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A way to construct quantum speed limits between arbitrary fixed states

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Abstract:
We shall first give an introduction to the subject of quantum speed limits and then present a more general quantum speed limit than what is currently known and which has been derived in an unconventional way. Finally, we generalise the process and find that there are infinite speed limits that can be constructed via a similar process.

Keywords: Quantum Information, Quantum computing, Quantum speed limits

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A way to construct quantum speed limits between arbitrary fixed states

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We shall first give an introduction to the subject of quantum speed limits and then present a more general quantum speed limit than what is currently known. Finally, we generalise the process and find that there are infinite speed limits that can be constructed via a similar process.

I. INTRODUCTION

Quantum speed limits are rigorous estimates on how fast a state of a quantum system can depart from the initial state in the course of quantum evolution.

We first look into what work has been established so far (without going through the derivations, we state the final results, which are the known quantum speed limits and under what conditions or approximations they hold true):

We describe important milestones, the Mandelstam-Tamm and the Margolus-Levitin bounds on the quantum speed limit.

1) The uncertainty relation of Mandelstam and Tamm: Mandelstam and Tamm proposed the first notion of a quantum speed limit :

The minimal time for a quantum system to evolve between two orthogonal states is determined by:

$$T \geq \hbar\pi/(2\Delta H) \tag{1}$$

(For the derivation, please refer reference 3).

Uffink realised that in many situations ΔH gives not a very reasonable measure for the speed of a quantum evolution. The lower bound in the inequality can become arbitrarily small as ΔH, can be infinite even if the average energy is finite.

2) The quantum speed limit of Margolus and Levitin:

To tackle this problem Margolus and Levitin proposed an alternative derivation of the quantum speed limit. They obtained the minimal evolution time between two orthogonal states as :

$$T \geq \hbar\pi/(2 < H >) \tag{2}$$

Note that they explicitly assume that the average energy, $< H >$, is non-negative.

(For the derivation, please refer reference 3)

3) The unified bound is tight:

It was finally assumed that the minimum time for time evolution between orthogonal states is given by:

$$\mathrm{T} \geq max(\hbar\pi/(2 < H >), \hbar\pi/(2\Delta H)) \tag{3}$$

However, it was Levitin and Toffoli who finally realized that the situation was not quite as simple.

They proved the following theorem:

Under the assumption that the ground state energy of a quantum systems is zero, the only state for which the Mandelstam-Tamm bound Eq. (1) as well as the Margolus-Levitin bound Eq. (2) are attained is given by

$$|\Psi >= (|E_0 > +|E_1 >)/\sqrt{2} \tag{4}$$

;where $H|E_k >= kE_k|E_k >$, for k=1,2.

, and E1 is the energy of the first excited state.

II.

Now we shall present a few results that may improve the current speed limits and also may be used in wider domains.

We first note that between two fixed states, there are infinite Hamiltonians that evolve one into the other. It happens in this way: Let Ψ and Ψ′ be the solutions of two different Schrödinger equations. It may happen that when we plug in a value of t=1 say , in Ψ and t=2 in Ψ′ the expressions of both the statevectors become the same.

(For example,

$$\Psi_1(x_1, t_1) = e^{-x_1} e^{i\sin(\pi t_1/3)} + e^{-x_1^2} e^{i\cos(\pi t_1/5)}$$

and

$$\Psi_2(x_2, t_2) = e^{-x_2} e^{i\cos(\pi t_2/12)} + e^{-x_2^2} e^{i\sin(3\pi t_2/20)}$$

These are such that they become the same expression at $t_1 = 1\,\mathrm{s}$ and $t_2 = 2\,\mathrm{s}$ respectively.)

Thus, when we are given two *fixed states* (they contain only the spacial coordinates), each of these states could have been formed by plugging in different values of the time coordinates present in different statevectors, which are solutions of different Schrödinger equations(formed by different Hamiltonians).

(For example,take any random Hamiltonian that forms a time dependent state vector. When values $t = t_1$ and $t = t_2$ are plugged in the statevector, two states are formed. Now consider another Hamiltonian which produces another statevector. When values $t = t_1'$ and $t = t_2'$ are plugged in this statevector, two states are formed.But these two states are the same states that were formed when values $t = t_1$ and $t = t_2$ were plugged in the first statevector).

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Thus there are infinite Hamiltonians that evolve one fixed state to another, but the time the evolution takes varies.

Now, let us have two fixed states $\Psi(x,t_1)$ and $\Psi(x,t_2)$ such that

$$\Psi(x,t_1) = \Psi'(x,t_1'), \Psi(x,t_2) = \Psi'(x,t_2') \tag{5}$$

where Ψ' is a sample statevector (it is such that it passes through the two fixed states). We note that

$||\Psi(x,t_1)$- $\Psi(x,t_2)||$ is a constant (as the Hamiltonians are varied).

Let us consider one particular path (one particular Hamiltonian) from one of the states to the other. We shall prove this inequality :

$$||\Psi(x,t_1)\text{-}\ \Psi(x,t_2)|| \leq \int_{t_1}^{t_2} \sqrt{< H^2 >_{\Psi(x,t)}}dt/\ \hbar \tag{6}$$

in two different ways

1)

Proof:

We have

$$\mathrm{i}\hbar\partial\Psi/\partial t = H\Psi \tag{7}$$

That is,

$$\Psi(t+dt) - \Psi(t) = dtH\Psi(t)/(i\hbar) \tag{8}$$

Taking norm on both sides, we get

$$||\Psi(t+dt) - \Psi(t)||\hbar = \sqrt{< H^2 >_{\Psi(x,t)}}dt \tag{9}$$

Then, for times t_1 and t_2

$||\Psi(t_2) - \Psi(t_1)|| \leq ||\Psi(t_2) - \Psi(t_2 - dt)|| + ...||\Psi(t_1 + dt) - \Psi(t_1)||$

$= \Sigma\sqrt{< H^2 >_{\Psi(x,t)}}dt \ / \ \hbar = \int_{t_1}^{t_2}\sqrt{< H^2 >_{\Psi(x,t)}}dt/\hbar$

Thus,

$$||\Psi(t_2) - \Psi(t_1)|| \leq \int_{t_1}^{t_2}\sqrt{< H^2 >_{\Psi(x,t)}}dt/\hbar \tag{10}$$

Hence proved.

2) We have

$$\mathrm{i}\hbar\partial\Psi/\partial t = H\Psi \tag{11}$$

By the taylor expansion , we have , $f(t + dt) = f(t) + f'(t)dt + ...$ Here we have $f(t) = \Psi(t)$ and $f'(t) = H\Psi(t)/(i\hbar)$. Thus, on expanding

$$\Psi(t+dt) = \Psi(t) + dtH\Psi(t)/(i\hbar) - (dt)^2H^2\Psi(t)/\hbar^2 + ... \tag{12}$$

Applying the bra vector $< \Psi(t)|$ on both sides of eq (12) and then taking it's real part into consideration, we have:

$$Re(< \Psi(t)|\Psi(t+dt) >) = 1 - (\mathrm{dt})^2 < H^2 >_{\Psi\ (t)}/\hbar^2 \tag{13}$$

Now, for any times t_1 and t_2 , if U is the unitary time evolver of the system,

$$||\Psi(t_1)\text{-}\ \Psi(t_2)||^2 = ||U(t_2,t_1)\Psi(t_1) - \Psi(t_1)||^2 \tag{14}$$

= after much simplification,

$$2 - 2Re(< \Psi(t_2)|\Psi(t_1) >) \tag{15}$$

In Eq(15), substituting t_2 for t+dt, t_1 for t , we combine Eqs.(13) and (15)

$$||\Psi(t+dt) - \Psi(t)||\hbar = \sqrt{< H^2 >_{\Psi(x,t)}}dt \tag{16}$$

Then, for times t_1 and t_2

$||\Psi(t_2) - \Psi(t_1)|| \leq ||\Psi(t_2) - \Psi(t_2 - dt)|| + ...||\Psi(t_1 + dt) - \Psi(t_1)||$

$= \Sigma\sqrt{< H^2 >_{\Psi(x,t)}}dt \ / \ \hbar = \int_{t_1}^{t_2}\sqrt{< H^2 >_{\Psi(x,t)}}dt/\hbar$

Thus,

$$||\Psi(t_2) - \Psi(t_1)|| \leq \int_{t_1}^{t_2}\sqrt{< H^2 >_{\Psi(x,t)}}dt/\hbar \tag{17}$$

Hence proved.

III. INFERENCES

1)

$||\Psi(x,t_1)$- $\Psi(x,t_2)|| \leq \int_{t_1}^{t_2} \sqrt{< H^2 >_{\Psi(x,t)}}dt/\ \hbar$

Now this inequality is an inequality satisfied by each path the initial state takes to evolve into the final state. The interesting thing to observe is that by Eq(5),

$\Psi(x,t_1) = \Psi'(x,t_1'), \Psi(x,t_2) = \Psi'(x,t_2')$ where the primes denote any other system which passes through the two fixed states in it's time evolution by the system's Hamiltonian. In this other system, the inequality is

$||\Psi'(x,t_1')$- $\Psi'(x,t_2')|| \leq \int_{t_1'}^{t_2'} \sqrt{< H^2 >_{\Psi'(x,t')}}dt'/\ \hbar$

That is, the other system's statevector is Ψ', it passes through the two fixed states in times t_1' and t_2' respectively.

Here,

$||\Psi(x,t_1)$- $\Psi(x,t_2)|| = ||\Psi'(x,t_1')$- $\Psi'(x,t_2')||$

That means, the right hand side of the inequality, the larger term, is always greater than a fixed number, as the Hamiltonians (and thus the right hand side of the inequality,the integrals) are varied. So if we choose a Hamiltonian such that it time evolves one of the states into the other in a short time dt', the integral will approximately become

$\sqrt{< H^2 >_{\Psi'(x,t')}}dt'/\ \hbar$

Even with the term dt' multiplied , this is larger than the constant. That means

$\sqrt{< H^2 >_{\Psi'(x,t')}}/\hbar$

has to be sufficiently large. The energy of the system, to some accuracy is given by this term, and thus, if we need the system to time evolve from one state to the other in a short time interval , we need the energy to be given to the system as sufficiently large.

2)
$\int_{t_1}^{t_2} \sqrt{< H^2 >_{\Psi(x,t)}} dt/(t_2 - t_1)$

may be defined as the time average of the energies in the evolving states. Thus it is sort of an average of many averages.

3)
The inequality when applied to dynamics of two state systems or the most general time dependent three state systems always seems to reduce to the inequality $|sin(x)| \leq |x|$. For more details on the dynamics of quantum three state systems, please refer reference 2.
4)
In the introduction, we had seen that the known quantum speed limits had certain drawbacks. Namely:
a) They were useful only when considering the the evolution between two orthogonal states.

b) They were correct only for states of the form given by Eq(4).

Now the inequality that has been derived overcomes these drawbacks and gives an accurate quantum speed limit between any two states(which need not be orthogonal, nor be of the form of Eq(4)).

The problem of finding the speed limit for transition from one state to another presents by it's very construction the idea of varying the Hamiltonians between two fixed states. When we vary the Hamiltonians, the time values of the time at which the states have been evolved into or from, change. We have seen this. Thus to find the minimum time a state would take to get transformed into another state, we may consider only those Hamiltonians that do this evolution in a very short time.
Then it is clear that the new quantum speed limit of transition from one fixed state to another is the liminf of:

$$||\Psi(x,t_1)\text{- } \Psi(x,t_2)||\ \hbar/ \sqrt{< H^2 >_{\Psi'(x,t')}} \tag{18}$$

Where liminf is the infimum value of all the numbers given in Eq(18). Eq(18) has numbers formed by Hamiltonians that do the time evolution of the two states very quickly. Each time the process of time evolution of the two states is done by a Hamiltonian H (quickly), the time for evolution by this particular H is greater than the expresssion in Eq(18). Thus the infimum value of all the numbers in Eq(18) would give the required minimum time for time evolution of one of the fixed states into the other. Note that this minimum time can be close to zero only if
$\sqrt{< H^2 >_{\Psi'(x,t')}}$
is very large for some H, where H is one of the Hamiltonians that does the time evolution.

IV. GENERALISATION

Note that from Eq(8) we have,

$$\Psi(t + dt) - \Psi(t) = dtH\Psi(t)/(i\hbar) \tag{19}$$

, where we have renamed Eq(8) as Eq(19).

In it's right hand side, we have the term dt. Our objective then is to somehow obtain an expression for dt as a variable so we may find what is the minimum value it takes. Now let the two fixed states $\Psi(t_2), \Psi(t_1)$ be called Ψ_2, Ψ_1 respectively. We know that if H is a matrix that time evolves one of the two fixed states into the other in a very short time interval, $\Psi(t+dt) = \Psi_2$ and $\Psi(t) = \Psi_1$. That is, within time dt, the states change considerably. Then,

$$|\Psi_2\rangle - |\Psi_1\rangle = dt\,|H\Psi_1\rangle/(i\hbar) \tag{20}$$

V. INFINITE SPEED LIMITS

Let $\langle\Phi|$ be any fixed row vector. Operating it on both sides of Eq(20), we get what we desire:

$$\langle\Phi|\,(|\Psi_2\rangle - |\Psi_1\rangle)\text{= dt } \langle\Phi|H\Psi_1\rangle/(i\hbar) \tag{21}$$

Rearranging, we get,

$$\text{dt= } (i\hbar)\ \ \langle\Phi|\,(|\Psi_2\rangle - |\Psi_1\rangle)\ /\ \langle\Phi|H\Psi_1\rangle \tag{22}$$

Where the Hamiltonians H are such that they do the required time evolution in a short time dt.

Since $\langle\Phi|$ was an arbitrary row vector, we can see that there are infinite ways in which we can construct Eq(22). Given each $\langle\Phi|$, by varying the Hamiltonians, we get a lower bound for the time interval during which the time evolution of the fixed states can happen (we compare dt for each H and the minimum value of dt obtained would be the lower bound).

VI. IS IT ULTIMATE?

We note that Eq(20) gave us an expression for the time interval dt because *we could convert both sides of the equation into pure numbers.* Clearly, using every possible row vector $\langle\Phi|$, we encompass all possible ways in which we can do so. Thus the procedure developed gives the ultimate speed limit of transition between the arbitrary fixed states Ψ_1 and Ψ_2.

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