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STRONGLY RECURRENT TRANSFORMATION GROUPS

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Abstract

We define the notions of strong and strict recurrency for actions of countable ordered groups on σ-finite non atomic measure spaces with quasi-invariant measures. We show that strong recurrency is equivalent to non existence of weakly wandering sets of positive measure. We also show that for certain p.m.p ergodic actions the system is not strictly recurrent, which shows that strong and strict recurrency are not equivalent.

Keywords: ordered groups, σ-finite spaces, strongly recurrent actions, strictly recurrent actions, weakly wandering sets.

Mathematics Subject Classification 2010: 22F10, 37A05

1. Introduction

In modern dynamics there is a tendency to replace classical dynamics with dynamics under a transformation group [8]. The basic idea is to replace the dynamics given by a non-singular transformation (the action of $\mathbb{Z}$) with the action of an arbitrary (discrete) group G. The next step is to find analogs of ergodic and dynamical notions in the new more general setting. Most of these notions are symmetric, in the sense that they behave the same by moving forward or backward in (discrete) time. However, there are notions which best fit with moving forward in time (that is, defined based on the asymptotic behavior under positive powers of the transformation). One of these notions is that of recurrency. The natural substitute of the positive powers of a single transformation in a transformation group is the semigroup action of the positive cone of the underlying group, so it is natural to work with ordered groups in such situations. This is the main focus of the current paper. Our goal is extend some of the classical results on recurrency and its relation with wandering sets to transformation groups with an acting ordered group.

Let (X, m) be a finite or σ-finite and non-atomic measure space and G be a countable ordered group acting on X such that m is G-quasi-invariant, in the sense that $m(tB) = 0$ for every null set B and $t \in G$. We say that a measure μ on X is G-invariant if $\mu(tB) = \mu(B)$ for every measurable set B and $t \in G$.

The notion of recurrent transformations is well studied in ergodic theory [3]. A measurable transformation T on a finite or σ-finite measure space (X, m) is called recurrent if for every measurable set A of positive measure, $T^n x \in A$ for infinitely many integers n, for almost all $x \in A$. Every measurable transformation which preserves a finite invariant measure μ equivalent to m is recurrent. The converse is not true in general, for example an ergodic transformation which preserves an infinite σ-finite measure is always recurrent yet it does not preserve a finite invariant equivalent measure.

The existence of an equivalent finite invariant measure is one of the central issues in ergodic theory [1], [7]. Hajian in [5] defined the notion of a strongly recurrent transformation and showed that a transformation T is strongly recurrent if and only if there exists a finite invariant measure μ equivalent to m [5, Theorem 2]. Hajian also showed that existence of strongly recurrent sets of positive measure is equivalent to non existence of weakly wandering sets of positive measure [5, Theorem 1] (for extensions, see [6], [9]).

Weakly wandering sets are introduced in [4] and the condition that a transformation T does not have any weakly wandering set of positive measure is strengthened in [4] and this stronger condition (called "condition (W)*" in [5]) is shown to be equivalent to the existence of a finite invariant measure equivalent to m [4, Theorem 1]. Hajian further showed that a similar strengthening for a strongly recurrent transformations is not working for a wide class of measure

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preserving transformations [5, Theorem 3].

In this paper we introduce the notion of strongly recurrent transformation groups and prove the following result.

Theorem A. *Let (X, m) be a σ-finite non-atomic measure space and G be a countable ordered group acting on X such that m is G-quasi-invariant, then the following are equivalent:*

(i) no measurable subset of positive measure is weakly wandering,

(ii) all measurable subsets of positive measure are strongly recurrent,

(iii) there is an equivalent finite invariant measure on X.

We also introduce the notion of strictly recurrent transformation groups and show that it is stronger than the condition of being strongly recurrent.

Theorem B. *Let (X, m) be a non-atomic probability measure space and (G, G_+, u) be a countable scaled ordered group, such that the interval $[e, u]$ is finite. If there is an ergodic p.m.p. action of G on X then the system (G, X, m) is not strictly recurrent.*

2. Recurrent transformation groups

In this section we introduce the basic concepts of this paper. In this paper we only work with discrete groups.

Definition 2.1. An *ordered group* G is a group with a distinguished subsemigroup G_+containing the identity element e of G such that:

(i) $G_+G_+^{-1} = G$,

(ii) $G_+ \cap G_+^{-1} = \{e\}$.

These conditions mean that each element $s \in G$ could be decomposed as $s = uv^{-1}$ with $u, v \in G_+$, but of course the decomposition is not unique as $x = (ut)(vt)^{-1}$, for each $t \in G_+$. We write $x \geq e$ whenever $x \in G_+$ and $x \geq y$ (or equivalently $y \leq x$) whenever $xy^{-1} \in G_+$. We also write $x > y$ (or $y < x$) when $x \geq y$ but $x \neq y$. We also use the notation

$$[x, y] := \{t \in G : x \leq t \leq y\}$$

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which is non-empty when $x \leq y$ (as it contains x and y).

An *order ideal* of G is a subgroup I such that for each $x, y \in G$, $x \leq y$ and $y \in I$ implies that $x \in I$. An element $u \in G_+$ is called an *order unit* if for every $x \in G$, there is $n > 0$ such that $u^n \geq x$, i.e., the order ideal generated by u is the whole G. If such an element u exists, we call (G, G_+, u) an *scaled* ordered group.

A subset S of G is called *cofinal* if for each $x \in G$, there is $s \in S$ with $s > x$. It is called *upward directed* if for each $s, t \in S$ there is $u \in S$ with $u \geq s$ and $u \geq t$. The notion of *downward directed* subset is defined similarly. A countable cofinal upward directed subset S of G is loosely called an *infinite increasing sequence* in G (loosely because we are not dealing with a total order here). When moreover, $S \subseteq G_+$, we say that S is an infinite increasing sequence in G_+. Similarly, one could define an *infinite decreasing sequence* in G or G_+. By an *infinite sequence*, we mean an infinite increasing or decreasing sequence.

For the rest of the paper, let (X, m) be a finite or σ-finite and non-atomic measure space and G be an ordered group acting on X such that m is G-quasi-invariant.

Definition 2.2. A measurable subset B of X is *strongly recurrent* if there is a finite subset $F \subseteq G_+$ such that for each $t \in G$, $m(tB \cap sB) > 0$, for some $s \in F$. We say that the transformation group (G, X, m) is strongly recurrent (or equivalently, the action of G on X is strongly recurrent) if every measurable set of positive measure in X is strongly recurrent.

Since m is quasi-invariant, the above condition is clearly equivalent to the condition that

$$m(B \cap \textstyle\bigcup_{s \in F} t^{-1} sB) > 0,$$

for all $t \in G$, or equivalently

$$m(tB \cap \textstyle\bigcup_{s \in F} sB) > 0,$$

for all $t \in G$. We call F a recurrency set for B.

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Lemma 2.3. *Let a measurable subset B of X be strongly recurrent. Then*

(i) If F is a recurrency set for B, so is xF for each $x \in G$,

(ii) If G is upward directed, then B has a recurrency set $F \subseteq G_+$ iff it has a recurrency set $F \subseteq G$.

Proof (i) If $m(tB \cap sB) > 0$, for all $t \in G$ and some $s \in F$, then by quasi-invariance, $m(xtB \cap xsB) > 0$, for all $t \in G$ and some $s \in F$, and changing t to $x^{-1}t$ we get $m(tB \cap xsB) > 0$, for all $t \in G$ and some $s \in F$, that is, xF is a recurrency set for B.

(ii) If B has a recurrency set $F \subseteq G$, the since G is upward directed, we may choose $x \in G$ with $x \geq s^{-1}$, for each $s \in F$, that is, $xs \in G_+$, thus $xF \subseteq G_+$ is a recurrency set for B by (i). □

Remark 2.4. Note that our model example is the classical system (i.e., $\mathbb{Z}$-actions) where $\mathbb{Z}$ is both upward and downward directed, and the second part of the above lemma suggests that for an arbitrary group G (without order) one could define strong recurrency by requiring the existence of a finite recurrency set in G for any set of positive measure.

The following notion extends that of Hajian and Kakutani [4] (c.f., [9]).

Definition 2.5. A measurable subset B of X is *weakly wandering* if there is an infinite set S in G such that the sets sB are mutually disjoint for $s \in S$. The transformation group (G, X, m) is called weakly wandering if every measurable set of positive measure in X is weakly wandering.

We also need the following notion, which is not explicitly defined in [5], but is used under the name "condition (S)*" [5, page 520] for the case of $\mathbb{Z}$-action.

Definition 2.6. The transformation group (G, X, m) is called *strictly recurrent* (or equivalently, the action of G on X is strictly recurrent) if for each $\varepsilon > 0$ there is a finite subset $F \subseteq G_+$ such that for each measurable subset B of X with $m(B) \geq \varepsilon$ and each $t \in G$, $m(tB \cap sB) > 0$, for some $s \in F$.

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3. Proof of the main results

In order to prove the main result of this paper, we first need a lemma, which extends [5, Lemma 1]. Let us define the notion of "vanishing of limit inferiors" on G.

Definition 3.1. Let A be a measurable subset of X, we write

$$\liminf_{t\in G} m(tA) = 0$$

if there is an infinite set $S = (s_k)$ in G such that

$$\lim_{k\to\infty} m(s_k A) = 0.$$

Note that in $(\mathbb{Z}, \mathbb{Z}_+)$, in the above situation we may always choose an infinite sequence inside S, and so we may assume that S is a sequence in the sense of the previous section. This is not possible for general ordered groups. However, as we would see, this weaker notion of limit inferior is enough for our purposes.

Lemma 3.2. *Let G be a countable ordered group and let A be a measurable set of positive measure such that*

$$\liminf_{t\in G} m(tA) = 0$$

Then given $0 < \varepsilon < m(A)$, there exists a measurable subset $A^{'}$ of A with $m(A^{'}) < \varepsilon$ such that the set $S := A - A^{'}$ is not strongly recurrent.

Proof Since G is countable, we may choose an increasing sequence F_k of finite subsets of G with union G. Given $s \in F_k$, the measure $sm(B) := m(s^{-1}B)$ is equivalent to m (by quasi-invariance) and we have the Radon-Nikodym derivative $f_s := [\frac{dsm}{dm}] \in L^1(X,m)_+$. Note that each F_k is a finite set and put

$$f_k := \max\{f_s : s \in F_k\} \quad (k = 1, 2, \cdots),$$

then $f_k \in L^1(X,m)$ and

$$m(s^{-1}B) \le \int_B f_k dm \quad (s \in F_k),$$

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for every measurable set B and each $k = 1, 2, \cdots$. Since f_k is integrable, for each sequence (ε_k) of positive reals there is a sequence (δ_k) of positive reals such that $\int_B f_k dm < \varepsilon_k$, for each measurable set B with $m(B) < \delta_k$.

By assumption we may choose an infinite increasing sequence $S = (s_k)$ in G such that $m(s_k A) < \delta_k$, for $k = 1, 2 \cdots$. Thus

$$m(s^{-1}s_k A) \leq \int_{s_k A} f_k dm < \varepsilon_k \quad (s \in F_k),$$

for $k = 1, 2 \cdots$. Put

$$A^{'} := \bigcup_{k\geq 1} \bigcup_{s\in F_k} (s^{-1}s_k A \cap A),$$

then for given $0 < \varepsilon < m(A)$ and $\varepsilon_k := \varepsilon/2^k |F_k|$,

$$m(A^{'}) \leq \sum_k \sum_{s\in F_k} m(s^{-1}s_k A) \leq \sum_k |F_k|\varepsilon_k = \varepsilon,$$

and for $S := A - A^{'}$,

$$s^{-1}s_k S \cap S \subseteq s^{-1}s_k A \cap (A - A^{'}) = \emptyset \quad (s \in F_k, k = 1, 2, \cdots).$$

If S is strongly recurrent with finite recurrency set F, then there is k with $F_k \supseteq F$. Thus for $t = s_k$,

$$s^{-1}tS \cap S = \emptyset \quad (s \in F),$$

which is a contradiction. □

Proof of Theorem A. By [6, Theorem 1] we have the equivalence of (i) and (iii) (without countability assumption on G). We need to check the other two implications.

(i)⇒(ii). Let B be a measurable set of positive measure which is not strongly recurrent. Then there is an infinite set $S = (s_k)$ in G such that

$$m(s_{k+1}B \cap (\bigcup_{i=1}^{k} s_i B)) = 0,$$

for $k = 1, 2, \cdots$. Let

$$N := B \cap (\bigcup_{k=1}^{\infty} \bigcup_{i=1}^{k-1} s_k s_i^{-1} B),$$

then $m(N) = 0$ and $B^{'} := B - N$ is of positive measure and weakly wandering.

Conversely, let W be a weakly wandering set of positive measure. Choose an infinite sequence $S = (s_k)$ such that $s_k W$'s are mutually disjoint. Since (X, m) is σ-finite, we may choose a finite measure equivalent to m (not necessarily invariant). Since being strongly recurrent or weakly wandering is not effected by replacing m with an equivalent measure, we may assume that m is finite. We have $\lim_{k\to\infty} m(s_k W) = 0$, since otherwise, passing to a subsequence, we may assume that $m(s_k W) > \varepsilon$, for some $\varepsilon > 0$ and each $k = 1, 2, \cdots$, which violates finiteness of m, thus

$$\liminf_{t\in G} m(tW) = 0.$$

By Lemma 3.2, there is a measurable subset B of W which is of positive measure, but not strongly recurrent. □

Definition 3.3. The transformation group (G, X, m) is called *ergodic* if for each G-invariant measurable subset B of X, $m(B) = 0$ or $m(X - B) = 0$.

Lemma 3.4. *If (G, G_+, u) is a countable scaled ordered group such that the interval $[e, u]$ is finite and (G, X, m) is ergodic, then for each $\varepsilon > 0$ and $N \geq 1$ there are measurable subsets C and E of X such that $m(C) \leq \varepsilon$, $u^k E$'s are mutually disjoint, for $0 \leq k \leq N-1$, and $C := X - \bigcup_{k=0}^{N-1} \bigcup_{s\in G(k)} s^{-1}E$.*

Proof Let $N_0 := |[e, u]|$. For each $i \geq 0$, we have $e \leq s \leq u$ iff $u^i \leq su^i \leq u^{i+1}$. Conversely, $u^i \leq t \leq u^{i+1}$ iff $e \leq tu^{-i} \leq u$.Thus all intervals $[u^i, u^{i+1}]$ have the same cardinal N_0. Given $\varepsilon > 0$ and $N \geq 1$, since (X, m) is non atomic, there is a measurable subset E_0 of X with $0 < m(E_0) \leq \varepsilon/NN_0$. Let $G(i) := [u^i, u^{i+1}]$, define A_k by $A_0 = E_0$ and

$$A_k := u^{-k}E_0 - \bigcup_{i=0}^{k-2} \bigcup_{s\in G(i)} s^{-1}E_0.$$

Then A_k's are mutually disjoint and

$$u^k A_n \subseteq A_{n-k} \quad (k = 1, \cdots, n; n = 0, 1, 2, \cdots).$$

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Set $E_i := A_{iN}$, then $u^k E_i \subseteq A_{iN-k}$ for $k \leq iN$, thus $u^k E_i$'s are mutually disjoint for $k \leq iN$ and $i = 1, 2, \cdots$. Put $E = \bigcup_{i \geq 1} E_i$ and $C := X - \bigcup_{k=0}^{N-1} \bigcup_{s \in G(k)} s^{-1}E$. Then $u^k E$'s are disjoint, for $k = 0, \cdots, N-1$.

Since $\tilde{E} := \bigcup_{s \in G} s^{-1}E = \bigcup_{k=-\infty}^{\infty} \bigcup_{s \in G(k)} s^{-1}E$ is G-invariant, $m(X - \tilde{E}) = 0$. For $D := \bigcup_{k=0}^{N-1} \bigcup_{s \in G(k)} s^{-1}E$ and $I(N) := \mathbb{Z} - \{0, \cdots, N-1\}$,

$$\begin{aligned} m(X - C) &= m(D) = m(\tilde{E} - D) \\ &= m\big(\textstyle\bigcup_{k \in I(N)} \bigcup_{s \in G(k)} s^{-1}E\big) \\ &\leq m\big(\textstyle\bigcup_{k \in I(N)} \bigcup_{s \in G(k)} \bigcup_{i=1}^{\infty} s^{-1}A_{iN}\big) \\ &= m\big(\textstyle\bigcup_{k \in I(N)} \bigcup_{s \in G(k)} \bigcup_{i=1}^{\infty} (s^{-1}u^{-iN}E_0 - \bigcup_{j=0}^{iN-2} \bigcup_{s \in G(j)} s^{-1}E_0)\big). \end{aligned}$$

Similarly, since $\tilde{E}_0 := \bigcup_{k=-\infty}^{\infty} \bigcup_{s \in G(k)} s^{-1}E_0$ is G-invariant, $m(X - \tilde{E}_0) = 0$. Now in the last line of the above estimate of $m(X - C)$, in

$$Z := \textstyle\bigcup_{k \in I(N)} \bigcup_{s \in G(k)} \bigcup_{i=1}^{\infty} (s^{-1}u^{-iN}E_0 - \bigcup_{j=0}^{iN-2} \bigcup_{s \in G(j)} s^{-1}E_0),$$

all terms of the form $s^{-1}E_0$ with $s \in G(\ell)$ for $\ell < 0$ or $\ell > N$ are already omitted, thus

$$Z \cap \tilde{E}_0 = Z \cap \textstyle\bigcup_{k=0}^{N-1} \bigcup_{s \in G(k)} s^{-1}E_0,$$

Therefore,

$$\begin{aligned} m(X - C) &= m(Z) \\ &= m(Z \cap \tilde{E}_0) \\ &= m(Z \cap \textstyle\bigcup_{k=0}^{N-1} \bigcup_{s \in G(k)} s^{-1}E_0) \\ &\leq m(\textstyle\bigcup_{k=0}^{N-1} \bigcup_{s \in G(k)} s^{-1}E_0) \\ &\leq N N_0 m(E_0) \leq \varepsilon. \end{aligned}$$

□

Proof of Theorem B. Let $N_0 := |[e, u]|$, and fix $q > 3$, $\varepsilon = \frac{1}{N_0(q+1)}$, $k > 1$, and $N = kq$. Apply Lemma 3.5 to get a measurable subset E of X with $u^i E$'s mutually disjoint, for $0 \leq i \leq N-1$, and

$$m\big(X - \textstyle\bigcup_{0 \leq i \leq N-1} \bigcup_{s \in G(i)} s^{-1}E\big) \leq \varepsilon,$$

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then by the proof of Lemma 3.5 we know that $|G(i)| = N_0$, for each i, thus

$$\begin{aligned} NN_0 m(E) &= \sum_{i=1}^{N-1} \sum_{s\in G(i)} m(s^{-1}E) \\ &\geq m\big(\bigcup_{0\leq i\leq N-1}\bigcup_{s\in G(i)} s^{-1}E\big) \\ &\geq 1-\varepsilon. \end{aligned}$$

therefore, $m(E) \geq \frac{1}{NN_0}(1-\varepsilon)$. Put $A := \bigcup_{i=0}^{k-1} u^i E$, then

$$m(A) = km(E) \geq \frac{k(1-\varepsilon)}{NN_0} = \frac{N_0(q+1)-1}{qN_0^2(q+1)} \geq \frac{N_0 q}{qN_0^2(q+1)} = \varepsilon.$$

Choose n_k with $2k < n_k < (q-1)k = N-k$, we have $n_k + k - 1 < N-1$, hence $u^i E$'s are mutually disjoint, for $0 \leq i \leq n_k + k - 1$. In particular,

$$m(u^{n_k} A \cap \bigcup_{i=0}^{k-1} u^i A) = m(\bigcup_{i=n_k}^{n_k+k-1} u^i E \cap \bigcup_{i=0}^{2k-2} u^i E) = 0,$$

therefore, (G, X, m) is not strictly recurrent. □

When (G, G_+, u) is a scaled ordered group, we say that the transformation group (G, X, m) is *u-ergodic* if for each measurable subset B of X with $m(uB) = m(B)$, we have $m(B) = 0$ or $m(X-B) = 0$. For the scaled ordered group $(\mathbb{Z}, \mathbb{Z}_+, 1)$, 1-ergodicity is equivalent to ergodicity. In general, u-ergodicity is stronger than ergodicity. One could modify the previous lemma to get a version of Theorem B for uncountable scaled ordered groups, under the stronger condition of u-ergodicity, for some scale u. We need the following lemma, which is proved essentially the same as Lemma 3.4, for which we only sketch the proof.

Lemma 3.5. *If (G, G_+, u) is a scaled ordered group and (G, X, m) is u-ergodic, then for each $\varepsilon > 0$ and $N \geq 1$ there are measurable subsets C and E of X such that $m(C) \leq \varepsilon$, $u^k E$'s are mutually disjoint, for $0 \leq k \leq N-1$, and $X - C = \bigcup_{0\leq k\leq N-1} u^k E$.*

Proof Given $\varepsilon > 0$ and $N \geq 1$, choose a measurable subset E_0 of X with $0 < m(E_0) \leq \varepsilon/N$. Define A_k inductively by $A_0 = E_0$ and

$$A_k := u^{-k}E_0 - \bigcup_{i=0}^{k-1} A_i = u^{-k}E_0 - \bigcup_{i=0}^{k-1} u^{-i}E_0.$$

Then A_k's are mutually disjoint and $u^k A_n \subseteq A_{n-k}$ for $1 \le k \le n$ Set $E_i := A_{iN}$, then $u^k E_i \subseteq A_{iN-k}$ for $k \le iN$, thus $u^k E_i$'s are mutually disjoint for each i and $k \le iN$. Put $E = \bigcup_{i \ge 1} E_i$ and $C := X - \bigcup_{k=0}^{N-1} u^k E$. Then $u^k E$'s are disjoint, for $k = 0, \cdots, N-1$. Since $\tilde{E} := \cup_{k=-\infty}^{\infty} u^k E$ is u-invariant, $m(X - \tilde{E}) = 0$, thus for $I(N) := \mathbb{Z} - \{0, \cdots, N-1\}$,

$$\begin{aligned} m(X - C) &= m\big(\bigcup_{k \in I(N)} \bigcup_{i=1}^{\infty} u^k A_i\big) \\ &\le m\big(\bigcup_{k \in I(N)} \bigcup_{i=1}^{\infty} A_{iN-k}\big) \\ &= m\big(\bigcup_{k \in I(N)} \bigcup_{i=1}^{\infty} (u^{k-iN} E_0 - \bigcup_{j=0}^{iN-k} u^{-j} E_0)\big). \end{aligned}$$

Let Z be the set whose measure appeared in the last line of the above estimate, then since $\tilde{E}_0 := \cup_{k=-\infty}^{\infty} u^k E_0$ is u-invariant, $m(X - \tilde{E}_0) = 0$. Therefore, in Z all terms of the form $u^\ell E_0$ for $\ell < 0$ or $\ell > N$ are already omitted, thus

$$\begin{aligned} m(X - C) = m(Z) = m(Z \cap \tilde{E}_0) &= m(Z \cap \bigcup_{k=0}^{N-1} u^k E_0) \\ &\le m(\bigcup_{k=0}^{N-1} u^k E_0) \le N m(E_0) \le \varepsilon. \quad \square \end{aligned}$$

Now if we modify the proof of Theorem B by taking $\varepsilon = \frac{1}{q+1}$, taking E as in Lemma 3.2, and $A := \bigcup_{i=0}^{k-1} u^i E$, we get $Nm(E) \ge 1 - \varepsilon$ and

$$m(A) = km(E) \ge \frac{k(1-\varepsilon)}{kq} = \frac{1}{q+1} = \varepsilon,$$

and the rest is done exactly as in the proof of Theorem B, giving the following version of Theorem B for the action of (possibly) uncountable ordered groups.

Proposition 3.6. *Let (X, m) be a non-atomic probability measure space and (G, G_+, u) be a scaled ordered group. If there is a u-ergodic p.m.p. action of G on X then the system (G, X, m) is not strictly recurrent.*

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