

Using a Toy Model to Improve the Quantization of Gravity and Field Theories

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Abstract

A half-harmonic oscillator, which gets its name because the coordinate is strictly positive, has been quantized and determined that it was a physically correct quantization. This positive result was found using affine quantization (AQ). The main purpose of this paper is to compare results of this new quantization procedure with those of canonical quantization (CQ). Using Ashtekar-like classical variables and CQ, we quantize the same toy model. While these two quantizations lead to different results, they both would reduce to the same classical Hamiltonian if $\hbar \rightarrow 0$. Since these two quantizations have differing results, only one of the quantizations can be physically correct.

Two brief sections illustrate how AQ can correctly help quantum gravity and the quantization of most field theory problems.

1 A Simple Model Problem

A frequent toy model to study, using $-\infty < p, q < \infty$, with the Poisson bracket $\{q, p\} = 1$, is defined by its classical Hamiltonian, $H = (p^2 + q^2)/2$,

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i.e., the harmonic oscillator. Using canonical quantization (CQ), the quantum Hamiltonian becomes $(P^2 + Q^2)/2$, and its eigenfunctions and eigenvalues are well known [1]. For example, the eigenvalues are $\hbar(n + 1/2)$, where $n = 0, 1, 2, \dots$. It is safe to say that this quantization is valid.

The basic classical affine variables are the dilation $d = pq$ and $q \neq 0$ because if $q = 0$, then $d = 0$ and p can not help. This leads to the quantum affine variables for an affine quantization (AQ) which are the dilation operator $D = (PQ + QP)/2$ and $Q \neq 0$, because if $Q = 0$ is allowed, then, just like the classical case, $D = 0$ and P can not help.¹ The half-harmonic oscillator has the same classical Hamiltonian, $(p^2 + q^2)/2 = (d^2/q^2 + q^2)/2$ with $q > 0$, and the quantum Hamiltonian becomes $(DQ^{-2}D + Q^2)/2$. That model has now been solved using affine quantization (AQ), and its eigenvalues are $2\hbar(n + 1)$, with $n = 0, 1, 2, \dots$ [2, 3]. Beyond that, the coordinate stopping point can be moved from $q > 0$ to $q > -b$, with $b > 0$, which, through computer calculations, have found that the eigenvalues continue to be equally spaced for all $b > 0$, and effectively found them to be the original eigenvalues (along with the original eigenfunctions) for the full harmonic oscillator when $b \rightarrow \infty$ [4]. It is recognized that all of this is part of a valid quantization.

A brief summary of the quantum formulation of the half-harmonic oscillator is presented here. First, for the full harmonic oscillator, where $-\infty < q < \infty$, the quantum Hamiltonian is $(P^2 + Q^2)/2$, while for the half-harmonic oscillator, where $q > 0$, the quantum Hamiltonian is $[P^2 + (3/4)\hbar^2/Q^2 + Q^2]/2 = (DQ^{-2}D + Q^2)/2$, which, as noted, also has equally spaced eigenvalues. A partial-harmonic oscillator, where $q > -b$ with $b > 0$, and the quantum Hamiltonian is $[P^2 + (3/4)\hbar^2/(Q + b)^2 + Q^2]/2$, also has equally spaced (and b -dependent) eigenvalues for all b . This is beautifully illustrated in Fig. 1, page 15, in [4].²

We now turn our attention to a different procedure of quantizing the half-harmonic oscillator which uses CQ.

¹If $Q \neq 0$ then $P^\dagger \neq P$. However, $P^\dagger Q = PQ$. In Sec. 2.2 we observe that $P^\dagger P + (3/4)\hbar^2/Q^2 + Q^2 = P^2 + (3/4)\hbar^2/Q^2 + Q^2$ since every eigenfunction requires that for very tiny $x > 0$ then $\psi(x) \propto x^{3/2}$ [3].

²The author of Fig. 1 used a shifted equation to simplify the analysis. This shift changed the eigenfunctions from $\psi(x)$ to $\psi(x - b)$, but that had no influence on the eigenvalues.

2 A Different Quantization Procedure for the Half-Harmonic Oscillator

2.1 A change of classical variables

We start again with the classical Hamiltonian for the half-harmonic oscillator which is still $H = (p^2 + q^2)/2$ and $q > 0$, but this time we will use different coordinates. To let our new coordinate variables span the whole real line, which makes them ‘Ashtekar-like’ [5], we choose $q = s^2$, where $-\infty < s < \infty$. Thus, s is the new coordinate. For the new momentum we choose $r = 2p\sqrt{q}$ because the Poisson bracket $\{s, r\} = \{\sqrt{q}, 2p\sqrt{q}\} = 1$.³ The classical Hamiltonian now becomes $H = (p^2 + q^2)/2 = (r^2/4s^2 + s^4)/2$.

2.2 Quantization with the new quantum operators

For quantization, the new variables use canonical quantum operators, $r \rightarrow R$ and $s \rightarrow S$, with $[S, R] = i\hbar\mathbb{1}$. Following the CQ rules, this leads to $\mathcal{H}_{CQ} = [RS^{-2}R + S^4]/2$. This quantum operator, using canonical operators where $[S, R] = i\hbar\mathbb{1}$, is quite different from the valid affine expression $\mathcal{H}_{AQ} = [DQ^{-2}D + Q^2]/2$, rearranged into canonical operators with $[Q, P] = i\hbar\mathbb{1}$, that becomes $\mathcal{H}_{AQ} = [P^2 + (3/4)\hbar^2/Q^2 + Q^2]/2$. It is evident that these two canonical quantum Hamiltonian operators, \mathcal{H}_{CQ} and \mathcal{H}_{AQ} , have different eigenfunctions and eigenvalues. While the CQ and AQ stories lead to different results, only one of them can be physically correct.

This example joins many others for which two classical Hamiltonians, expressed in suitably different classical variables, are equal in their values, but that the two quantum Hamiltonians lead to different results.

2.3 A missing point

An issue that arose in Sec. 2.1 asked if $q = s^2 > 0$ or $q = s^2 \geq 0$ should be adopted for the half-harmonic oscillator in order to achieve a valid half-harmonic oscillator quantization.

Should we care if a *single point* in a whole space is missing, i.e., does $s \neq 0$ really matter? In classical mechanics it might only be nothing but a nuisance.

³It may be noticed that $0 < q = s^2$, hence $s \neq 0$, but instead we ignore this single point issue for now. *Do not miss Sec. 2.3 where ‘single points’ are examined further.*

However, as we will find out, in quantum mechanics it really can matter. We start by using many half-harmonic oscillators, leading to $H = \sum_{n=1}^N (p_n^2 + q_n^2)/2$, with $q_n > 0$ for all n . This equation can be interpreted differently as an N -dimensional vector, such as \vec{p} for which $\vec{p}^2 = p_1^2 + p_2^2 + \dots$ and \vec{q} for which $\vec{q}^2 = q_1^2 + q_2^2 + \dots$, with $q_n > 0$ for all n . This implies that $\vec{q}^2 > 0$, in which *a single point in an N dimensional space has been removed (imagine that for $N = 10,000!$). How could that matter? **It matters because the quantum theory of this ‘toy model’ is $\mathcal{H} = [\vec{P}^2 + (3/4)\hbar^2/\vec{Q}^2 + \vec{Q}^2]/2$, which is dramatically sensitive to a single missing point where $\vec{Q}^2 = 0$.***

By missing just *one point* in the entire coordinate region can lead to an incorrect quantization, as was shown by our toy model. Could that be likely to have any influence on the quantization of gravity?

A possible reply to that question may be found in the next section.

3 A Valid Quantum Gravity in a Nutshell

Before trying to solve a problem you should correctly formulate it!

Physics says that the distance between two different, but very close, points in space is given by $ds(x)^2 = g_{ab}(x) dx^a dx^b > 0$. This requirement ensures that $g(x) \equiv \det[g_{ab}(x)] > 0$. Likewise, from a purely mathematical view, and in preparation for an AQ quantization, the proper configuration space comes from $J(x) = C^a(x) g_{ab}(x) C^b(x)$, and all non-identically zero ‘vectors’ with components $C^a(x)$. While mathematically $J(x)$ could be positive, zero, or negative, we choose only those metrics $g_{ab}(x)$ for which $J(x) > 0$. That leads to the desired physically correct configuration space.

Instead of choosing the classical functions $\pi^{ab}(x)$ and $g_{cd}(x)$, we choose the dilation field (also known as the momentric field) $\pi_b^a(x) \equiv \pi^{ac}(x) g_{bc}(x)$ along with the metric field $g_{ab}(x)$. The Hamiltonian function (ignoring the cosmological constant) is given by

$$H(\pi, g) = \int \{g(x)^{-1/2} [\pi_b^a(x) \pi_a^b(x) - \frac{1}{2} \pi_a^a(x) \pi_b^b(x)] + g(x)^{1/2} R(x)\} d^3x, \quad (1)$$

where $R(x)$ is the 3-dimensional coordinate Ricci scalar [7].

For an AQ quantization, which uses Schrödinger’s representation, the basic quantum operators are the metric quantum operator $\hat{g}_{ab}(x) = g_{ab}(x)$ and the dilation quantum operator $\hat{\kappa}_b^a(x) = [\hat{\pi}^{ac}(x)^\dagger g_{bc}(x) + g_{bc}(x) \hat{\pi}^{ac}(x)]/2$, in which $\hat{\pi}^{ac}(x)^\dagger g_{bc}(x) = \hat{\pi}^{ac}(x) g_{bc}(x)$. These operators lead to the commutation rules

$$\begin{aligned}
[\hat{\pi}_b^a(x), \hat{\pi}_d^c(x')] &= i \frac{1}{2} \hbar \delta^3(x - x') [\delta_d^a \hat{\pi}_b^c(x) - \delta_b^c \hat{\pi}_d^a(x)] , \\
[g_{ab}(x), \hat{\pi}_d^c(x')] &= i \frac{1}{2} \hbar \delta^3(x - x') [\delta_d^c g_{ab}(x) + \delta_b^c g_{ad}(x)] , \\
[g_{ab}(x), g_{cd}(x')] &= 0 ,
\end{aligned} \tag{2}$$

and the quantum Hamiltonian operator is

$$\begin{aligned}
\mathcal{H}(\hat{\pi}, g) &= \int \{ [\hat{\pi}_b^a(x) g(x)^{-(1/2)} \hat{\pi}_a^b(x) - \frac{1}{2} \hat{\pi}_a^a(x) g(x)^{-(1/2)} \hat{\pi}_b^b(x)] \\
&\quad + g(x)^{(1/2)} R(x) \} d^3x .
\end{aligned} \tag{3}$$

While the quantum Hamiltonian of gravity is only part of the overall task, an incorrect version of that aspect is unlikely to use any further elements, like constraints, etc., to render a correct final quantization . Several articles pertaining to gravity by the author are [2, 8, 9, 10].

4 Multiple Valid Quantum Field Theories in a Nutshell

Let us consider a scalar field $\varphi(x)$, where x refers to an s -dimensional spatial field, as well as a momentum field $\pi(x)$. These two classical fields lead to the dilation field $\kappa(x) = \pi(x)\varphi(x)$, which is used instead of $\pi(x)$, along with $\varphi(x)$. Since $\kappa(x)$ would vanish if $\varphi(x) = 0$, it is essential to require that $\varphi(x) \neq 0$. Moreover, if $\pi(x)$ or $\varphi(x)$ were infinite, they could not properly render $\kappa(x)$, and therefore we require that $|\pi(x)| + |\varphi(x)| < \infty$, which then implies that $|\kappa(x)| < \infty$ as well. The presence of $(\vec{\nabla}\varphi(x))^2$ as part of the Hamiltonian density allows us to accept both positive and negative signs of $\varphi(x) \neq 0$, as if the field was still continuous. Using affine variables, the classical Hamiltonian density is given by

$$H(x) = \frac{1}{2} [\kappa(x)^2 / \varphi(x)^2 + (\vec{\nabla}\varphi(x))^2 + m^2 \varphi(x)^2] + g \varphi(x)^p . \tag{4}$$

Now we find that if $1/\varphi(x)^2 = 0$, then $\kappa(x) = 0$. To prevent that from happening, we require that $|\varphi(x)| < \infty$, which reinforces that it has already been adopted. So not only must $0 < \varphi(x)^{-2} < \infty$, it automatically requires that $0 < |\varphi(x)|^p < \infty$. Hence, it is fair to say that $0 \leq H(x) < \infty$. Finally, the classical Hamiltonian becomes

$$H(\kappa, \varphi) = \int \{ [\frac{1}{2} [\kappa(x)^2 / \varphi(x)^2 + (\vec{\nabla}\varphi(x))^2 + m^2 \varphi(x)^2] / 2 + g \varphi(x)^p \} d^s x , \tag{5}$$

which still must require that $H(\kappa, \varphi) < \infty$ in order to eliminate fields like $\varphi(x) = 1$ over any infinite space.

Using AQ and Schrödinger's representation for the dilation operator ,

$$\hat{\kappa}(x) = [\hat{\pi}(x)^\dagger \varphi(x) + \varphi(x) \hat{\pi}(x)]/2 , \quad (6)$$

and the quantum Hamiltonian becomes

$$\mathcal{H}(\hat{\kappa}, \varphi) = \int \{ [\frac{1}{2} [\hat{\kappa}(x)(\varphi(x))^{-2} \hat{\kappa}(x) + (\vec{\nabla} \varphi(x))^2 + m^2 \varphi(x)^2] + g \varphi(x)^p \} d^s x . \quad (7)$$

Using Monte Carlo, this last expression has already given positive results for the models φ_3^{12} and φ_4^4 , where φ_n^p refers to the interaction power p and the spacetime number $n = s + 1$ [11, 12].

5 Summary

This paper has largely been devoted to see if ‘Ashtekar-like’ coordinates can lead to physically correct quantizations by examining a toy model in order to find whether or not such variables might also lead to a physically correct quantization of gravity. It appears that a canonical quantization of the toy model, made possible by using Ashtekar-like variables, lead to a different quantum Hamiltonian from the known correct canonical equation. Thus, using canonical quantization and Ashtekar-like coordinates, have failed to lead to the physically correct quantization of the toy model.

In addition, using affine quantization procedures, and with various coordinate space removals, we have also given a highly realistic quantum version of quantum gravity, as well as for many realistic quantum examples of conventional quantum field theory.

If you use the right tools, you may solve a problem fairly easily. If you use the wrong tools, you may never solve the problem.

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