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The Theory of Quantum Uncertainties and Quantum Measurements

harsh.anand@tifr.res.in

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1 Author

Harsh Anand

2 Affiliation

Tata Institute of Fundamental Research, Mumbai

3 Correspondence

India, Telangana, Hyderabad, Tarnaka, Nagarjuna Nagar, 12-13-696, Ph No. 7659912238

4 Email Id

harsh.anand@tifr.res.in

5 Orcid Id

0000-0001-6903-4864

6 Abstract

1) We shall discuss what modern interpretations say about the Heisenberg's uncertainties. These interpretations explain that a quantity begins to 'lose' meaning when a conjugate property begins to 'acquire' definite meaning. We know that a quantity losing meaning means that it has no fixed value and has an uncertainty . In this paper we look deeper into this interpretation and the outcome reveals more evidence of the quantity losing meaning. Newer insights appear.

2) We consider two extreme cases of hypothetical processes nature undergoes, without interference by a measurement: One, a system collapses to an energy eigenstate under the influence of a Hamiltonian instantaneously at a time t. This is thus what would happen if we would measure the system's energy. Next, when a particle becomes localised to a point at a time t_0 under the influence of a Hamiltonian. This is thus what would happen if we would measure the system's position. We shall prove that both these situations cannot arise under ordinary circumstances and thus measurement processes cannot be modelled by physical Hamiltonians.

7 Keywords

Quantum information, Foundations of Quantum mechanics, Quantum time evolution, Quantum measurements

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Abstract

A quantum system has some inherent amounts of hidden data. This data can be determined only via the act of measurement and thus via disturbing the system. An example of such a measurement process could be simply the experiment* of measuring the energy, position of a wave . This experiment is in general a discontinous process in which the wave doesn't evolve continously. In this experiment, after we conduct it, a)the energy states collapse instantaneously to a eigenstate, b) the wave spread all across collapses to a single point,

Here we want to study a case where the system is under the effect of a Hamiltionian that performs a and b and thus mimics the act of the experiment* but does it without disturbing the system via the act of measurement. That is, if it were possible to mimic the act of measurement by including the presence of a Hamiltonian in the system, the Hamiltonian being such that it performs a,b while evolving the states analytically ; what physical characteristics of the final wave would be observed? From these results, we conclude that without disturbing the system (and thus enact the measurement) : 1) A general normalised statefunction (that is a linear superposition of energy eigenstates which has probability p_i to collapse into the ith energy eigenstate) cannot time evolve into into the ith eigenstate instantaneously (like happens when we measure the energy) (or infact cannot time evolve instantaneously into any other general statefunction). 2) A general normalised statefunction (that is, a linear combination of continous position eigenstates) cannot instantaneously collapse into a particular position coordinate x_0 on the real line , (like happens when we measure it's position in the lab) , when it is under the effect of a Hamiltonian which analytically time evolves and space evolves the state.

In this way, we conclude that the process of measuring a system via experiments in the quantum world cannot be theoretically modelled using any conceivable Hamiltonian.

We divide this paper into three parts:

1) Rapid building up of energy of a system when one state evolves into another in an infinitesimal time interval.

2) Rapid building up of momentum of a system during rapid localising of a wave.

3) In part 3, in the conclusion, we look into the modern interpretations of the Heisenberg uncertainties; which state that the uncertainties in the certain quantities in the quantum regime do not arise due to disturbing a system via measurement. These uncertainties are simply inherent, they are actually there in the quantities! *Quantum physics states that whether we look/measure or not, some quantities exist that simply have no definite value* (for example, the position of a particle isn't a single value, it's a probabilistic random variable).

This interpretation is predicted, starting from the definitions of the Hamiltonian operator and the momentum operator in this paper.

1 Rapid building up of energy of a system when one state evolves into another in an infinitesimal time interval

1.1

We first note that between two fixed states, there are infinite Hamiltonians that evolve one into the other. It happens in this way: Let Ψ and Ψ' be the solutions of two different Schrödinger equations. It may happen that when we plug in a value of t=1 say , in Ψ and t=2 in Ψ' the expressions of both the statevectors become the same.

(For example,

$$\Psi_1(x_1,t_1) = e^{-x_1}e^{i\sin(\pi t_1/3)} + e^{-x_1^2}e^{i\cos(\pi t_1/5)}$$

and

$$\Psi_2(x_2,t_2) = e^{-x_2}e^{i\cos(\pi t_2/12)} + e^{-x_2^2}e^{i\sin(3\pi t_2/20)}$$

These are such that they become the same expression at $t_1 = 1\,\mathrm{s}$ and $t_2 = 2\,\mathrm{s}$ respectively.)

Thus, when we are given two *fixed states* (they contain only the spacial coordinates), each of these states could have been formed by plugging in different values of the time coordinates present in different statevectors, which are solutions of different Schrödinger equations(formed by different Hamiltonians).

(For example,take any random Hamiltonian that forms a time dependent state vector. When values $t = t_1$ and $t = t_2$ are plugged in the statevector, two states are formed. Now consider another Hamiltonian which produces another statevector. When values $t = t'_1$ and $t = t'_2$ are plugged in this statevector, two states are formed.But these two states are the same states that were formed when values $t = t_1$ and $t = t_2$ were plugged in the first statevector).

Thus there are infinite Hamiltonians that evolve one fixed state to another, but the time the evolution takes varies.

Now, let us have two fixed states $\Psi(x, t_1)$ and $\Psi(x, t_2)$ such that

$$\Psi(x, t_1) = \Psi'(x, t'_1), \Psi(x, t_2) = \Psi'(x, t'_2) \tag{1}$$

where Ψ' is a sample statevector (it is such that it passes through the two fixed states). We note that

$||\Psi(x, t_1)$- $\Psi(x, t_2)||$ is a constant (as the Hamiltonians are varied).

Let us consider one particular path (one particular Hamiltonian) from one of the states to the other. We shall prove this inequality :

$$||\Psi(x, t_1)\text{-}\ \Psi(x, t_2)|| \leq \int_{t_1}^{t_2} \sqrt{\langle H^2 \rangle}_{\Psi(x,t)} dt/\ \hbar \tag{2}$$

Proof:

We have

$$\mathrm{i}\hbar\partial\Psi/\partial t = H\Psi \tag{3}$$

That is,

$$\Psi(t + dt) - \Psi(t) = dtH\Psi(t)/(i\hbar) \tag{4}$$

Taking norm on both sides, we get

$$||\Psi(t + dt) - \Psi(t)||\hbar = \sqrt{\langle H^2 \rangle}_{\Psi(x,t)} dt \tag{5}$$

Then, for times t_1 and t_2

$||\Psi(t_2) - \Psi(t_1)|| \leq ||\Psi(t_2) - \Psi(t_2 - dt)|| + \ldots.||\Psi(t_1 + dt) - \Psi(t_1)||$

$= \Sigma\sqrt{\langle H^2 \rangle}_{\Psi(x,t)} dt\ /\ \hbar = \int_{t_1}^{t_2} \sqrt{\langle H^2 \rangle}_{\Psi(x,t)} dt/\hbar$ Thus,

$$||\Psi(t_2) - \Psi(t_1)|| \leq \int_{t_1}^{t_2} \sqrt{\langle H^2 \rangle}_{\Psi(x,t)} dt/\hbar \tag{6}$$

Hence proved.

1.2 Inferences

1)

$||\Psi(x, t_1)$- $\Psi(x, t_2)|| \leq \int_{t_1}^{t_2} \sqrt{\langle H^2 \rangle}_{\Psi(x,t)} dt/\ \hbar$

Now this inequality is an inequality satisfied by each path the initial state takes to evolve into the final state. The interesting thing to observe is that by Eq(5),

$\Psi(x, t_1) = \Psi'(x, t'_1), \Psi(x, t_2) = \Psi'(x, t'_2)$ where the primes denote any other system which passes through the two fixed states in it's time evolution by the system's Hamiltonian. In this other system, the inequality is

$||\Psi'(x, t'_1)$- $\Psi'(x, t'_2)|| \leq \int_{t'_1}^{t'_2} \sqrt{\langle H^2 \rangle}_{\Psi'(x,t')} dt'/\ \hbar$

That is, the other system's statevector is Ψ', it passes through the two fixed states in times t_1' and t_2' respectively. Here,

$||\Psi(x,t_1)$- $\Psi(x,t_2)|| = ||\Psi'(x,t_1')$- $\Psi'(x,t_2')||$

That means, the right hand side of the inequality, the larger term, is always greater than a fixed number, as the Hamiltonians (and thus the right hand side of the inequality,the integrals) are varied. So if we choose a Hamiltonian such that it time evolves one of the states into the other in a short time dt', the integral will approximately become

$\sqrt{\langle H^2\rangle}_{\Psi'(x,t')}dt'/\ \hbar$

Even with the term dt' ***multiplied , this is larger than the constant. That means*** $\sqrt{\langle H^2\rangle}_{\Psi'(x,t')}/\hbar$

has to be sufficiently large. The energy of the system, to some accuracy is given by this term, and thus, if we need the system to time evolve from one state to the other in a short time interval , we need the energy to be given to the system as sufficiently large.

In different words, say we claim that we have indeed found a particular system's particular state that evolves to another state via the given Hamiltonian, in infinitesimal time. **Then**, the state has extremely high energy.

2 Rapid building up of momentum of a system during rapid localising of a wave

2.1 Question

Let $f_1(x,t)$ and $f_2'(x',t)$ be two complex valued functions, each of two real variables x,t and x',t respectively. Let them be constructed such that $f_1(x=1,t) = f_2'(x'=2,t) = a_1(t)$ and $f_1(x=3,t) = f_2'(x'=2+\epsilon,t) = a_2(t)$. The question is will there exist such a function f_2', which is analytic, for every ϵ. Clearly it will also depend on $a_2(t)$. For our discussion here, we shall explicitly assume that such functions exist and also use such functions [See section 4].

The question:

Let $\Psi(x,t)$ be a statefunction. Say this particle is majorly spread over coordinates x=1 and x=2 during the time interval we are interested in. We want to change this statefunction such that the new statefunction is majorly spread only over a small interval,(x_1',x_2'). The new wavefunction is $\Psi'(x',t)$. The question is then: Find a quantity depending on the magnitude of the momentum of the particle that needs to be optimised during the time interval in which the process is happening .

In all our discussions below, $\Psi(x)$ shall mean $\Psi(x,t)$.

2.2 Definitions

Now we shall define the quantities.

$$\langle P\rangle_N = (\hbar/i)\textstyle\int_0^{t_0}\ \Psi^*|_{x1}(\partial\Psi/\partial x)|_{x1}\mathrm{dt} \tag{7}$$

$$||\Psi(x_1)||_N^2 = \textstyle\int_0^{t_0}\ \Psi^*(x_1)\Psi(x_1)dt \tag{8}$$

and

$$\langle\Psi(x_1)|\Psi(x_2)\rangle_N = \textstyle\int_0^{t_0}\ \Psi^*(x_1)\Psi(x_2)dt \tag{9}$$

where, the number t_0 is a constant positive number.

2.3 Calculations

We first need to prove that the inner product defined in Eq(9) is a valid inner product. We then need to prove that the norm defined in Eq(8) is a valid norm.

2.3.1 Eq 9, the new inner product:

$\langle\Psi(x_1)|\Psi(x_2)\rangle_N = \int_0^{t_0} \Psi^*(x_1)\Psi(x_2)dt$

It can be easily seen that there is a conjugate symmetry between the vectors, the inner product is linear in it's second argument and $\langle\Psi(x_1)|\Psi(x_1)\rangle_N$ is always real(by conjugate symmetry) and positive, if $\Psi(x_1,t)$ is not 0 for all t uptil t_0.

Thus it is a valid inner prooduct.

2.3.2 Eq 8, the new norm

$||\Psi(x_1)||_N^2 = \int_0^{t_0} \Psi^*(x_1)\Psi(x_1)dt$

Here it can be easily verified that $||\Psi(x_1)||_N^2$ is always greater than 0, is equal to 0 only if $\Psi(x_1,t)$ is 0 for all time uptil t_0, in which case we would say that $\Psi(x,t)$ is 0 at x_1 for all t uptil t_0 but may or may not be zero for all x. The scaling property is also easily verified. However the triangle inequality can be proved only by using the Shwartz inequality on the new inner product defined in 3.1.1 .

Proof that the new norm satisfies the triangle inequality:

$||\Psi(x_1)+\Psi(x_2)||_N^2 = \int dt(\Psi(x_1)^*+\Psi(x_2)^*)(\Psi(x_1)+\Psi(x_2)) = \int dt[(\Psi(x_1)^*\Psi(x_1))+(\Psi(x_2)^*\Psi(x_2))+(\Psi(x_1)^*\Psi(x_2))+(\Psi(x_2)^*)\Psi(x_1))] = ||\Psi(x_1)||_N^2 + ||\Psi(x_2)||_N^2 + 2|\langle\Psi(x_1)|\Psi(x_2)\rangle_N| \leq ||\Psi(x_1)||_N^2 + ||\Psi(x_2)||_N^2 + 2||\Psi(x_1)||_N||\Psi(x_2)||_N = (||\Psi(x_1)||_N + ||\Psi(x_2)||_N)^2$

2.3.3

$$\text{-}i\hbar\partial\Psi/\partial x = P\Psi \tag{10}$$

That is,

$$\Psi(x+dx) - \Psi(x) = dxP\Psi(x)/(-i\hbar) \tag{11}$$

Taking the new norm on both sides, we get

$$||\Psi(x+dx) - \Psi(x)||_N\hbar = \sqrt{\langle P^2\rangle}_N dx \tag{12}$$

Then, for points x_1 and x_2

$||\Psi(x_2) - \Psi(x_1)||_N \leq ||\Psi(x_2) - \Psi(x_2 - dx)||_N + ...||\Psi(x_1 + dx) - \Psi(x_1)||_N$

$= \Sigma\sqrt{\langle P^2\rangle}_N dx \ / \ \hbar$

$=$

$\int_{x_1}^{x_2}\sqrt{\langle P^2\rangle}_N \mathrm{dx}/\hbar$. Thus,

$$||\Psi(x_2) - \Psi(x_1)||_N \leq \int_{x_1}^{x_2}\sqrt{\langle P^2\rangle}_N dx/\hbar \tag{13}$$

Caution: In equations 12 and 13, the $\langle P^2\rangle_N$ is the newly defined expectation value of P^2 ; from Eq(7). Similarly the new norm is denoted by $_N$

2.4 Answer

We drop the time coordinate t from our notations but include them where used.We have $\Psi(x,t)$ to begin with. It shall be transformed to $\Psi'(x',t)$ by an unspecified process. From time 0 to t_0, both of the wavefunctions are majorly spread over the space intervals $(1,2)$ and (x_1', x_2') respectively. $(x_2' - x_1')$ is an infinitesimal as mentioned in Section 1. This means that the value of the wavefunctions outside these intervals is negligibly small in the specified time interval.

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We mentioned that Ψ' has been transformed from Ψ. However we do not know of any other way to relate them. The process happened under a random unknown Hamiltonian. However the one assumption we may make is:

$\Psi(x=1) = \Psi'(x'=x'_1) = \mathrm{f}$

$\Psi(x=2) = \Psi'(x'=x'_2) = \mathrm{g}$

We know:

$||f-g||_N \leq \int_1^2 \sqrt{\langle P^2\rangle}_N dx/\hbar$

We thus also have :

$||f-g||_N \leq \int_{x'_1}^{x'_2} \sqrt{\langle P^2\rangle}_N dx'/\hbar$

.
Considering the latter; we get:

$||\Psi'(x'_1)$ - $\Psi'(x'_2)||_N\hbar \leq \int_{x'_1}^{x'_2} \sqrt{\langle P^2\rangle}_N dx' \approx \sqrt{\langle P^2\rangle}_N (x'_2 - x'_1) = \sqrt{\langle P^2\rangle}_N dx'$

Thus, we have;

$$||\Psi'(x'_1) \text{ - } \Psi'(x'_2)||_N\hbar \leq \sqrt{\langle P^2\rangle}_N dx' \tag{14}$$

where,

$$\langle P^2\rangle_N = (\hbar/i)^2 \int_0^{t_0} \Psi'^*|_{x'_1} (\partial^2\Psi'/\partial x'^2)|_{x'_1} \mathrm{dt} \tag{15}$$

;
where x'_1 is taken just for convenience. It could be any number between (x'_1, x'_2)
Now, we have

$\Psi'^*|_{x'_1} (\partial^2\Psi'/\partial x'^2)|_{x'_1} = [\ \Psi'^*(\partial^2\Psi'/\partial x'^2)]|_{x'_1}$

$= d(\int_{-\infty}^{x} \mathrm{dx'} \Psi'^*(\partial^2\Psi'/\partial x'^2))\ /dx\ |_{x'_1} = (dy/dx)|_{x'_1}$

where,

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$y= \int_{-\infty}^{x} dx'\Psi'^{*}(\partial^2\Psi'/\partial x'^2)$

Thus, if u is a small enough number ,

$dy/dx|_{x'_1} = (y(x'_1+u) - y(x'_1))/u$

$\approxeq_{(1)}$

$(1/u)\int_{x'_1}^{x'_2}[dx'(\Psi'^{*}(\partial^2\Psi'/\partial x'^2)]$

$\approxeq_{(2)}$ $(1/u)\int_{-\infty}^{\infty}[dx'(\Psi'^{*}(\partial^2\Psi'/\partial x'^2)] = (i/\hbar)^2\langle P^2\rangle/u$

where $\langle P^2\rangle$ is the usual average value of P^2.
Note:
1) $\approxeq_{(1)}$ is because x'_1+u is $\approxeq$x'$_2$, since we are considering the wavefunction present in a very narrow region.
2) $\approxeq_{(2)}$ is because outside (x'_1, x'_2), the wavefunction is almost 0.
Thus,

$\langle P^2\rangle_N = (\hbar/i)^2\int_0^{t_0}$ dt $(i/\hbar)^2\langle P^2\rangle/u$

Also, the small number u is approximately $= dx' = x'_2 - x'_1$.
Thus squaring Eq(14) gives:

$(||\Psi'(x'_1) - \Psi'(x'_2)||_N\hbar)^2 \leq \langle P^2\rangle_N\ (dx')^2$

$= (\int_0^{t_0}[\text{dt}\ \ \langle P^2\rangle/dx'])\ (dx')^2$

Thus we have , when t_0 is small, with

$$(||\Psi'(x'_1) - \Psi'(x'_2)||_N\hbar)^2 = C_0\ t_0 \tag{16}$$

$\Longrightarrow$

$$\text{dx'}\geq \text{C}_0\ t_0\ /(\ \int_0^{t_0}[\text{dt}\ \ \langle P^2\rangle]\ ) \tag{17}$$

$\Longrightarrow$

$$\text{dx'}\geq \text{C}_0/\langle P^2\rangle \tag{18}$$

where C_0 is a constant. Eq. 16 is because
$||\Psi'(x'_1) - \Psi'(x'_2)||^2{}_N$
$= \int_0^{t_0}\ |\Psi'(x'_1) - \Psi'(x'_2)|^2 dt$
$\approxeq |\Psi'(x'_1) - \Psi'(x'_2)|^2\text{t}_0$, when t_0 is small and

$C_0 = |\Psi'(x_1') - \Psi'(x_2')|^2$
Eq.18 is because
$\int_0^{t_0}[\mathrm{dt}\ \langle P^2 \rangle] \approxeq \langle P^2 \rangle t_0$.
Thus we have, from Eq(18);
$\langle P^2 \rangle dx' \geq \mathrm{C}_0$;
which means that the momentum squared, or the kinetic energy is an infinity if the particle exists in an infinitesimal region.

2.5 Remark

We in our discussion, assumed that the only connection between the waves Ψ and Ψ' was that their expressions were respectfully equal at the respective end points. We did not use the properties of the Hamiltonian that time evolves them. By including this information from well and suitably defined Hamiltonians, we shall be able to conclude our hypothesis.

3 Conclusions

1) From section 1, we have obtained that for any state Ψ in the presence of a known Hamiltonian to instantaneously jump into the being of an eigenstate of the Hamiltonian, which is what happens in a measurement process, the energy of the state would have to be an infinity which contradicts the fact that the expectation value of it's energy is infact finite.

2) From section 2, we have gathered that any state $\Psi(x,t)$, in the presence of a known Hamiltonian, simply cannot "*be*" localised to a point under ordinary conditions. It can "*be found*" to be localised to a point under ordinary conditions, but for it to be localised somewhere in space, it would have to have an unusually large amount of kinetic energy.

3.1

We note that the Heisenberg's uncertainties say this: 1) The time energy uncertainty says that the more precisely we know the particle's time of evolution, the more uncertain is it's energy. 2) The position momentum uncertainty says that the more precisely we know the particle's position, the less precisely we know it's momentum.

Further, from relatively new experiments, we know that this doesn't need to invoke measurement processes. At a fundamental level, we know that if the particle *is* (*whether we know it or not*) localised in a small region, it's momentum will infact not be defined. This means that the momentum will have no definite value, it will have no meaning . Similarly, if the particle time evolves into another state in infinitesimal time, it's energy will be undefined, it will have no meaning.

3.2

However, we have arrived at more illuminating and novel conclusions: In our discussion, we have considered two extreme cases.

One, when a state evolves into another in infinitesimal time. The Heisenberg uncertainty says regarding this that the energy of this state is then undefined. The value of energy should have been one number, because the state must have one value of it's energy. But, against this intuition, we know that the energy value now has a large uncertainty associated with it, which means the energy of the state is no longer fixed; basically we cannot speak of an entity called energy. We derived in this paper that not only is the uncertainty in the energy of the state large, but also that " the range of values in which the energy lies is large, an infinity." **We claim here that this is the reason why there is no longer a definition of energy, that it is no longer meaningful.**

Two, when a particle is localised at a point. The Heisenberg uncertainty says regarding this that the momentum of this particle is then undefined. The value of momentum should have been one number, because the state must have one value of it's momentum. But against this intuition, we know that the momentum value now has a large uncertainty associated with it, which means the momentum of the state is no longer fixed; basically we cannot speak of an entity called momentum. We derived in this paper that not only is the uncertainty in the momentum of the state large, but also that " the range of values in which the momentum lies is large, an infinity." **We claim here that this is the reason why there is no longer a definition of momentum, that it is no longer meaningful.**

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