respectively. The method for generating these files is straightforward. Initially, the values of the sum of divisors function are computed and stored to memory. Then these values are read from memory and the simplified Dirichlet products are computed and stored to memory. The procedure is repeated for whatever order of  $r_{n,i}$  is desired. Normally, the maximum  $n$  value is set to 2 million or larger. Euler’s totient function is computed using the Möbius function (see Theorem 2.3 of Apostol’s book).

### 4 Conclusion

More efficient algorithms are needed to compute  $r_{n,i}$  values for  $n > 2000000$ . Even computing  $r_{n,i}$  for  $n$  up to 8 million only gives 2 or 3 more superabundant numbers for a given  $i$  value. One of Alaoglu and Erdős’ lemmas could then be investigated. Their Lemma 4 is

**Lemma 1** *If  $q$  is the greatest prime of exponent  $k$ , and if  $q^{1-\theta} > \log p$ , then all primes between  $q$  and  $q + q^\theta$  have exponent  $k - 1$ .*

If Riemann’s hypothesis is true any  $\theta > 1/2$  can be used.

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```
unsigned int newsum;
unsigned int temp,iters;
double temp1,maxt;
FILE *Outfp;
Outfp=fopen("out7jb.dat","w");
maxt=0.0;
for (i=1; i<=max; i++) {
    newsum=0;
    if (order==2) {
        for (j=1; j<=i; j++) {
            if (i==((i/j)*j))
                newsum=newsum+numdiv(j,2);
        }
    }
    if (order==3) {
        for (j=1; j<=i; j++) {
            if (i==((i/j)*j))
                newsum=newsum+n2[j-1];
        }
    }
    if (order==4) {
        for (j=1; j<=i; j++) {
            if (i==((i/j)*j))
                newsum=newsum+n3[j-1];
        }
    }
    if (order==5) {
        for (j=1; j<=i; j++) {
            if (i==((i/j)*j))
                newsum=newsum+n4[j-1];
        }
    }
    if (order==6) {
        for (j=1; j<=i; j++) {
            if (i==((i/j)*j))
                newsum=newsum+n5[j-1];
        }
    }
    temp=phi(i,table,tsize);
    temp1=(double)newsum/(double)i;
    iters=order-1;
    for (k=1; k<=iters; k++)
        temp1=temp1*(double)temp/(double)i;
    if (temp1>maxt) {
        printf(" %d %d %d %e \n",i,newsum,temp,temp1);
        if (select==1)
            fprintf(Outfp, " %e \n",temp1);
        else
            printf(" %d \n",i);
        maxt=temp1;
    }
}
return;
}
```