Sums and Partial Sums of Horadam Sequences: The Sum Formulas of $\sum_{k=0}^n x^k W_k$ and $\sum_{k=n}^{n+m} x^k W_k$ via Generating Functions

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Abstract. In this paper, we present sums and partial sums of Horadam sequences via generating functions which extends a recent result of Prodinger [6].

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1. Introduction

The generalized Fibonacci sequence (or generalized (r, s) -sequence or Horadam sequence or 2-step Fibonacci sequence) $\{W_n(W_0, W_1; r, s)\}_{n \geq 0}$ (or shortly $\{W_n\}_{n \geq 0}$) is defined (by Horadam [2]) as follows:

$$W_n = rW_{n-1} + sW_{n-2}, \quad W_0 = a, W_1 = b, \quad n \geq 2 \quad (1.1)$$

where W_0, W_1 are arbitrary complex (or real) numbers and r, s are real numbers, see also Horadam [1], [3] and [4].

The sequence $\{W_n\}_{n \geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{r}{s}W_{-(n-1)} + \frac{1}{s}W_{-(n-2)}$$

for $n = 1, 2, 3, \dots$ when $s \neq 0$. Therefore, recurrence (1.1) holds for all integer n .

For more information on Horadam numbers, see for example, [8] and [9]. For some specific values of a, b, r and s , it is worth presenting these special Horadam numbers in a table as a specific name. In literature, for example, the following names and notations (see Table 1 and Table 2) are used for the special cases of r, s and initial values.

Table 1 A few special case of generalized Fibonacci sequence.

No	Sequences (Numbers)	Notation
1	Generalized Fibonacci	$\{W_n\} = \{W_n(W_0, W_1; 1, 1)\}$
2	Generalized Pell	$\{W_n\} = \{W_n(W_0, W_1; 2, 1)\}$
3	Generalized Jacobsthal	$\{W_n\} = \{W_n(W_0, W_1; 1, 2)\}$
4	Generalized 2-primes	$\{W_n\} = \{W_n(W_0, W_1; 2, 3)\}$
5	Generalized Mersenne	$\{W_n\} = \{W_n(W_0, W_1; 3, -2)\}$
6	Generalized p-Mersenne	$\{W_n\} = \{W_n(W_0, W_1; p, -(p-1))\}$
7	Generalized balancing	$\{W_n\} = \{W_n(W_0, W_1; 6, -1)\}$
8	Generalized Oresme	$\{W_n\} = \{W_n(W_0, W_1; 1, -\frac{1}{4})\}$
9	Generalized p-Oresme	$\{W_n\} = \{W_n(W_0, W_1; 1, -\frac{1}{p^2})\}$

Table 2. Notation tables of a few special case of generalized (r,s) (generalized Fibonacci) sequence

No	Name of sequence	Notation: $W_n(W_0, W_1; r, s)$	OEIS [7]
1	Fibonacci	$F_n = W_n(0, 1; 1, 1)$	A000045
2	Lucas	$L_n = W_n(2, 1; 1, 1)$	A000032
3	Pell	$P_n = W_n(0, 1; 2, 1)$	A000129
4	Pell-Lucas	$Q_n = W_n(2, 2; 2, 1)$	A002203
5	Jacobsthal	$J_n = W_n(0, 1; 1, 2)$	A001045
6	Jacobsthal-Lucas	$j_n = W_n(2, 1; 1, 2)$	A014551
7	2-primes	$G_n = W_n(1, 2; 2, 3)$	A015518
8	Lucas 2-primes	$H_n = W_n(2, 2; 2, 3)$	A102345
9	Modified 2-primes	$E_n = W_n(1, 1; 2, 3)$	A046717
10	Mersenne	$M_n = W_n(0, 1; 3, -2)$	A000225
11	Mersenne-Lucas	$H_n = W_n(2, 3; 3, -2)$	A000051
12	Balancing	$B_n = W_n(0, 1; 6, -1)$	A001109
13	modified Lucas-balancing	$H_n = W_n(2, 6; 6, -1)$	A003499
14	Lucas-balancing	$C_n = W_n(1, 3; 6, -1)$	A001541
15	Modified Oresme	$G_n = W_n(0, 1; 1, -\frac{1}{4})$	
16	Oresme-Lucas	$H_n = W_n(2, 1; 1, -\frac{1}{4})$	
17	Oresme	$O_n = W_n(0, \frac{1}{2}; 1, -\frac{1}{4})$	

Here, OEIS stands for On-line Encyclopedia of Integer Sequences.

Binet’s formula of generalized Fibonacci sequence can be calculated using its characteristic equation (the quadratic equation) which is given as

$$x^2 - rx - s = 0. \tag{1.2}$$

The roots of characteristic equation are

$$\alpha = \frac{r + \sqrt{r^2 + 4s}}{2}, \quad \beta = \frac{r - \sqrt{r^2 + 4s}}{2}, \quad (1.3)$$

Binet's formula can be given as follows:

THEOREM 1. *The general term of the generalized Fibonacci sequence W_n can be presented by the following Binet formula:*

$$W_n = \begin{cases} \frac{W_1 - \beta W_0}{\alpha - \beta} \alpha^n - \frac{W_1 - \alpha W_0}{\alpha - \beta} \beta^n & , \quad \text{if } \alpha \neq \beta \text{ (Distinct Roots Case)} \\ (nW_1 - \alpha(n-1)W_0)\alpha^{n-1} & , \quad \text{if } \alpha = \beta \text{ (Single Root Case)} \end{cases}. \quad (1.4)$$

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} W_n x^n$ of the sequence $\{W_n\}$.

LEMMA 2. *Suppose that $f_{W_n}(x) = \sum_{n=0}^{\infty} W_n x^n$ is the ordinary generating function of the generalized Fibonacci sequence $\{W_n\}_{n \geq 0}$. Then, $\sum_{n=0}^{\infty} W_n x^n$ is given by*

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - rW_0)x}{1 - rx - sx^2}. \quad (1.5)$$

Now we define two special cases of the sequence $\{W_n\}$. (r, s) sequence $\{G_n(0, 1; r, s)\}_{n \geq 0}$ and Lucas (r, s) sequence $\{H_n(2, r; r, s)\}_{n \geq 0}$ are defined, respectively, by the second-order recurrence relations

$$G_{n+2} = rG_{n+1} + sG_n, \quad G_0 = 0, G_1 = 1, \quad (1.6)$$

$$H_{n+2} = rH_{n+1} + sH_n, \quad H_0 = 2, H_1 = r. \quad (1.7)$$

The sequences $\{G_n\}_{n \geq 0}$ and $\{H_n\}_{n \geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} G_{-n} &= -\frac{r}{s}G_{-(n-1)} + \frac{1}{s}G_{-(n-2)}, \\ H_{-n} &= -\frac{r}{s}H_{-(n-1)} + \frac{1}{s}H_{-(n-2)}, \end{aligned}$$

for $n = 1, 2, 3, \dots$ respectively. Therefore, recurrences (1.6) and (1.7) hold for all integer n .

Some special cases of (r, s) sequence $\{G_n(0, 1; r, s)\}_{n \geq 0}$ and Lucas (r, s) sequence $\{H_n(2, r; r, s)\}_{n \geq 0}$ are as follows:

- (1) $G_n(0, 1; 1, 1) = F_n$, Fibonacci sequence,
- (2) $H_n(2, 1; 1, 1) = L_n$, Lucas sequence,
- (3) $G_n(0, 1; 2, 1) = P_n$, Pell sequence,
- (4) $H_n(2, 2; 2, 1) = Q_n$, Pell-Lucas sequence,
- (5) $G_n(0, 1; 1, 2) = J_n$, Jacobsthal sequence,
- (6) $H_n(2, 1; 1, 2) = j_n$, Jacobsthal-Lucas sequence.
- (7) $G_n(1, 2; 2, 3) = G_n$, 2-primes sequence,
- (8) $H_n(2, 2; 2, 3) = H_n$, Lucas 2-primes sequence,

- (9) $G_n(0, 1; 3, -2) = M_n$, Mersenne sequence,
- (10) $H_n(2, 3; 3, -2) = H_n$, Mersenne-Lucas sequence,
- (11) $G_n(0, 1; 6, -1) = B_n$, balancing sequence,
- (12) $H_n(2, 6; 6, -1) = H_n$, modified Lucas-balancing sequence,
- (13) $G_n(0, 1; 1, -\frac{1}{4}) = G_n$, modified Oresme sequence,
- (14) $H_n(2, 1; 1, -\frac{1}{4}) = H_n$, Oresme-Lucas-Lucas sequence.

Binet's formulas of (r, s) and Lucas (r, s) numbers are given as

$$G_n = \begin{cases} \frac{\alpha^n - \beta^n}{\alpha - \beta} & , \text{ if } \alpha \neq \beta \text{ (Distinct Roots Case)} \\ n\alpha^{n-1} & , \text{ if } \alpha = \beta \text{ (Single Root Case)} \end{cases}$$

and

$$H_n = \begin{cases} \alpha^n + \beta^n & , \text{ if } \alpha \neq \beta \text{ (Distinct Roots Case)} \\ 2\alpha^n & , \text{ if } \alpha = \beta \text{ (Single Root Case)} \end{cases}.$$

Lemma 2 gives the following results as particular examples (generating functions of (r, s) and Lucas (r, s) numbers).

COROLLARY 3. *Generating functions of (r, s) and Lucas (r, s) numbers are*

$$\begin{aligned} \sum_{n=0}^{\infty} G_n x^n &= \frac{x}{1 - rx - sx^2}, \\ \sum_{n=0}^{\infty} H_n x^n &= \frac{2 - rx}{1 - rx - sx^2}, \end{aligned}$$

respectively.

Note that if

$$K_n = \sum_{k=0}^n W_k$$

then

$$\sum_{k=n}^{n+m} W_k = \sum_{k=0}^{n+m} W_k - \sum_{k=0}^{n-1} W_k = K_{n+m} - K_{n-1}.$$

In a quite recent preprint, Prodinger [6] proved the following Theorem via generating functions which gives complete answer the sum problem addressed in [5].

THEOREM 4. (Prodinger) *For $n \geq 0$, we have*

$$K_n = \frac{W_0 + W_1 - W_0 r}{1 - r - s} - \frac{(2W_0 s + W_1 r + 2W_1 s - W_0 r s)}{2(1 - r - s)} G_n - \frac{W_1 + W_0 s}{2(1 - r - s)} H_n$$

and

$$K_{n+m} - K_{n-1} = -\frac{(2W_0 s + W_1 r + 2W_1 s - W_0 r s)}{2(1 - r - s)} (G_{m+n} - G_{n-1}) - \frac{(W_1 + W_0 s)}{2(1 - r - s)} (H_{m+n} - H_{n-1}).$$

In the next sections, we extend the results of Prodinger. Note that Prodinger does not consider the case $1 - r - s = 0$ which we also deal with this case as well. For example, for the generalized Mersenn numbers (the case $r = 3$ and $s = -2$), we get $1 - r - s = 0$.

2. Extension of Prodinger's Theorem 4: The Sum Formulas of $\sum_{k=0}^n x^k W_k$ and $\sum_{k=n}^{n+m} x^k W_k$ via Generating Functions

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} x^n W_n z^n$ of the sequence $\{x^n W_n\}$.

LEMMA 5. Suppose that $f_{x^n W_n}(z) = \sum_{n=0}^{\infty} x^n W_n z^n$ is the ordinary generating function of the sequence $\{x^n W_n\}_{n \geq 0}$. Then, $\sum_{n=0}^{\infty} x^n W_n z^n$ is given by

$$\sum_{n=0}^{\infty} x^n W_n z^n = \frac{W_0 + x(W_1 - rW_0)z}{1 - rxz - sx^2z^2}. \quad (2.1)$$

Proof. Note that

$$x^n W_n = x^n (rW_{n-1} + sW_{n-2}).$$

Using the definition of generalized Fibonacci numbers, and subtracting $rxz \sum_{n=0}^{\infty} x^n W_n z^n$ and $sx^2z^2 \sum_{n=0}^{\infty} x^n W_n z^n$ from $\sum_{n=0}^{\infty} x^n W_n z^n$ we obtain

$$\begin{aligned} (1 - rxz - sx^2z^2) \sum_{n=0}^{\infty} x^n W_n z^n &= \sum_{n=0}^{\infty} x^n W_n z^n - rxz \sum_{n=0}^{\infty} x^n W_n z^n - sx^2z^2 \sum_{n=0}^{\infty} x^n W_n z^n \\ &= \sum_{n=0}^{\infty} x^n W_n z^n - r \sum_{n=0}^{\infty} x^{n+1} W_n z^{n+1} - s \sum_{n=0}^{\infty} x^{n+2} W_n z^{n+2} \\ &= \sum_{n=0}^{\infty} x^n W_n z^n - r \sum_{n=1}^{\infty} x^n W_{n-1} z^n - s \sum_{n=2}^{\infty} x^n W_{n-2} z^n \\ &= (W_0 + xW_1z) - rxW_0z + \sum_{n=2}^{\infty} x^n (W_n - rW_{n-1} - sW_{n-2}) z^n \\ &= W_0 + x(W_1 - rW_0)z. \end{aligned}$$

Rearranging above equation, we obtain (2.1). $\square$

Lemma 5 gives the following results as particular examples.

COROLLARY 6. Generating functions $\sum_{n=0}^{\infty} x^n G_n z^n$ and $\sum_{n=0}^{\infty} x^n H_n z^n$ are

$$\begin{aligned} \sum_{n=0}^{\infty} x^n G_n z^n &= \frac{xz}{1 - rxz - sx^2z^2}, \\ \sum_{n=0}^{\infty} x^n H_n z^n &= \frac{2 - rxz}{1 - rxz - sx^2z^2}, \end{aligned}$$

respectively.

Let

$$S_n = \sum_{k=0}^n x^k W_k.$$

The following theorem presents some sum formulas of generalized Fibonacci (Horadam) numbers with positive subscripts.

THEOREM 7. *Let x be a nonzero complex (or real) number.*

(a): *If $1 - rx - sx^2 \neq 0$ then*

$$\begin{aligned} S_n &= \frac{W_0 + x(W_1 - rW_0)}{1 - rx - sx^2} - \frac{2W_1sx^2 + 2W_0sx + W_1rx - W_0rsx^2}{2(1 - rx - sx^2)}x^n G_n - \frac{W_1x + W_0sx^2}{2(1 - rx - sx^2)}x^n H_n \\ &= \frac{\Theta(x)}{2(1 - rx - sx^2)} \end{aligned} \quad (2.2)$$

where

$$\Theta(x) = 2(W_0 + (W_1 - rW_0)x) - (s(2W_1 - rW_0)x + (rW_1 + 2sW_0))x^{n+1}G_n - (W_1 + sxW_0)x^{n+1}H_n.$$

(b): *If $1 - rx - sx^2 = u(x - a)(x - b) = 0$ for some $u, a, b \in \mathbb{C}$ with $u \neq 0$ and $a \neq b$, i.e., $x = a$ or $x = b$, then*

$$S_n = \frac{\Theta_1(x)}{-2(r + 2sx)}$$

where

$$\Theta_1(x) = 2(W_1 - rW_0) + (-(r + nr + 4sx + 2nsx)W_1 + s(-2n + 2rx + nrx - 2)W_0)x^n G_n - x^n((n + 1)W_1 + sx(n + 2)W_0)H_n.$$

(c): *If $1 - rx - sx^2 = u(x - c)^2 = 0$ for some $u, c \in \mathbb{C}$ with $u \neq 0$, i.e., $x = c$, then*

$$S_n = \frac{\Theta_2(x)}{4s}$$

where

$$\Theta_2(x) = (n + 1)((nr + 4sx + 2nsx)W_1 + s(2n - 2rx - nrx)W_0)x^{n-1}G_n + (n + 1)(nx^{n-1}W_1 + sx^n(n + 2)W_0)H_n.$$

Proof.

(a): Note that using generating functions, we get

$$\begin{aligned}
 S(z) &= \sum_{n=0}^{\infty} S_n z^n = \frac{1}{1-z} \frac{W_0 + x(W_1 - rW_0)z}{1 - rxz - sx^2 z^2} \\
 &= \frac{W_0 + x(W_1 - rW_0)}{1 - rx - sx^2} \frac{1}{1-z} - \frac{2W_1 sx^2 + 2W_0 sx + W_1 rx - W_0 r sx^2}{2(1 - rx - sx^2)} \frac{xz}{1 - rxz - sx^2 z^2} \\
 &\quad - \frac{W_1 x + W_0 sx^2}{2(1 - rx - sx^2)} \frac{2 - rxz}{1 - rxz - sx^2 z^2} \\
 &= \frac{W_0 + x(W_1 - rW_0)}{1 - rx - sx^2} \sum_{n=0}^{\infty} z^n - \frac{2W_1 sx^2 + 2W_0 sx + W_1 rx - W_0 r sx^2}{2(1 - rx - sx^2)} \sum_{n=0}^{\infty} x^n G_n z^n \\
 &\quad - \frac{W_1 x + W_0 sx^2}{2(1 - rx - sx^2)} \sum_{n=0}^{\infty} x^n H_n z^n \\
 &= \sum_{n=0}^{\infty} \left(\frac{W_0 + x(W_1 - rW_0)}{1 - rx - sx^2} - \frac{2W_1 sx^2 + 2W_0 sx + W_1 rx - W_0 r sx^2}{2(1 - rx - sx^2)} x^n G_n - \frac{W_1 x + W_0 sx^2}{2(1 - rx - sx^2)} x^n H_n \right) z^n.
 \end{aligned}$$

Comparing on both sides leads to (2.2).

(b): We use (2.2). For $x = a$ and $x = b$, the right hand side of the above sum formula (2.2) is an indeterminate form. Now, we can use L'Hospital rule. Then we get (b) by using

$$\sum_{k=0}^n a^k W_{mk+j} = \frac{\frac{d}{dx} \Theta(x)}{\frac{d}{dx} (2(1 - rx - sx^2))} \Big|_{x=a}$$

and similarly

$$\sum_{k=0}^n b^k W_{mk+j} = \frac{\frac{d}{dx} \Theta(x)}{\frac{d}{dx} (2(1 - rx - sx^2))} \Big|_{x=b}.$$

(c): We use (2.2). For $x = c$, the right hand side of the above sum formula (2.2) is an indeterminate form. Now, we can use L'Hospital rule (twice). Then we get (c) by using

$$\sum_{k=0}^n c^k W_{mk+j} = \frac{\frac{d^2}{dx^2} \Theta(x)}{\frac{d^2}{dx^2} (2(1 - rx - sx^2))} \Big|_{x=c}. \square$$

We now concentrate on finding expressions for the partial sums

$$\sum_{k=n}^{n+m} x^k W_k = \sum_{k=0}^{n+m} x^k W_k - \sum_{k=0}^{n-1} x^k W_k = S_{n+m} - S_{n-1}.$$

COROLLARY 8. Let x be a non-zero complex (or real) number.

(a): If $1 - rx - sx^2 \neq 0$ then

$$S_{n+m} - S_{n-1} = \frac{\Gamma_1}{2(1 - rx - sx^2)}$$

where

$$\Gamma_1 = -(s(2W_1 - rW_0)x + (rW_1 + 2sW_0))(x^{m+1}G_{n+m} - G_{n-1})x^n - (W_1 + sxW_0)(x^{m+1}H_{n+m} - H_{n-1})x^n.$$

(b): If $1 - rx - sx^2 = u(x - a)(x - b) = 0$ for some $u, a, b \in \mathbb{C}$ with $u \neq 0$ and $a \neq b$, i.e., $x = a$ or $x = b$, then

$$S_{n+m} - S_{n-1} = \frac{\Gamma_2}{-2(r + 2sx)}$$

where

$$\begin{aligned} \Gamma_2 = & (- (r + (n+m)r + 4sx + 2(n+m)sx)W_1 + s(-2(n+m) + 2rx + (n+m)rx - 2)W_0)x^{n+m}G_{n+m} - \\ & (- (r + (n-1)r + 4sx + 2(n-1)sx)W_1 + s(-2(n-1) + 2rx + (n-1)rx - 2)W_0)x^{n-1}G_{n-1} - \\ & (((n+m) + 1)W_1 + sx((n+m) + 2)W_0)x^{n+m}H_{n+m} + (((n-1) + 1)W_1 + sx((n-1) + 2)W_0)x^{n-1}H_{n-1}. \end{aligned}$$

(c): If $1 - rx - sx^2 = u(x - c)^2 = 0$ for some $u, c \in \mathbb{C}$ with $u \neq 0$, i.e., $x = c$, then

$$S_{n+m} - S_{n-1} = \frac{\Gamma_3}{4s}$$

where

$$\begin{aligned} \Gamma_3 = & ((n+m)+1)((n+m)r + 4sx + 2(n+m)sx)W_1 + s(2(n+m) - 2rx - (n+m)rx)W_0)x^{n+m-1}G_{n+m} - \\ & ((n-1)+1)((n-1)r + 4sx + 2(n-1)sx)W_1 + s(2(n-1) - 2rx - (n-1)rx)W_0)x^{n-2}G_{n-1} + \\ & ((n+m)+1)((n+m)x^{n+m-1}W_1 + sx^{n+m}((n+m) + 2)W_0)H_{n+m} - ((n-1)+1)((n-1)x^{n-2}W_1 + \\ & sx^{n-1}((n-1) + 2)W_0)H_{n-1}. \end{aligned}$$

3. The Sum Formulas of $\sum_{k=0}^n kx^k W_k$ and $\sum_{k=n}^{n+m} kx^k W_k$

Let

$$Y_n = \sum_{k=0}^n kx^k W_k.$$

The following theorem presents some sum formulas of generalized Fibonacci (Horadam) numbers with positive subscripts.

THEOREM 9. Let x be a non-zero complex (or real) number. Then

(a): If $1 - rx - sx^2 \neq 0$ then

$$\begin{aligned} Y_n &= \frac{x(1 - rx - sx^2)\Theta'(x) + x(r + 2sx)\Theta(x)}{2(1 - rx - sx^2)^2} \\ &= \frac{\Delta_1}{2(1 - rx - sx^2)^2} \end{aligned} \quad (3.1)$$

where $\Theta(x)$ is as in Theorem 7 (a) and $\Theta'(x)$ denotes the derivative of $\Theta(x)$ with respect to x , and

$$\begin{aligned} \Delta_1 = & 2x((sx^2 + 1)W_1 - sx(rx - 2)W_0) + ((2ns^2x^3 + rsx^2 + 3nr sx^2 + nr^2x - 2nsx - 4sx - r - \\ & nr)W_1 - s(nr sx^3 + 2sx^2 + r^2x^2 - 2nsx^2 + nr^2x^2 - 2rx - 3nr x + 2n + 2)W_0)x^{n+1}G_n + ((nsx^2 - \\ & sx^2 + nr x - n - 1)W_1 + sx(nsx^2 + nr x - n + rx - 2)W_0)x^{n+1}H_n. \end{aligned}$$

(b): If $1 - rx - sx^2 = u(x - a)(x - b) = 0$ for some $u, a, b \in \mathbb{C}$ with $u \neq 0$ and $a \neq b$, i.e., $x = a$ or $x = b$, then

$$\begin{aligned} Y_n &= \frac{4s\Theta(x) + 2sx\Theta'(x) - (4sx^2 + 3rx - 2)\Theta''(x) + x(1 - rx - sx^2)\Theta'''(x)}{4(-2s + 6s^2x^2 + r^2 + 6rsx)} \\ &= \frac{\Delta_2}{4(-2s + 6s^2x^2 + r^2 + 6rsx)} \end{aligned}$$

where $\Theta''(x)$ and $\Theta'''(x)$ denote the second and third derivatives of $\Theta(x)$, respectively, with respect to x , and

$$\begin{aligned} \Delta_2 = & 12sxW_1 - 4s(3rx - 2)W_0 + ((-nr(n+1)^2 + x((n+2)(n+1)(nr^2 - 2ns - 4s) + sx(n+3)(2n^2sx + \\ & 3n^2r + 8nsx + 7nr + 2r)))W_1 + s(-2n(n+1)^2 + x(r(3n+2)(n+2)(n+1) - x(n+3)(n^2rsx + 3nr^2 - 2n^2s + \\ & n^2r^2 + 4nr sx - 2ns + 2r^2 + 4s)))W_0)x^{n-1}G_n + ((-n(n+1)^2 + x(n+2)(n^2r + n^2sx + 2nsx + nr - 3sx)) \\ & W_1 + sx(-(n+1)(n+2)^2 + x(n+3)(n^2sx + n^2r + 4nsx + 2r + 3nr))W_0)x^{n-1}H_n. \end{aligned}$$

(c): If $1 - rx - sx^2 = u(x - c)^2 = 0$ for some $u, c \in \mathbb{C}$ with $u \neq 0$, i.e., $x = c$, then

$$\begin{aligned} Y_n &= \frac{-4(2r + 5sx)\Theta'''(x) - (10sx^2 + 7rx - 4)\Theta''''(x) + x((1 - rx - sx^2)\Theta''''(x))}{48s^2} \\ &= \frac{\Delta_3}{12s^2} \end{aligned}$$

where $\Theta''''(x)$ and $\Theta'''''(x)$ denote the fourth and fifth derivatives of $\Theta(x)$, respectively, with respect to x , and

$$\begin{aligned} \Delta_3 = & x^{-2}(((nx^n(2r + 5sx)(n+1)(-r + nr + 4sx + 2nsx) + 546x^{12}(630s^2x^3 + 187r^2x + 760rsx^2 - \\ & 450sx - 154r))W_1 + s(nx^n(n+1)(2n - 2rx - nr x - 2)(2r + 5sx) - 546x^{12}(315rsx^3 - 440sx^2 + \\ & 270r^2x^2 - 599rx + 308))W_0)G_n + ((nx^n(n-1)(n+1)(2r + 5sx) + 6006x^{12}(20sx^2 + 17rx - 14))W_1 + \\ & sx(nx^n(n+2)(n+1)(2r + 5sx) + 24570x^{12}(7sx^2 + 6rx - 5))W_0)H_n). \end{aligned}$$

Proof.

(a): We know from Theorem 7 that

$$S_n = \sum_{k=0}^n x^k W_k = \frac{\Theta(x)}{2(1 - rx - sx^2)}$$

where

$$\Theta(x) = 2(W_0 + (W_1 - rW_0)x) - (s(2W_1 - rW_0)x + (rW_1 + 2sW_0))x^{n+1}G_n - (W_1 + sxW_0)x^{n+1}H_n.$$

By taking the derivative of the both sides of the above formulas with respect to x , we get

$$\sum_{k=0}^n kx^{k-1}W_k = \frac{(1 - rx - sx^2)\Theta'(x) + (r + 2sx)\Theta(x)}{2(1 - rx - sx^2)^2}$$

i.e.

$$\begin{aligned} Y_n &= \sum_{k=0}^n kx^k W_k = \frac{x(1 - rx - sx^2)\Theta'(x) + x(r + 2sx)\Theta(x)}{2(1 - rx - sx^2)^2} \\ &= \frac{\Delta_1}{2(1 - rx - sx^2)^2}. \end{aligned}$$

(b): We use (a). For $x = a$ and $x = b$, the right hand side of the above sum formula 3.1) is an indeterminate form. Now, we can use L'Hospital rule (twice). Then we get (b) by using

$$\begin{aligned} Y_n &= \sum_{k=0}^n k a^k W_k = \frac{\frac{d^2}{dx^2}(x(1-rx-sx^2)\Theta'(x) + x(r+2sx)\Theta(x))}{\frac{d^2}{dx^2}(2(1-rx-sx^2)^2)} \Big|_{x=a} \\ &= \frac{4s\Theta(a) + 2sa\Theta'(a) - (4sa^2 + 3ra - 2)\Theta''(a) + a(1-ra-sa^2)\Theta'''(a)}{4(-2s + 6s^2a^2 + r^2 + 6rsa)} \\ &= \frac{\Delta_2}{4(-2s + 6s^2a^2 + r^2 + 6rsa)} \end{aligned}$$

and similarly

$$\begin{aligned} Y_n &= \sum_{k=0}^n k a^k W_k \\ &= \frac{4s\Theta(b) + 2sb\Theta'(b) - (4sb^2 + 3rb - 2)\Theta''(b) + b(1-rb-sb^2)\Theta'''(b)}{4(-2s + 6s^2b^2 + r^2 + 6rsb)}. \end{aligned}$$

(c): We use (a). For $x = c$, the right hand side of the above sum formula 3.1) is an indeterminate form. Now, we can use L'Hospital rule (four times). Then we get (c) by using

$$\begin{aligned} Y_n &= \sum_{k=0}^n k a^k W_k = \frac{\frac{d^4}{dx^4}(x(1-rx-sx^2)\Theta'(x) + x(r+2sx)\Theta(x))}{\frac{d^4}{dx^4}(2(1-rx-sx^2)^2)} \Big|_{x=a} \\ &= \frac{-4(2r + 5sx)\Theta'''(x) - (10sx^2 + 7rx - 4)\Theta''''(x) + x((1-rx-sx^2)\Theta''''(x))}{48s^2} \\ &= \frac{\Delta_3}{12s^2}. \square \end{aligned}$$

We now concentrate on finding expressions for the partial sums

$$\sum_{k=n}^{n+m} k x^k W_k = \sum_{k=0}^{n+m} k x^k W_k - \sum_{k=0}^{n-1} k x^k W_k = Y_{n+m} - Y_{n-1}.$$

COROLLARY 10. Let x be a non-zero complex (or real) number. If $1 - rx - sx^2 \neq 0$ then

$$Y_{n+m} - Y_{n-1} = \frac{\Psi}{2(1 - rx - sx^2)^2}$$

where

$$\begin{aligned} \Psi &= 2x((sx^2 + 1)W_1 - sx(rx - 2)W_0) + ((2(n+m)s^2x^3 + rsx^2 + 3(n+m)rsx^2 + (n+m)r^2x - 2(n+m)sx - \\ &4sx - r - (n+m)r)W_1 - s((n+m)rsx^3 + 2sx^2 + r^2x^2 - 2(n+m)sx^2 + (n+m)r^2x^2 - 2rx - 3(n+m)rx + 2(n+m) + 2) \\ &W_0)x^{n+m+1}G_{n+m} - (2x((sx^2 + 1)W_1 - sx(rx - 2)W_0) + ((2(n-1)s^2x^3 + rsx^2 + 3(n-1)rsx^2 + (n-1)r^2x - 2(n-1) \\ &1)sx - 4sx - r - (n-1)r)W_1 - s((n-1)rsx^3 + 2sx^2 + r^2x^2 - 2(n-1)sx^2 + (n-1)r^2x^2 - 2rx - 3(n-1)rx + 2(n-1) + 2) \\ &W_0)x^{n-1+1}G_{n-1}) + (((n+m)sx^2 - sx^2 + (n+m)rx - (n+m) - 1)W_1 + sx((n+m)sx^2 + (n+m)rx - (n+m) \\ &+ rx - 2)W_0)x^{n+m+1}H_{n+m} - (((n-1)sx^2 - sx^2 + (n-1)rx - (n-1) - 1)W_1 + sx((n-1)sx^2 + (n-1)rx - (n-1) \\ &+ rx - 2)W_0)x^{n-1+1}H_{n-1}). \end{aligned}$$

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