Article

The Singularity of Legendre Functions of the First Kind as a Consequence of the Symmetry of Legendre's Equation

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Abstract: Legendre's equation is key in various branches of physics. Its general solution is a linear function space, spanned by the Legendre functions of first and second kind. In physics however, commonly the only acceptable members of this set are the Legendre polynomials. Quantization of the eigenvalues of Legendre's operator is a consequence of this. We present and explain a stand-alone, in-depth argument for rejecting all solutions of Legendre's equation, but the polynomial ones, in physics. We show that the combination of the linearity, the mirror symmetry and the signature of the regular singular points of Legendre's equation is quintessential to the argument. We demonstrate that the evenness or oddness of the Legendre polynomials is a consequence of the same ingredients.

Keywords: Legendre's Equation; Legendre Functions; Legendre Polynomials ; Singularities; Symmetry

1. Introduction

1.1. Motivation

From the point of view of mathematical physics, it is a curious fact that the full solution of Legendre's equation, i.e. a myriad of algorithms for evaluation of all Legendre functions, has been meticulously documented in the solid, special functions literature [1–5], while what is really needed in many crucial applications in physics is a clear argument explaining why only the Legendre *polynomials* – and these don't need any specialized literature to document how to evaluate them at all – are *acceptable* solutions. As it seems, such an argument and explanation are not at all easily found in the literature. It is the aim of the present manuscript to unearth such an argument.

1.2. Singularities; Fuchsian and Legendre Equations.

The common argument to *reject* Legendre functions, both of the first and second kind, as acceptable functions in a given context in physics, is that they are *singular*, and *unbounded* indeed on the application domain. Indeed, the possibility of singular solutions is an important feature of Fuchsian differential equations, of which Legendre's equation is a particularly important example.

Fuchsian 2$^{\text{nd}}$ order differential equations [6–8] are key in many subdisciplines of mathematical physics [3]. Regular singular points of these equations are commonly associated with special points of curvilinear coordinate systems, and hence with the geometry and symmetry of the situation of application. In this manuscript we address Legendre's equation, for dependent variable $y = y(\xi)$ as a function of independent variable ξ:

$$(1 - \xi^2)\, y'' - 2\,\xi\, y' + \nu\,(\nu + 1)\, y = 0 \ . \tag{1}$$

This equation emerges [3,5] by separation of variables from Laplace's operator in spheroidal, including spherical, coordinates. Coordinate ξ then is associated with a latitudinal coordinate, its domain is the closed interval $[-1, 1]$ and the regular singular points $\xi = \pm 1$ are associated with the poles of the spheroidal coordinate system. In the context of Laplace's

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operator, equation (1) actually is a special case of the general, or *associated* Legendre equation [5]. The solutions of this more general equation are actually easily expressed in terms of solutions of equation (1), and unboundedness really arises if and only if the involved solutions of equation (1) are unbounded.

The unboundedness of the Legendre functions of the *first* kind is a *crucial* argument, in many application domains of mathematical physics, for *rejecting* these functions. It is the very reason why only the Legendre *polynomials* remain as the *sole* physically acceptable solutions of the (2^{nd} order!) Legendre equation (1). A direct implication, but one that is even more important for physics, is that *the parameter ν, and hence the eigenvalues of the Legendre operator*

$$L[y(\xi),\,\xi] \;=\; -\frac{d}{d\xi}\left(\,(1-\xi^2)\,\frac{d}{d\xi}\,y\,\right)\;, \tag{2}$$

become quantized; it seems hardly an exaggeration to mention that this is at the foundation of our understanding of the periodic system of chemical elements. Indeed, in the quantum mechanics of atoms, the discrete integer values of the parameter ν are the quantum numbers of orbital angular momentum [9,10]. In geophysical fluid dynamics, they label the fundamental modes of the atmosphere, i.e. the planetary Rossby-Haurwitz waves [11]. In as far as there is value in understanding *why* such quantization occurs, the value of any argument that helps explaining it can hardly be exaggerated. Therefore, it is certainly of value to unearth arguments that imply and explain the unboundedness of the Legendre functions $P_\nu(\xi)$, as we aim to do in this manuscript.

In this study we focus on Legendre's equation (1) and its solutions, but our reasoning and some of the results have broader validity.

1.3. Frobenius' Theory

It is well-known that solutions of Fuchsian differential equations about regular singular points can have singularities. The character of selected solutions *about regular singular points* can be diagnosed by Frobenius' theory [12,13]. Frobenius' theory indeed renders generalized series solutions about regular singular points, including possibly singular factors of the form ξ^r, in which r is some number, and, also possibly, logarithmic factors.

In case of Legendre's equation (1), about $\xi = 1$, by means of Frobenius' method, two solutions are readily found, one analytical (Legendre function $P_\nu(\xi)$ of the first kind) and another (Legendre function $Q_\nu(\xi)$ of the second kind) with a logarithmic singularity. This easily establishes the singularity, the unboundedness indeed, of Legendre's functions of the *second* kind, so that *these* mathematically well-defined solutions $Q_\nu(\xi)$ can be rejected, based on *physical* arguments, in important physical application domains, such as in quantum mechanics (atomic physics) [9], electro-magnetism [14, e.g.] and in geo- and astrophysical domains, e.g. gravitation and fluid dynamics [11,15].

For the Legendre functions $P_\nu(\xi)$ of the *first* kind however, although they are analytical about $\xi = 1$, so that they may *seem* to be more straightforward, the situation *actually* is *less* trivial. The decision about whether or not these functions are physically acceptable commonly depends on whether or not they are bounded at the *other* regular singular point, $\xi = -1$, and Frobenius' theory offers no direct solace in this respect. Only when ν takes integer values n, the series expansions of the $P_\nu(\xi)$ can be shown to have only a finite number of non-zero terms, so they are actually the Legendre *polynomials* and hence physically acceptable solutions indeed.

In all other cases, one can show that the series expansion about $\xi = 1$ rendered by Frobenius' method for the $P_\nu(\xi)$ does not converge at $\xi = -1$. However, then the *hurdle* arises that – although some sources [14,16] seem to suggest some argument along this line of thought – *non-convergence* of a series *in itself*, at some point, provides no *convincing* argument for e.g. the being-unbounded of the function it aims to represent. Specifically, in the case at hand, mere divergence, at $\xi = -1$, of their series expansions about $\xi = 1$ does

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not provide a truly solid argument, at least not at any elementary level, for rejecting the functions $P_\nu(\xi)$ on physical grounds.

1.4. Aim and prospect

The aim of this manuscript is to present an argument, at a level that is as elementary as possible, that shows that, *and explains why*, Legendre functions of the first kind, $P_\nu(\xi)$, are *unbounded* at $\xi = -1$. We shall show that the reasons include the symmetry of Legendre's equation (1) about $\xi = 0$, and that the singularity of $P_\nu(\xi)$ at $\xi = -1$ is in that sense a consequence, in some sense a mirror image indeed, of the singularity of $Q_\nu(\xi)$ at $\xi = 1$. We shall, as a by-catch, see that *the fact that Legendre polynomials are either even or odd* is also *implied* by the symmetry of Legendre's equation, but only because this symmetry is *combined* with the fact that $Q_\nu(\xi)$ is unbounded.

Hence, explanation will be found in the symmetry of the situation and in the signature and its consequences of the regular singular points. This latter aspect resonates with Gray's [8] recognition of Fuchs, as having been the first to see the decisive importance of regular singular points.

2. Form Invariance of an Equation and Implied Transformation Properties of Solutions.

Under the coordinate transform

$$\xi = -\eta, \tag{3}$$

Legendre's equation (1) transforms into

$$(1 - \eta^2)\, y'' - 2\,\eta\, y' + \nu\,(\nu + 1)\, y = 0 \;; \tag{4}$$

comparison of representations (1) and (4) reveals that Legendre's equation is *form invariant* under transformation (3). In the remainder of this manuscript, we shall refer to this symmetry as *mirror symmetry*. Now, let

$$y = y(\xi) := f(\xi) \;, \tag{5}$$

represent a solution of equation (1). In expressions (5), the sign "=" means, "the *value* of y is calculated as a function of ξ (*without specifying what the functional relationship between y and ξ would be, nor how it would be called*). In the second part of expressions (5), the symbol ":=" specifies that "this value is calculated by some functional expression f"; (*note that the symbol f in itself does not specify which variable would obtain the calculated value*). For example, if y would be the sine of x, f would be 'sin'.

Using this notation, we can describe any solution of equation (4) as

$$y = y(\eta) := g(\eta) \;. \tag{6}$$

Using (5) however, i.e. the fact that $y := f(\xi)$ solves (1), we see, by (3) and using the chain rule, that a special solution of equation (4) can be found as

$$y = y(\eta) := f(\xi(\eta)) := f(-\eta) \;. \tag{7}$$

Because equations (4) and (1) are equal in *form* however, we can conclude that, if $y := f(-\eta)$ solves (4), then $y := f(-\xi)$ solves (1).

In summary: the form invariance of Legendre's equation under transformation (3) implies that, whenever $y := f(\xi)$ is a solution of Legendre's equation (1), then so is $y := f(-\xi)$. N.B: it does *not* follow that the solutions themselves are form invariant: we may *not* conclude that solutions must be even, i.e. it is not *implied* (nor excluded) that $f(\xi) = f(-\xi)$.

The argument applies to any differential equation that has the same symmetry. For example, $y'' - y = 0$ is form invariant under transformation (3); so the fact that $y = \exp(x)$

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is a solution implies that $y = \exp(-x)$ is also a solution; but neither of these functions is even.

3. Mirror symmetric Fuchsian, 2^{nd} order ODE's, with regular singular points at $\xi = \pm 1$.

Now consider a Fuchsian, 2^{nd} order differential equation, with regular singular points at $\xi = 1$ and $\xi = -1$. Because the equation is linear, its general solution is the span of a fundamental set $\{y_1 := f(\xi), y_2 := g(\xi)\}$. Mirror symmetry, i.e. form invariance under transformation (3), then implies that

$$f(-\xi) \in \mathrm{span}\{\, y_1 := f(\xi), y_2 := g(\xi) \,\} \ , \tag{8}$$

or

$$f(-\xi) \;=\; \alpha\, f(\xi) \;+\; \beta\, g(\xi) \ \ , \ \text{ for some numbers } \alpha, \beta. \tag{9}$$

We now further confine cases to equations for which, as for Legendre's equation, y_2 is unbounded, while y_1 is bounded, for $\xi \uparrow 1$; so $f(1)$ is assumed to be finite.

Then, if $f(\xi)$ would be bounded at *both* regular singular points $\xi = \pm 1$, then, considering (9) in the limit $\xi \uparrow 1$ implies $\beta = 0$, because $g(\xi)$ is unbounded in this limit. Hence, under the adopted restrictions, relation (9) reduces to

$$f(-\xi) \;=\; \alpha\, f(\xi) \ , \tag{10}$$

from which we deduce

$$f(-\xi) \;=\; f(-(-\xi)) \;=\; \alpha\, f(-\xi) \;=\; \alpha^2\, f(\xi) \ , \tag{11}$$

so that we must have $\alpha^2 = 1$, so $\alpha = \pm 1$. Hence, with (10), we arrive at the following

Lemma 1. *If a mirror symmetric, 2^{nd} order Fuchsian ordinary differential equation with regular singular points at $\xi = \pm 1$ has a fundamental solution y_2 that is unbounded at the regular singular point $\xi = \pm 1$, while the other fundamental solution y_1 is bounded at* both *regular singular points, then y_1 is either even or odd.*

Corollary 1. *Legendre polynomials must be either even or odd.*

Reversely, it follows that,

Corollary 2. *If a mirror symmetric, 2^{nd} order Fuchsian ordinary differential equation with regular singular points at $\xi = \pm 1$ has a fundamental solution y_2 such that $\lim_{\xi \uparrow 1} y_2(\xi)$ is unbounded, while a first solution $y_1(\xi)$ is finite at $\xi = 1$, then, unless $y_1(\xi)$ is either even or odd, $\lim_{\xi \downarrow -1} y_1(\xi)$ is unbounded.*

Note that thus such a singularity of $y_1(\xi)$ at $\xi = -1$ is a *consequence* of the mirror symmetry, as well as of the linearity, of the Fuchsian differential equation *and* of the singularity of the *other* fundamental solution y_2 at the *other* regular singular point $\xi = 1$.

As we shall see, this argument applies to Legendre Functions of the first kind, and hence demonstrates and explains their unboundedness at $\xi = -1$.

4. Legendre functions of the first kind are neither even, nor odd.

From our result in the previous section it follows that the singularity (unboundedness) of Legendre functions of the first kind, $P_\nu(\xi)$, would be *implied* by the fact that, for non-integer values of ν, their curves are not mirror symmetric in the vertical axis, nor point-mirror symmetric in the origin. That is, to prove that these functions are unbounded at $\xi = -1$, is suffices to show that the functions are neither even nor odd. This absence of evenness and oddness can be confirmed from their series expansions about $\xi = 0$. This we shall explore in this section.

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A technical complication is rooted in the fact that Legendre functions of the first kind, $P_\nu(\xi)$, are *defined* as those solutions of Legendre's equation (1) that take a finite value at the regular singular point $\xi = 1$. The functions are conventionally normalized as

$$P_\nu(1) = 1 \ , \tag{12}$$

which sets the leading coefficient of their power series expansion about $\xi = 1$ equal to $a_0 = 1$. As a consequence however, finding their exact value $P_\nu(0)$ *at the origin* $\xi = 0$ is not a trivial affair, and hence neither is finding their series expansions about $\xi = 0$, from scratch, as we wish to do here, for the sake of offering a self-contained treatment of our topic.

Indeed, because $P_\nu(\xi)$ is *defined* as the non-singular solution of Legendre's equation (1), that obeys condition (12), following Frobenius [12] we start by looking for generalized power series solutions

$$y(x) \ = \ \sum_{n=0}^{\infty} a_n \, x^{n+r} \ , \tag{13}$$

of equation (1), as rewritten in terms of a shifted coordinate x

$$x \ = \ \xi - 1 \ . \tag{14}$$

In terms of x, equation (1) takes the form

$$-\, x\,(x\,+\,2)\,y'' \,-\, 2\,(x\,+\,1)\,y' \,+\, \nu\,(\nu\,+\,1)\,y \ = \ 0 \ . \tag{15}$$

Following Frobenius' method [12,13,16–20], we readily find that r needs to satisfy the indicial equation [6,7]

$$F(r) \ = \ 0 \quad \text{with} \quad F(r) \ = \ 2\,r^2 \ . \tag{16}$$

From Frobenius' theory and the fact that the indicial equation (16) has a double root, $r_1 = r_2 = 0$, it immediately follows that equation (15) has one analytical solution $y_1(x)$ about the regular singular point $x = 0$, with $y_1(x)|_{x=0} \neq 0$, while *consequently* the second, linearly independent solution is *unbounded* in the limit $x \to 0$, due to a logarithmic singularity.

Hence, we see that the characters of the Legendre functions of first and second kind about the regular singular point $x = 0$, i.e. $\xi = 1$, are immediate from the indicial equation. The same is implied for *any* Fuchsian differential equation that has (16) as an indicial equation.

Proceeding with Frobenius' method, for the recurrence relation of the coefficients a_n for $y_1(x)$ as in (13) we readily find

$$a_{n+1} \ = \ -\frac{n\,(n\,+\,1) - \nu\,(\nu\,+\,1)}{F(n+1)}\,a_n \ ; \tag{17}$$

from this, with $a_0 = 1$, all a_n can be obtained, in principle. The resulting series are the series expansions of the Legendre functions of the first kind, $P_\nu(x)$, about the regular singular point $x = 0$, or $\xi = 1$.

From recurrence relation (17) it follows that the Legendre functions $P_\nu(x)$ of the first kind are polynomials $P_N(x)$, (the Legendre polynomials indeed), if and only if ν takes an integer value N. Indeed, negative values for such N would not add any independent solutions, that are not already obtained for positive N, while for $\nu = N$ and $0 \leq N$, relation (17) implies $a_n = 0$ for all n, $N < n$.

To decide whether or not the Legendre functions of the first kind are even, or odd, we need their representation in terms of ξ, so we substitute (14) for x, together with $r = 0$, into (13) and expand binomially, to find

$$P_\nu(\xi) \ = \ \sum_{n=0}^{\infty} \sum_{m=0}^{n} (-1)^{n-m} \binom{n}{n-m} a_n\,\xi^m \ , \tag{18}$$

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which can be rearranged to

$$P_\nu(\xi) \;=\; \sum_{m=0}^{\infty}\;\sum_{n=m}^{\infty}(-1)^{n-m}\left(\begin{array}{c} n \\ m \end{array}\right) a_n\,\xi^m \quad , \tag{19}$$

so that we have

$$P_\nu(\xi) \;=\; \sum_{m=0}^{\infty} c_m\,\xi^m \quad , \tag{20}$$

with

$$c_k \;=\; \sum_{j=k}^{\infty}(-1)^{j-k}\left(\begin{array}{c} j \\ k \end{array}\right) a_j \;\; ; \tag{21}$$

in expressions (18) to (21) we have used the notation

$$\left(\begin{array}{c} n \\ i \end{array}\right) \;=\; \frac{n!}{i!\,(n-i)!} \quad . \tag{22}$$

5. $P_\nu(\xi)$ is neither odd nor even when n is non-integer.

Because the graph of

$$f(\nu) \;=\; \nu\,(\nu+1) \tag{23}$$

is a parabola, with a minimum for $\nu = -\frac{1}{2}$, $f(\nu)$ takes all its possible (real) values for $-\frac{1}{2} \leq \nu$, so we need to consider $P_\nu(\xi)$ only for these values for ν:

$$-\frac{1}{2} \leq \nu \quad . \tag{24}$$

The function $f(\nu)$ then is strictly increasing, as a function of ν, so that

$$m < \nu \;\;\text{implies}\;\; m\,(m+1) - \nu\,(\nu+1) < 0 \quad . \tag{25}$$

According to the recurrence relation (17) then, with (16), i.e. $0 < F(n+1)$ for all n, and given that $P_\nu(x)|_{x=0} = a_0 = 1$, we have

$$0 < a_m \quad \text{for all } m < \nu \quad . \tag{26}$$

Now, assume that ν is not an integer and let M be the smallest integer, such that $\nu < M$, then from (17), (25) and (26) we find the following sign pattern for the coefficients a_n:

$$\begin{array}{cccccc} a_0, & \ldots & a_M, & a_{M+1}, & a_{M+2}, & \ldots \\ + & + & + & - & + & \text{(alternating)} \end{array} \quad , \tag{27}$$

that is: up until and including a_M all coefficients a_n will be positive, a_{M+1} will be negative, and from then on, the signs of the coefficients will alternate. Furthermore, we may conclude from the recurrence relation (17) that, for non-integer ν, none of the coefficients a_m will take the value zero:

$$0 \neq a_n \quad , \quad \text{whenever } \nu \text{ is non-integer.} \tag{28}$$

As a consequence, because (21) implies

$$c_M \;=\; \sum_{j=M}^{\infty}(-1)^{j-M}\left(\begin{array}{c} j \\ M \end{array}\right) a_j \quad , \tag{29}$$

we may conclude from (27) and (28) that $0 < c_M$. Similarly, from

$$c_{M+1} = \sum_{j=M+1}^{\infty} (-1)^{j-(M+1)} \begin{pmatrix} j \\ M+1 \end{pmatrix} a_j \ , \tag{30}$$

combined with (27) and (28) we conclude that $c_{M+1} < 0$.

Therefore, we have found *two subsequent*, non-zero coefficients c_n, and hence the Legendre function of the first kind, $P_\nu(\xi)$, for non-integer ν, is neither odd nor even.

Combined with corollary 2, this completes our proof that $P_\nu(\xi)$ is unbounded in the limit $\xi \downarrow -1$, i.e. at the opposite regular singular point.

6. Conclusion

To summarize our focus and line of reasoning, we list the following observations.

In important applications in physics, such as atomic physics, electro-magnetism, classical gravitation, and in astro- and geophysical fluid dynamics, in particular when Laplace's operator is involved in spheroidal coordinates, Legendre's equation (1) is key.

The literature about special functions offers extensive and detailed documentation of the general solution of this equation, in terms of the Legendre functions of the first and second kind.

The *complete* problem statement in physics however often does not merely consist of Legendre's equation, but rather of Legendre's equation *supplemented* with the requirement that we are looking for functions that solve this equation *while they remain finite throughout the domain of application*. A consequence of this condition is that all of the Legendre functions, except Legendre polynomials, are unacceptable, not as solutions of Legendre's equation, but as solutions of the problem statement in physics.

From the point of view of physics therefore, a theoretical treatment that includes detailed documentation of all Legendre functions is at least uneconomical, and it risks missing quintesential arguments.

In the present manuscript we offer an alternative, in the form of an argument, in as elementary terms as possible, that shows, and explains *why*, only the Legendre *polynomials* are bounded, hence *acceptable*, solutions of the stated problem. As a by-catch, we found that these polynomials *must* be either even or odd.

There seems to be added value in that our argument shows that these results are all consequences of a mirror symmetry of the situation, but only if and because this symmetry is *combined* with the signature of the regular singular points of Legendre's equation, as it can be readily obtained from Frobenius' theory. That the signature of the regular singular points has such a decisive role in the argument is fully in accordance with the classical works of Fuchs on the class of differential equations that are now named after him.

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