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Article

On the nonlocal problems in time for time-fractional subdiffusion equations

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Abstract: The nonlocal boundary value problem, $d_t^\rho u(t) + Au(t) = f(t)$ $(0 < \rho < 1, 0 < t \leq T)$, $u(\xi) = \alpha u(0) + \varphi$ (α is a constant and $0 < \xi \leq T$), in an arbitrary separable Hilbert space H with the strongly positive selfadjoint operator A, is considered. The operator d_t on the left hand side of the equation expresses either the Caputo derivative or the Riemann-Liouville derivative; naturally, in the case of the Riemann - Liouville derivatives, the nonlocal boundary condition should be slightly changed. Existence and uniqueness theorems for solutions of the problems under consideration are proved. The influence of the constant α on the existence of a solution to problems is investigated. Inequalities of coercivity type are obtained and it is shown that these inequalities differ depending on the considered type of fractional derivatives. The inverse problems of determining the right-hand side of the equation and the function φ in the boundary conditions are investigated.

Keywords: Nonlocal problems, the Riemann-Liouville and the Caputo derivatives, subdiffusion equation, inverse problems.

MSC: Primary 35R11; Secondary 34A12.

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1. Introduction

Let H be a separable Hilbert space with the scalar product $(\cdot,\cdot)$ and the norm $||\cdot||$ and $A : H \to H$ be an arbitrary unbounded positive selfadjoint operator in H. Suppose that A has a complete in H system of orthonormal eigenfunctions $\{v_k\}$ and a countable set of nonnegative eigenvalues λ_k. It is convenient to assume that the eigenvalues do not decrease as their number increases, i.e. $0 < \lambda_1 \leq \lambda_2 \cdots \to +\infty$.

Using the definitions of a strong integral and a strong derivative, fractional analogues of integrals and derivatives can be determined for vector-valued functions (or simply functions) $h : \mathbb{R}_+ \to H$, while the well-known formulae and properties are preserved (see, for example, [1]). Recall that the fractional integration of order $\sigma < 0$ of the function $h(t)$ defined on $[0,\infty)$ has the form

$$\partial_t^\sigma h(t) = \frac{1}{\Gamma(-\sigma)} \int_0^t \frac{h(\xi)}{(t-\xi)^{\sigma+1}} d\xi, \quad t > 0, \tag{1}$$

provided the right-hand side exists. Here $\Gamma(\sigma)$ is Euler's gamma function. Using this definition one can define the Riemann - Liouville fractional derivative of order ρ, $0 < \rho < 1$, as

$$\partial_t^\rho h(t) = \frac{d}{dt} \partial_t^{\rho-1} h(t).$$

If in this definition we interchange differentiation and fractional integration, then we get the definition of the regularized derivative, that is, the definition of the fractional derivative in the sense of Caputo:

$$D_t^\rho h(t) = \partial_t^{\rho-1} \frac{d}{dt} h(t).$$

Note that if $\rho = 1$, then fractional derivatives coincides with the ordinary classical derivative of the first order: $\partial_t h(t) = D_t h(t) = \frac{d}{dt} h(t)$.

Let $\rho \in (0, 1)$ be a fixed number and let $C((a, b); H)$ stand for a set of continuous functions $u(t)$ of $t \in (a, b)$ with values in H.

The subject of this work is the following two nonlocal boundary value problems:

$$\begin{cases} D_t^\rho u(t) + Au(t) = f(t), \quad 0 < t \leq T; \\ u(\xi) = \alpha u(0) + \varphi, \quad 0 < \xi \leq T \end{cases} \tag{2}$$

and

$$\begin{cases} \partial_t^\rho u(t) + Au(t) = g(t), \quad 0 < t \leq T; \\ \partial_t^{\rho-1} u(t)\Big|_{t=\xi} = \alpha \lim_{t\to 0} \partial_t^{\rho-1} u(t) + \phi, \quad 0 < \xi \leq T, \end{cases} \tag{3}$$

where $f(t), g(t) \in C((0, T]; H)$, $\varphi, \phi \in H$ and α is a constant. These problems are also called *the forward problems*.

Definition 1. *A function $u(t) \in C([0, T]; H)$ with the properties $D_t^\rho u(t), Au(t) \in C((0, T); H)$ and satisfying conditions (2) is called **the solution** of the nonlocal problem (2).*

The definition of the solution to the nonlocal problem (3) is introduced in a similar way.

If $\alpha = 0$ (and $\xi = T$), then these problems are called *the backward problems*. The backward problems in case (2) were studied in detail in [5] - [7]. The work [8] is devoted to the study of the backward problem in case (3). Therefore, in what follows we only consider the case

$$\alpha \neq 0. \tag{4}$$

The backward problems for the diffusion process are of great importance in engineering fields and are aimed at determining the previous state of a physical field (for example, at $t = 0$) based on its current information (see, for example, [6] and for the classical head equation see [9]). However, regardless of the fact that the Riemann-Liouville or Caputo derivative is taken into the equation, this problem is ill-possed in the sense of Hadamard. In other words, a small change of $u(T)$ in the norm of space H leads to large changes in the initial data. As can be seen from the main results of papers [5] - [7] and [8] (note, in these works $0 < \rho < 1$), the situation changes if we take sufficiently smooth function $u(T)$. In the case $\rho = 1$ these problems are also called (see, for example, [9], p. 214) the inverse heat conduction problem with inverse time (*retrospective inverse problem*). It should also be noted, that in this case even the smoothness of the function $u(T)$ does not guarantee the stability of the solution (see, for example, Chapter 8.2 of [9]).

The following nonlocal boundary value problem for the classical diffusion equation

$$\begin{cases} u'(t) + Au(t) = f(t), \quad 0 < t \leq T; \\ u(\xi) = u(0) + \varphi, \quad 0 < \xi \leq T \end{cases} \tag{5}$$

in an arbitrary Banach space E with the strongly positive operator A, has been extensively studied by numerous researchers (see, for example, A. O. Ashyralyev et al. [10] - [11]). As shown in these papers, in contrast to the retrospective inverse problem, the problem (5) is coercively solvable in some spaces of differentiable functions. It should also be noted that various nonlocal boundary value problems for parabolic equations reduce to the boundary value problem (5) (see [12], Chapter 1).

In the present paper we prove the existence and uniqueness theorems for solutions of problems (2) and (3). Next, we will study the dependence of the existence of a solution on the value of the parameter α. We will also prove, in contrast to the backward problems, the solutions of problems (2) and (3) continuously depend on the right-hand side of the equation and on function φ. Inequalities of coercivity type are obtained and it is shown that these inequalities differ depending on the considered type of fractional derivatives. The inverse problems of determining the right-hand side of the equation and function φ in the boundary conditions are investigated.

The remainder of this paper is composed of four sections. In the next section, we introduce the Hilbert space associated with the degree of operator A and recall some properties of the Mittag-Leffler functions. Section 3 is devoted to the study of the nonlocal problem (2). Here, we first investigate problem (2) for the homogeneous equation, and then move on to the main problem. In Section 4, we study the inverse problem of determining the right-hand side of equation (2). In this case, we assume that the unknown function f does not depend on t. The next section is devoted to the study of the inverse problem for the determination of the boundary function φ. Since problems (2) and (3) are studied in a similar way, in Section 6 we present only the main points of the proof of the theorem on the existence and uniqueness of the solution to problem (3). Inverse problems for equation (3) are considered in the same way as inverse problems for equation (2). Therefore, we omit these details.

2. Preliminaries

In this section, we introduce the Hilbert space of "smooth" functions related to the degree of operator A and recall some properties of the Mittag-Leffler functions, which we will use in what follows.

Let τ be an arbitrary real number. We introduce the power of operator A, acting in H according to the rule

$$A^\tau h = \sum_{k=1}^{\infty} \lambda_k^\tau h_k v_k,$$

where h_k is the Fourier coefficients of a function $h \in H$: $h_k = (h, v_k)$. Obviously, the domain of this operator has the form

$$D(A^\tau) = \{h \in H : \sum_{k=1}^{\infty} \lambda_k^{2\tau} |h_k|^2 < \infty\}.$$

For elements of $D(A^\tau)$ we introduce the norm

$$||h||_\tau^2 = \sum_{k=1}^{\infty} \lambda_k^{2\tau} |h_k|^2 = ||A^\tau h||^2,$$

and together with this norm $D(A^\tau)$ turns into a Hilbert space.

For $0 < \rho < 1$ and an arbitrary complex number μ, by $E_{\rho,\mu}(z)$ we denote the Mittag-Leffler function with two parameters:

$$E_{\rho,\mu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\rho n + \mu)}. \tag{6}$$

If the parameter $\mu = 1$, then we have the classical Mittag-Leffler function: $E_\rho(z) = E_{\rho,1}(z)$.

In what follows we need the asymptotic estimate of the Mittag-Leffler function with a sufficiently large negative argument. The well known estimate has the form (see, for example, [13], p. 136)

$$|E_{\rho,\mu}(-t)| \le \frac{C}{1+t}, \quad t>0, \tag{7}$$

where μ is an arbitrary complex number. This estimate essentially follows from the following asymptotic estimate (see, for example, [13], p. 134):

$$E_{\rho,\mu}(-t) = \frac{t^{-1}}{\Gamma(\mu-\rho)} + O(t^{-2}). \tag{8}$$

For the Mittag-Leffler function with two parameters $E_{\rho,\rho}(-t)$ one can get a better estimate than (7). Indeed, using the asymptotic estimate (see, for example, [13], p. 134)

$$E_{\rho,\rho}(-t) = -\frac{t^{-2}}{\Gamma(-\rho)} + O(t^{-3}), \tag{9}$$

and the fact that $E_{\rho,\rho}(t)$ is real analytic, we can obtain the following inequality [8]

$$|E_{\rho,\rho}(-t)| \le \frac{C}{1+t^2}, \quad t>0. \tag{10}$$

We will also use a coarser estimate with positive number λ and $0<\varepsilon<1$:

$$|t^{\rho-1}E_{\rho,\rho}(-\lambda t^{\rho})| \le \frac{Ct^{\rho-1}}{1+(\lambda t^{\rho})^2} \le C\lambda^{\varepsilon-1}t^{\varepsilon\rho-1}, \quad t>0, \tag{11}$$

which is easy to verify. Indeed, let $t^{\rho}\lambda<1$, then $t<\lambda^{-1/\rho}$ and

$$t^{\rho-1} = t^{\rho-\varepsilon\rho}t^{\varepsilon\rho-1} < \lambda^{\varepsilon-1}t^{\varepsilon\rho-1}.$$

If $t^{\rho}\lambda \ge 1$, then $\lambda^{-1}\le t^{\rho}$ and

$$\lambda^{-2}t^{-\rho-1} = \lambda^{-1+\varepsilon}\lambda^{-1-\varepsilon}t^{-\rho-1} \le \lambda^{\varepsilon-1}t^{\varepsilon\rho-1}.$$

Proposition 1. *Let $0<\rho<1$. Then*

$$E_{\rho}(x) > 0, \quad \frac{d}{dx}E_{\rho}(x) > 0, \quad x\in\mathbb{R}. \tag{12}$$

Proof. For $x\ge 0$ this is obvious; estimates (12) follow from definition (6).

For $x<0$ we use the integral representation (see, for example, [2], p. 54)

$$E_{\rho}(x) = \frac{\sin\rho\pi}{\pi}\int_0^{\infty}\frac{e^{-t|x|^{1/\rho}}}{1+2t^{\rho}\cos\rho\pi+t^{2\rho}}\,t^{\rho}\,dt > 0.$$

Then

$$\frac{d}{dx}E_{\rho}(x) = |x|^{(1-\rho)/\rho}\cdot\frac{\sin\rho\pi}{\rho\pi}\int_0^{\infty}\frac{e^{-t|x|^{1/\rho}}}{1+2t^{\rho}\cos\rho\pi+t^{2\rho}}\,t^{\rho+1}\,dt > 0.$$

□

Using Proposition 1, by virtue of estimates (12) and equality $E_\rho(0) = 1$, we arrive at (see [2], p. 47).

Proposition 2. *The Mittag-Leffler function of negative argument $E_\rho(-x)$ is monotonically decreasing function for all $0 < \rho < 1$ and*

$$0 < E_\rho(-x) < 1. \tag{13}$$

Proposition 3. *Let $\rho > 0$ and $\lambda > 0$. Then for all positive $t > 0$ one has [6]*

$$\int_0^t \eta^{\rho-1} E_{\rho,\rho}(-\lambda\eta^\rho)d\eta = \frac{1}{\lambda}\big(1 - E_\rho(-\lambda t^\rho)\big).$$

Proof. First, we calculate the derivative of the Mittag-Leffler function

$$\frac{d}{dt}E_\rho(-\lambda t^\rho) = -\rho\lambda t^{\rho-1}\sum_{n=1}^{\infty}\frac{n(-\lambda t^\rho)^{n-1}}{\Gamma(\rho n+1)} = -\rho\lambda t^{\rho-1}\sum_{k=0}^{\infty}\frac{(k+1)(-\lambda t^\rho)^k}{\Gamma(\rho(k+1)+1)} =$$

(since $\Gamma(x+1) = x\Gamma(x)$)

$$= -\lambda t^{\rho-1}\sum_{k=0}^{\infty}\frac{(-\lambda t^\rho)^k}{\Gamma(\rho k+\rho)} = -\lambda t^{\rho-1}E_{\rho,\rho}(-\lambda t^\rho).$$

Note that here the series is termwise differentiable in $\mathbb{R}$.

Now, by virtue of the equality

$$\int_0^t \eta^{\rho-1}E_{\rho,\rho}(-\lambda\eta^\rho)d\eta = -\frac{1}{\lambda}\int_0^t \frac{d}{d\eta}E_\rho(-\lambda\eta^\rho)d\eta,$$

we get the required result. □

Proposition 4. *Let $0 < \rho < 1$ and $\lambda > 0$. Then*

$$\frac{d}{dt}\Big[t^\rho E_{\rho,\rho+1}(-\lambda t^\rho)\Big] > 0,\ t > 0,$$

i.e. $t^\rho E_{\rho,\rho+1}(-\lambda t^\rho)$ strictly increases as a function of $t > 0$.

Proof. Using (6) and term-by-term integration we arrive at (see [2], formula (4.4.4))

$$\int_0^t \eta^{\rho-1}E_{\rho,\rho}(-\lambda\eta^\rho)d\eta = t^\rho E_{\rho,\rho+1}(-\lambda t^\rho), \tag{14}$$

or by Proposition 3,

$$t^\rho E_{\rho,\rho+1}(-\lambda t^\rho) = \frac{1}{\lambda}\big(1 - E_\rho(-\lambda t^\rho)\big).$$

It remains to apply Proposition 1. □

Proposition 5. *Let* $0<\rho<1$ *and* $\lambda>0$. *Then for all positive* t *one has*

$$\partial_t^{\rho-1}\left(t^{\rho-1}E_{\rho,\rho}(-\lambda t^\rho)\right)=E_\rho(-\lambda t^\rho).$$

Proof. By definition of the fractional integration (1) we have

$$\partial_t^{\rho-1}\left(t^{\rho-1}E_{\rho,\rho}(-\lambda t^\rho)\right)=\frac{1}{\Gamma(1-\rho)}\int\limits_0^t\frac{\xi^{\rho-1}E_{\rho,\rho}(-\lambda\xi^\rho)}{(t-\xi)^\rho}d\xi=$$

$$=\frac{1}{\Gamma(1-\rho)}\sum_{j=0}^\infty\frac{(-\lambda)^j}{\Gamma(\rho j+\rho)}\int\limits_0^t\frac{\xi^{\rho-1+\rho j}}{(t-\xi)^\rho}d\xi=$$

$$=\frac{1}{\Gamma(1-\rho)}\sum_{j=0}^\infty\frac{(-\lambda)^j}{\Gamma(\rho j+\rho)}t^{\rho j}\int\limits_0^1 s^{\rho-1+\rho j}(1-s)^{-\rho}ds.$$

On the other hand, using the properties of Euler's beta function $B(a,b)$, we obtain

$$B(\rho+\rho j,1-\rho)=\int\limits_0^1 s^{\rho-1+\rho j}(1-s)^{-\rho}ds=\frac{\Gamma(\rho+\rho j)\Gamma(1-\rho)}{\Gamma(\rho j+1)}.$$

By virtue of the definition of the Mittag-Leffler function $E_\rho(z)$ this implies the statement of the proposition. □

3. Well-posedness of the problem (2)

To solve problem (2), we divide it into two auxiliary problems:

$$\begin{cases} D_t^\rho\omega(t)+A\omega(t)=f(t), \quad 0<t\leq T;\\ \omega(0)=0\end{cases} \tag{15}$$

and

$$\begin{cases} D_t^\rho w(t)+Aw(t)=0, \quad 0<t\leq T;\\ w(\xi)=\alpha w(0)+\psi, \quad 0<\xi\leq T,\end{cases} \tag{16}$$

where $\psi\in H$ is a given vector.

Problem (16) is a special case of problem (2), and the solution to problem (15) is defined similarly to Definition 1.

If $\psi=\varphi-\omega(\xi)$ and $\omega(t)$ and $w(t)$ are the corresponding solutions, then it is easy to verify that function $u(t)=\omega(t)+w(t)$ is a solution to problem (2). Therefore, it is sufficient to solve the auxiliary problems.

For problem (15) we have the following statement.

Theorem 1. *Let $f(t) \in C([0,T]; D(A^\varepsilon))$ for some $\varepsilon \in (0,1)$. Then problem (15) has a unique solution and this solution has the representation*

$$\omega(t) = \sum_{k=1}^{\infty} \left[\int_0^t \eta^{\rho-1} E_{\rho,\rho}(-\lambda_k \eta^\rho) f_k(t-\eta) d\eta \right] v_k. \tag{17}$$

Moreover, there is a constant $C_\varepsilon > 0$ such that the following coercive type inequality holds:

$$||D_t^\rho \omega(t)||^2 + ||\omega(t)||_1^2 \le C_\varepsilon \max_{t\in[0,T]} ||f||_\varepsilon^2, \quad 0 < t \le T. \tag{18}$$

Proof of Theorem 1. It is not hard to verify that the series (17) is a formal solution to problem (15) (see, for example, [2], p. 173). In order to prove that function (17) is actually a solution to the problem, it remains to substantiate this formal statement, i.e. to show that the operators A and D_t^ρ can be applied term-by-term to series (17).

Let $S_j(t)$ be the partial sum of series (17). Then

$$AS_j(t) = \sum_{k=1}^{j} \left[\int_0^t \eta^{\rho-1} E_{\rho,\rho}(-\lambda_k \eta^\rho) f_k(t-\eta) d\eta \right] \lambda_k v_k. \tag{19}$$

Due to the Parseval equality we may write

$$||AS_j(t)||^2 = \sum_{k=1}^{j} \lambda_k^2 \left| \int_0^t \eta^{\rho-1} E_{\rho,\rho}(-\lambda_k \eta^\rho) f_k(t-\eta) d\eta \right|^2 .$$

Then, by inequality (11) for $0 < \varepsilon < 1$ one has

$$||AS_j(t)||^2 \le C \sum_{k=1}^{j} \left[\int_0^t \eta^{\varepsilon\rho-1} \lambda_k^\varepsilon |f_k(t-\eta)| d\eta \right]^2,$$

or, by virtue of the generalized Minkowski inequality,

$$||AS_j(t)||^2 \le C \left[\int_0^t \eta^{\varepsilon\rho-1} \left(\sum_{k=1}^{j} |\lambda_k^\varepsilon f_k(t-\eta)|^2 \right)^{\frac{1}{2}} d\eta \right]^2 \le C_\varepsilon \max_{t\in[0,T]} ||f(t)||_\varepsilon^2.$$

Hence, we obtain $A\omega(t) \in C([0,T]; H)$ and in particular $\omega(t) \in C([0,T]; H)$.

Further, from equation (2) one has $D_t^\rho S_j(t) = -AS_j(t) + \sum_{k=1}^{j} f_k(t) v_k$, $t > 0$. Therefore, from the above reasoning, we have $D_t^\rho \omega(t) \in C((0,T]; H)$ and

$$||D_t^\rho S_j(t)||^2 \le C_\varepsilon \max_{t\in[0,T]} ||f(t)||_\varepsilon^2 + ||f(t)||^2, \quad t > 0.$$

Thus, we have completed the rationale that (17) is a solution to problem (15). The last two inequalities imply the estimate (18).

The uniqueness of the solution can be proved by the standard technique based on completeness of the set of eigenfunctions $\{v_k\}$ in H (see, for example, [8]).

Theorem 1 is completely proved.

If f does not depend on t, then the statement of Theorem 1 is true for all $f \in H$.

Corollary 1. *Let $f \in H$. Then problem (15) has a unique solution and this solution has the representation*

$$\omega(t) = \sum_{k=1}^{\infty} f_k t^{\rho} E_{\rho,\rho+1}(-\lambda_k t^{\rho}) v_k. \tag{20}$$

Moreover, there is a positive constant C such that the following coercive type inequality holds:

$$||D_t^{\rho}\omega(t)||^2 + ||\omega(t)||_1^2 \leq C||f||^2, \quad 0 < t \leq T. \tag{21}$$

Proof. Since f does not depend on t, then we have the following form for the Fourier coefficients of ω (see (17))

$$\omega_k(t) = f_k \int_0^t \eta^{\rho-1} E_{\rho,\rho}(-\lambda_k \eta^{\rho}) d\eta. \tag{22}$$

Application of formula (14) to the integral shows that the formal solution to problem (15) has the form (20).

Let $S_j(t)$ be the partial sum of series (20). Then by virtue of estimate (7), we get

$$||AS_j(t)||^2 \leq C \sum_{k=1}^{j} \left| \frac{\lambda_k t^{\rho} f_k}{1 + \lambda_k t^{\rho}} \right|^2 \leq C||f||^2.$$

Now, using this estimate and repeating the arguments similar to the proof of Theorem 1, it is easy to check that (20) is indeed a solution to problem (15) and estimate (21) holds true.

□

We now turn to problem (16). In accordance with the Fourier method, we will look for a solution to problem (16) in the form of a series:

$$w(t) = \sum_{k=1}^{\infty} T_k(t) v_k,$$

where $T_k(t)$, $k \geq 1$, are solutions of the nonlocal problems:

$$\begin{cases} D_t^{\rho} T_k(t) + \lambda_k T_k(t) = 0, & 0 < t \leq T; \\ T_k(\xi) = \alpha T_k(0) + \psi_k, \end{cases} \tag{23}$$

where ψ_k is the Fourier coefficients of function $\psi \in H$.

Let us denote $T_k(0) = b_k$. Then the unique solution to the differential equation (23) with this initial condition has the form $T_k(t) = b_k E_{\rho}(-\lambda_k t^{\rho})$ (see, for example, [2], p.174, [3], [4], p. 17). From the nonlocal conditions of (23) we obtain the following equation to find the unknown numbers b_k:

$$b_k E_{\rho}(-\lambda_k \xi^{\rho}) = \alpha b_k + \psi_k. \tag{24}$$

By virtue of property (13) of the Mittag-Leffler function, $E_\rho(-\lambda_k\xi^\rho) \neq \alpha$ for all $\alpha \geq 1$ and $\alpha < 0$ (note, $\xi > 0$ and $\lambda_k > 0$). Therefore, from (24) we have

$$b_k = \frac{\psi_k}{E_\rho(-\lambda_k\xi^\rho) - \alpha}, \quad |b_k| \leq C_\alpha |\psi_k|, \quad k \geq 1, \quad \text{and} \quad \alpha \geq 1 \quad \text{or} \quad \alpha < 0, \tag{25}$$

here and below, by C_β we will denote a constant depending on β, not necessarily the same one.

If $\alpha = 0$, then $E_\rho(-\lambda_k\xi^\rho) \neq 0$, but the Mittag-Leffler function can asymptotically tend to zero (see (8)). Therefore, in this case one has:

$$b_k = \frac{\psi_k}{E_\rho(-\lambda_k\xi^\rho)}, \quad |b_k| \leq C_\rho \lambda_k \xi^\rho |\psi_k|.$$

This case, as noted above (see (4)), has been studied in detail in [5] - [7].

Let $0 < \alpha < 1$. Then according to Proposition 2, there is a unique $\lambda_0 > 0$ such that $E_\rho(-\lambda_0\xi^\rho) = \alpha$. If $\lambda_k \neq \lambda_0$ for all $k \geq 1$, then the estimate in (25) holds with some constant $C_\alpha > 0$.

Thus, if $\alpha \notin (0,1)$ or $\alpha \in (0,1)$, but $\lambda_k \neq \lambda_0$ for all $k \geq 1$, then the formal solution of problem (16) has the form

$$w(t) = \sum_{k=1}^{\infty} \frac{\psi_k}{E_\rho(-\lambda_k\xi^\rho) - \alpha} E_\rho(-\lambda_k t^\rho) v_k. \tag{26}$$

Finally, let $0 < \alpha < 1$ and $\lambda_k = \lambda_0$ for $k = k_0, k_0 + 1, ..., k_0 + p_0 - 1$, where p_0 is the multiplicity of the eigenvalue λ_{k_0}. Then the nonlocal problem (23) has a solution if the boundary function $\psi(x)$ satisfies the following orthogonality conditions

$$\psi_k = (\psi, v_k) = 0, \ k \in K_0; \ K_0 = \{k_0, k_0 + 1, ..., k_0 + p_0 - 1\}, \tag{27}$$

and for these $k \in K_0$ arbitrary numbers b_k are solutions of equation (24). For all other k we have

$$b_k = \frac{\psi_k}{E_\rho(-\lambda_k\xi^\rho) - \alpha}, \quad |b_k| \leq C_\alpha |\psi_k|, \quad k \notin K_0. \tag{28}$$

Thus, the formal solution of problem (16) in this case has the form

$$w(t) = \sum_{k \notin K_0} \frac{\psi_k}{E_\rho(-\lambda_k\xi^\rho) - \alpha} E_\rho(-\lambda_k t^\rho) v_k + \sum_{k \in K_0} b_k E_\rho(-\lambda_k t^\rho) v_k. \tag{29}$$

Throughout what follows we will assume that whenever $0 < \alpha < 1$ and $\lambda_k = \lambda_0$, then orthogonality condition (27) is satisfied.

Let us show that the operators A and D_t^ρ can be applied term-by-term to series (26); for series (29) this question is considered in a completely similar way.

Let $S_j(t)$ be the partial sum of series (26). Then

$$AS_j(t) = \sum_{k=1}^{j} \lambda_k \frac{\psi_k}{E_\rho(-\lambda_k\xi^\rho) - \alpha} E_\rho(-\lambda_k t^\rho) v_k. \tag{30}$$

Due to the Parseval equality we may write

$$||AS_j(t)||^2 \leq \sum_{k=1}^{j} \lambda_k^2 \left| \frac{\psi_k}{E_\rho(-\lambda_k\xi^\rho) - \alpha} E_\rho(-\lambda_k t^\rho) \right|^2.$$

Using estimates (7), (25) and (28) we obtain

$$||AS_j(t)||^2 \leq C_\alpha \sum_{k=1}^{j} \lambda_k^2 \left| \frac{\psi_k}{1+\lambda_k t^\rho} \right|^2 \leq C_\alpha t^{-2\rho} \sum_{k=1}^{j} |\psi_k|^2. \tag{31}$$

Therefore if $\psi \in H$, then $Au(t) \in C((0,T];H)$. From equation (16) one has $D_t^\rho u(t) = -Au(t)$, $t > 0$, and the above estimates imply

$$||D_t^\rho w(t)||^2 \leq C_\alpha t^{-2\rho} \sum_{k=1}^{j} |\psi_k|^2, \tag{32}$$

which means $D_t^\rho w(t) \in C((0,T];H)$.

For $S_j(t)$, taking into account estimate (7), we obtain

$$||S_j(t)||^2 \leq C_\alpha \sum_{k=1}^{j} |\psi_k|^2. \tag{33}$$

Hence $w(t) \in C([0,T];H)$, which was required by the definition of the solution to problem (16).

Let us investigate the uniqueness of the solution to problem (16). Suppose we have two solutions: $w_1(t)$, $w_2(t)$ and set $w(t) = w_1(t) - w_2(t)$. Then we have

$$\begin{cases} D_t^\rho w(t) + Aw(t) = 0, \quad 0 < t \leq T; \\ w(\xi) = \alpha w(0), \quad 0 < \xi \leq T. \end{cases} \tag{34}$$

Let $w_k(t) = (w(t), v_k)$. Since the operator A is self-adjoint, one has

$$D_t^\rho w_k(t) = (D_t^\rho w(t), v_k) = -(Aw(t), v_k) = -(w(t), Av_k) = -\lambda_k w_k(t) \tag{35}$$

and the nonlocal condition implies

$$w_k(\xi) = \alpha w_k(0). \tag{36}$$

Let us denote $w_k(0) = b_k$. Then the unique solution to the differential equation (35) with this initial condition has the form $w_k(t) = b_k E_\rho(-\lambda_k t^\rho)$ (see, for example, [2], p.174, [3], [4], p. 17). From the nonlocal conditions of (36) we obtain the following equation to find the unknown numbers b_k:

$$b_k E_\rho(-\lambda_k \xi^\rho) = \alpha b_k. \tag{37}$$

Let first $\alpha \notin (0,1)$ or $\alpha \in (0,1)$, but $\lambda_k \neq \lambda_0$ for all $k \geq 1$. Then $E_\rho(-\lambda_k \xi^\rho) \neq \alpha$ for all k. Consequently, in this case all b_k are equal to zero (therefore $w_k(t) = 0$), and by virtue of completeness of the set of eigenfunctions $\{v_k\}$, we conclude that $w(t) \equiv 0$. Thus, problem (16) in this case has a unique solution.

Now suppose that $\alpha \in (0,1)$ and $\lambda_k = \lambda_0$, $k \in K_0$. Then $E_\rho(-\lambda_k \xi^\rho) = \alpha$, $k \in K_0$ and therefore equation (37) has the following solution: $b_k = 0$ if $k \notin K_0$ and b_k is an arbitrary number for $k \in K_0$. Thus, in this case, there is no uniqueness of the solution to problem (16).

Thus we obtain the following statement:

Theorem 2. *Let $\psi \in H$.*

If $\alpha \notin (0,1)$ or $\alpha \in (0,1)$, but $\lambda_k \neq \lambda_0$ for all $k \geq 1$, then problem (16) has a unique solution and this solution has the form (26).

If $\alpha \in (0,1)$ and $\lambda_k = \lambda_0$, $k \in K_0$, then we assume that the orthogonality conditions (27) are satisfied. The solution of problem (16) has the form (29) with arbitrary coefficients b_k, $k \in K_0$.

Moreover, there is a constant $C_\alpha > 0$ such that the following coercive type inequality holds:

$$||D_t^\rho w(t)||^2 + ||w(t)||_1^2 \leq C_\alpha t^{-2\rho} ||\psi||^2, \quad 0 < t \leq T. \tag{38}$$

Note that the proof of the coercive type inequality (38) follows from the estimates (31) and (32).

Now let us move on to solving the main problem (2). Let $\varphi \in H$ and $f(t) \in C([0,T]; D(A^\varepsilon))$ for some $\varepsilon \in (0,1)$. As noted above, if we put $\psi = \varphi - \omega(\xi) \in H$ and $\omega(t)$ and $w(t)$ are the corresponding solutions of problems (15) and (16), then function $u(t) = \omega(t) + w(t)$ is a solution to problem (2). Therefore, if $\alpha \notin (0,1)$ or $\alpha \in (0,1)$, but $\lambda_k \neq \lambda_0$ for all $k \geq 1$, then

$$u(t) = \sum_{k=1}^{\infty} \left[\frac{\varphi_k - \omega_k(\xi)}{E_\rho(-\lambda_k \xi^\rho) - \alpha} E_\rho(-\lambda_k t^\rho) + \omega_k(t) \right] v_k, \tag{39}$$

where

$$\omega_k(t) = \int_0^t \eta^{\rho-1} E_{\rho,\rho}(-\lambda_k \eta^\rho) f_k(t-\eta) d\eta.$$

The uniqueness of the function $u(t)$ follows from the uniqueness of the solutions $\omega(t)$ and $w(t)$.

If $\alpha \in (0,1)$ and $\lambda_k = \lambda_0$, $k \in K_0$, then

$$u(t) = \sum_{k \notin K_0} \left[\frac{\varphi_k - \omega_k(\xi)}{E_\rho(-\lambda_k \xi^\rho) - \alpha} E_\rho(-\lambda_k t^\rho) + \omega_k(t) \right] v_k + \sum_{k \in K_0} b_k E_\rho(-\lambda_k t^\rho) v_k. \tag{40}$$

The corresponding orthogonality conditions have the form

$$(\varphi, v_k) = (\omega(\xi), v_k), \; k \in K_0; \; K_0 = \{k_0, k_0+1, ..., k_0 + p_0 - 1\}. \tag{41}$$

In particular, if

$$(\varphi, v_k) = 0, \; (f(t), v_k) = 0, \; \text{for all } t > 0, \; k \in K_0; \; K_0 = \{k_0, k_0+1, ..., k_0 + p_0 - 1\}, \tag{42}$$

then the orthogonality conditions (41) are satisfied.

Thus we have proved the main result of this section:

Theorem 3. *Let $\varphi \in H$ and $f(t) \in C([0,T]; D(A^\varepsilon))$ for some $\varepsilon \in (0,1)$.*

If $\alpha \notin (0,1)$ or $\alpha \in (0,1)$, but $\lambda_k \neq \lambda_0$ for all $k \geq 1$, then problem (2) has a unique solution and this solution has the form (39).

If $\alpha \in (0,1)$ and $\lambda_k = \lambda_0$, $k \in K_0$, then we assume that the orthogonality conditions (42) are satisfied. The solution of problem (16) has the form (40) with arbitrary coefficients b_k, $k \in K_0$.

Moreover, there are constants $C_\alpha > 0$ and $C_\varepsilon > 0$ such that the following coercive type inequality holds:

$$||D_t^\rho u(t)||^2 + ||u(t)||_1^2 \leq C_\alpha t^{-2\rho} ||\varphi||^2 + C_\varepsilon \max_{t \in [0,T]} ||f||_\varepsilon^2, \quad 0 < t \leq T. \tag{43}$$

4. Inverse problem of determining the heat source density

The inverse problems of determining the right-hand side (the heat source density) of various subdiffusion equations have been considered by a number of authors (see, e.g. [14] - [26] and the bibliography therein). Let us mention only some of these works. The case of subdiffusion equations, the elliptic part A of which is an ordinary differential expression, considered in [14] - [20]. Authors of the papers [21] - [25], considered subdiffusion equations, in which the elliptic part A is either the Laplace operator or a second-order operator. The paper [26] studied the inverse problem for the subdiffusion equation (2) with the Cauchy condition. In this article [26] and most other articles, including [21] - [24], the Caputo derivative is used as a fractional derivative. The recent article [27] - [28] is devoted to the inverse problem for the subdiffusion equation with Riemann-Liouville derivatives. In [15] and [25], the fractional derivative in the subdiffusion equation is a two-parameter generalized Hilfer fractional derivative; this type of fractional derivative contains a parameter belonging to the interval [0, 1], and its extreme values correspond to the Caputo and Riemann-Liouville derivatives. Various models of applied problems leading to Hilfer fractional derivatives are investigated in [29]. Note also that the papers [15], [21], [24] contain a survey of papers dealing with inverse problems of determining the right-hand side of the subdiffusion equation.

In [30] the authors of this paper considered an inverse problem for the simultaneous determination of the order of the Riemann-Liouville fractional derivative and the source function in the subdiffusion equations. Using the classical Fourier method, the authors proved the uniqueness and existence theorem for this inverse problem.

In [31] - [32], the authors investigated the inverse problem of determining the order of the fractional derivative in the subdiffusion equation and in the wave equation, respectively.

It should be noted that in all of the listed works, the Cauchy conditions in time are considered. In the present paper, for the best of our knowledge, inverse problems for subdiffusion equations with nonlocal conditions in time are considered for the first time.

Let us consider *the inverse problem*

$$\begin{cases} D_t^\rho u(t) + Au(t) = f, \quad 0 < t \leq T; \\ u(\xi) = \alpha u(0) + \varphi, \quad 0 < \xi \leq T, \end{cases} \tag{44}$$

with the additional condition

$$u(\tau) = \Psi, \quad 0 < \tau \leq T, \quad \tau \neq \xi, \tag{45}$$

in which the unknown element $f \in H$, characterizing the action of heat sources, does not depend on t and $\Psi, \varphi \in H$ are given elements, α is an arbitrary given constant.

Note that if $\tau = \xi$, then the nonlocal condition in (44) coincides with the Cauchy condition $u(0) = \varphi_1$ (see (4)). In this case, this inverse problem was studied in [26].

Definition 2. *A pair $\{u(t), f\}$ of function $u(t) \in C([0,T]; H)$ and $f \in H$ with the properties $D_t^\rho u(t), Au(t) \in C((0,T]; H)$ and satisfying conditions (44), (45) is called* ***the solution*** *of the inverse problem (44), (45).*

In what follows we shall deal only with the case $\alpha \geq 1$, since in this case the uniqueness of the solution is relatively easy to prove.

Theorem 4. *Let $\varphi, \Psi \in D(A)$ and $\alpha \geq 1$. If $\tau > \frac{\xi}{2}$ $(\tau \neq \xi)$, then the inverse problem (44), (45) has a unique solution $\{u(t), f\}$ and this solution has the following form*

$$f = \sum_{k=1}^{\infty} \left[\frac{\alpha - E_\rho(-\lambda_k \xi^\rho)}{E_\rho(-\lambda_k \tau^\rho)\xi^\rho E_{\rho,\rho+1}(-\lambda_k \xi^\rho) + \tau^\rho E_{\rho,\rho+1}(-\lambda_k \tau^\rho)[\alpha - E_\rho(-\lambda_k \xi^\rho)]} \Psi_k + \right. \tag{46}$$

$$\left. + \frac{E_\rho(-\lambda_k \tau^\rho)}{E_\rho(-\lambda_k \tau^\rho)\xi^\rho E_{\rho,\rho+1}(-\lambda_k \xi^\rho) + \tau^\rho E_{\rho,\rho+1}(-\lambda_k \tau^\rho)[\alpha - E_\rho(-\lambda_k \xi^\rho)]} \varphi_k \right] v_k,$$

$$u(t) = \sum_{k=1}^{\infty} \left[\frac{E_\rho(-\lambda_k t^\rho)}{E_\rho(-\lambda_k \xi^\rho) - \alpha} [\varphi_k - f_k \xi^\rho E_{\rho,\rho+1}(-\lambda_k \xi^\rho)] + f_k t^\rho E_{\rho,\rho+1}(-\lambda_k t^\rho) \right] v_k. \tag{47}$$

.

Remark 1. *It obviously follows from the proof of uniqueness that if $\tau \leq \frac{\xi}{2}$, then the uniqueness of the function f will not hold for all $\alpha \geq 1$. For example, if $\tau = \frac{\xi}{2}$, then the uniqueness of the function f does not hold for $\alpha = 1$.*

Proof of Theorem 4. Existence. If f is known, then the unique solution of problem (44) has the form (39), and since f does not depend on t, then, thanks to formulas (22) and (14), it is easy to verify that the formal solution of problem (44) has the form (47).

By virtue of additional condition (45) and completeness of the system $\{v_k\}$ we obtain:

$$\frac{E_\rho(-\lambda_k \tau^\rho)}{E_\rho(-\lambda_k \xi^\rho) - \alpha} [\varphi_k - f_k \xi^\rho E_{\rho,\rho+1}(-\lambda_k \xi^\rho)] + f_k \tau^\rho E_{\rho,\rho+1}(-\lambda_k \tau^\rho) = \Psi_k.$$

After simple calculations, we get

$$f_k = \frac{\alpha - E_\rho(-\lambda_k \xi^\rho)}{E_\rho(-\lambda_k \tau^\rho)\xi^\rho E_{\rho,\rho+1}(-\lambda_k \xi^\rho) + \tau^\rho E_{\rho,\rho+1}(-\lambda_k \tau^\rho)[\alpha - E_\rho(-\lambda_k \xi^\rho)]} \Psi_k + \tag{48}$$

$$+ \frac{E_\rho(-\lambda_k \tau^\rho)}{E_\rho(-\lambda_k \tau^\rho)\xi^\rho E_{\rho,\rho+1}(-\lambda_k \xi^\rho) + \tau^\rho E_{\rho,\rho+1}(-\lambda_k \tau^\rho)[\alpha - E_\rho(-\lambda_k \xi^\rho)]} \varphi_k \equiv f_{k,1} + f_{k,2}.$$

With these Fourier coefficients we have the above formal series (46) for the unknown function f: $f = \sum_{k=1}^{\infty} (f_{k,1} + f_{k,2}) v_k$.

Let us show the convergence of series (46). If F_j the partial sums of series (46), then by virtue of the Parseval equality we may write

$$||F_j||^2 = \sum_{k=1}^{j} [f_{k,1} + f_{k,2}]^2 \leq 2 \sum_{k=1}^{j} f_{k,1}^2 + 2 \sum_{k=1}^{j} f_{k,2}^2 \equiv 2I_{1,j} + 2I_{2,j}. \tag{49}$$

Since $\xi > 0$, then $E_\rho(-\lambda_k \tau^\rho)\xi^\rho E_{\rho,\rho+1}(-\lambda_k \xi^\rho) > 0$. Therefore,

$$I_{1,j} \leq \sum_{k=1}^{j} \left| \frac{\alpha - E_\rho(-\lambda_k \xi^\rho)}{\tau^\rho E_{\rho,\rho+1}(-\lambda_k \tau^\rho)[\alpha - E_\rho(-\lambda_k \xi^\rho)]} \right|^2 |\Psi_k|^2 = \sum_{k=1}^{j} \frac{|\Psi_k|^2}{|\tau^\rho E_{\rho,\rho+1}(-\lambda_k \tau^\rho)|^2}.$$

Using the asymptotic estimate (see (8))

$$E_{\rho,\rho+1}(-t) = t^{-1} + O(t^{-2}), \tag{50}$$

we get

$$I_{1,j} \leq \sum_{k=1}^{j} \frac{\lambda_k^2 |\Psi_k|^2}{\big(1+O((-\lambda_k \tau^\rho)^{-1})\big)^2} \leq C \sum_{k=1}^{j} \lambda_k^2 |\Psi_k|^2 \leq C||\Psi||_1^2.$$

Since $\tau > 0$ and $E_\rho(-\lambda_k \xi^\rho) \neq \alpha$, then $\tau^\rho E_{\rho,\rho+1}(-\lambda_k \tau^\rho)[1 - E_\rho(-\lambda_k \xi^\rho)] > 0$. Therefore,

$$I_{2,j} \leq \sum_{k=1}^{j} \left| \frac{E_\rho(-\lambda_k \tau^\rho)}{E_\rho(-\lambda_k \tau^\rho)\xi^\rho E_{\rho,\rho+1}(-\lambda_k \xi^\rho)} \right|^2 |\varphi_k|^2 = \sum_{k=1}^{j} \frac{|\varphi_k|^2}{\big|\xi^\rho E_{\rho,\rho+1}(-\lambda_k \xi^\rho)\big|^2}.$$

By virtue of (50),

$$I_{2,j} \leq \sum_{k=1}^{j} \frac{\lambda_k^2 |\varphi_k|^2}{\big(1+O((-\lambda_k \xi^\rho)^{-1})\big)^2} \leq C \sum_{k=1}^{j} \lambda_k^2 |\varphi_k|^2 \leq C||\varphi||_1^2.$$

Thus, if $\varphi, \Psi \in D(A)$, then from estimates of $I_{i,j}$ and (49) we obtain $f \in H$.

After finding the unknown function $f \in H$, the fulfillment of the conditions of Definition 2 for function $u(t)$, defined by the series (47) is proved in exactly the same way as with Corollary 1 and Theorem 2.

Uniqueness. Suppose we have two solutions: $\{u_1(t), f_1\}$ and $\{u_2(t), f_2\}$. It is required to prove $u(t) \equiv u_1(t) - u_2(t) \equiv 0$ and $f \equiv f_1 - f_2 = 0$. Since the problem is linear, to determine $u(t)$ and f we have the problem:

$$D_t^\rho u(t) + Au(t) = f, \quad t > 0; \tag{51}$$

$$u(\xi) = \alpha u(0), \quad 0 < \xi \leq T, \tag{52}$$

$$u(\tau) = 0, \quad \tau > \frac{\xi}{2} \quad (\tau \neq \xi). \tag{53}$$

Let $u(t)$ be a solution to this problem and $u_k(t) = (u(t), v_k)$. Then, by virtue of equation (51) and the selfadjointness of operator A,

$$D_t^\rho u_k(t) = (D_t^\rho u(t), v_k) = -(Au(t), v_k) + (f, v_k) = -(u(t), Av_k) + (f, v_k) = \tag{54}$$

$$-(u(t), \lambda_k v_k) + f_k = -\lambda_k (u(t), v_k) + f_k = -\lambda_k u_k(t) + f_k, \quad t > 0.$$

Thus, taking into account (53), we have the following problem

$$D_t^\rho u_k(t) + \lambda_k u_k(t) + f_k = 0, \quad t > 0; \quad u(\tau) = 0.$$

If $t \geq \tau$, then the solution to this problem has the form (see, for example, [2], p.174, [3], [4], p. 17)

$$u_k(t) = f_k \int_\tau^t (t-\xi)^{\rho-1} E_{\rho,\rho}(-\lambda_k (t-\xi)^\rho) d\xi = f_k \int_0^{t-\tau} \eta^{\rho-1} E_{\rho,\rho}(-\lambda_k \eta^\rho) d\eta,$$

or (see (14))

$$u_k(t) = f_k \cdot (t-\tau)^\rho \, E_{\rho,\rho+1}(-\lambda_k (t-\tau)^\rho).$$

If $t \leq \tau$, then

$$u_k(t) = -f_k \int_\tau^t (\xi - t)^{\rho-1} E_{\rho,\rho}(-\lambda_k (\xi - t)^\rho) d\xi = f_k \cdot (\tau - t)^\rho \, E_{\rho,\rho+1}(-\lambda_k (\tau - t)^\rho).$$

Therefore, the nonlocal condition (52) implies:

$$f_k\big[|\tau-\xi|^\rho E_{\rho,\rho+1}(-\lambda_k|\tau-\xi|^\rho)-\alpha\,\tau^\rho E_{\rho,\rho+1}(-\lambda_k\tau^\rho)\big]=0,\ \xi<2\tau,\ \alpha\geq 1.$$

Since $\tau>\frac{\xi}{2}$, then $|\tau-\xi|<\tau$. Hence, due to the monotonicity of the Mittag-Leffler function $t^\rho E_{\rho,\rho+1}(-\lambda t^\rho)$, $t>0$ (see Proposition 4), one has $f_k=0$, for all $k\geq 1$. Therefore, from the completeness of the system of eigenfunctions $\{v_k\}$, we finally obtain $f=0$ and $u(t)\equiv 0$, as required.

5. The inverse problem of determining the boundary function φ

Consider the problem (2) and assume that, together with function $u(t)$, function φ in the nonlocal condition $u(\xi)=\alpha u(0)+\varphi$ is also unknown. To solve *this inverse problem*, we need an additional condition, and as such we again take the condition that was used in the previous inverse problem:

$$u(\tau)=\Psi,\quad 0<\tau\leq T,\quad \tau\neq\xi. \tag{55}$$

If $\tau=\xi$, then the nonlocal condition $u(\xi)=\alpha u(0)+\varphi$ coincides with the Cauchy condition $u(0)=\varphi_1$ (see (4)) and we have the inverse problem, considered in [5] - [7].

Definition 3. *A pair $\{u(t),\varphi\}$ of function $u(t)\in C([0,T];H)$ and $\varphi\in H$ with the properties $D_t^\rho u(t), Au(t)\in C((0,T);H)$ and satisfying conditions (2), (55) are called **the solution** of **the inverse problem** (2), (55).*

Again, as in the previous inverse problem, an additional condition is imposed on α, which simplifies the proof of the uniqueness of the solution.

Theorem 5. *Let $\Psi\in D(A)$, $f\in C([0,T];D(A^\varepsilon))$ for some $\varepsilon\in(0,1)$ and $E_\rho(-\lambda_k\xi^\rho)\neq\alpha$ for all k. Then the inverse problem (2), (55) has a unique solution $\{u(t),\varphi\}$ and this solution has the form*

$$\varphi=\sum_{k=1}^\infty\left[\frac{E_\rho(-\lambda_k\xi^\rho)-\alpha}{E_\rho(-\lambda_k\tau^\rho)}[\Psi_k-\omega_k(\tau)]+\omega_k(\xi)\right]v_k, \tag{56}$$

$$u(t)=\sum_{k=1}^\infty\left[\frac{\varphi_k-\omega_k(\xi)}{E_\rho(-\lambda_k\xi^\rho)-\alpha}\,E_\rho(-\lambda_k t^\rho)+\omega_k(t)\right]v_k, \tag{57}$$

where

$$\omega_k(t)=\int_0^t\eta^{\rho-1}E_{\rho,\rho}(-\lambda_k\eta^\rho)f_k(t-\eta)d\eta.$$

Proof of Theorem 5. Existence. If φ is known, then the solution of problem (3) has the form (57) (see Theorem 3). Condition (55) implies:

$$u(\tau)=\sum_{k=1}^\infty\left[\frac{\varphi_k-\omega_k(\xi)}{E_\rho(-\lambda_k\xi^\rho)-\alpha}\,E_\rho(-\lambda_k t^\rho)+\omega_k(\tau)\right]v_k=\Psi.$$

Let us expand the function $\Psi\in H$ in a Fourier series in the system $\{v_k\}$. Then

$$\frac{\varphi_k-\omega_k(\xi)}{E_\rho(-\lambda_k\xi^\rho)-\alpha}\,E_\rho(-\lambda_k t^\rho)+\omega_k(\tau)=\Psi_k,\ k\geq 1,$$

or

$$\varphi_k = \frac{E_\rho(-\lambda_k \xi^\rho) - \alpha}{E_\rho(-\lambda_k \tau^\rho)} [\Psi_k - \omega_k(\tau)] + \omega_k(\xi).$$

Therefore, equality (56) is formally established. It remains to prove that $\varphi \in H$.

Let Φ_j be the partial sum of series (56). Then due to Parseval's equality

$$||\Phi_j||^2 = \sum_{k=1}^{j} \left| \frac{E_\rho(-\lambda_k \xi^\rho) - \alpha}{E_\rho(-\lambda_k \tau^\rho)} [\Psi_k - \omega_k(\tau)] + \omega_k(\xi) \right|^2 \leq \tag{58}$$

$$\leq 3 \sum_{k=1}^{j} \left[\left| \frac{E_\rho(-\lambda_k \xi^\rho) - \alpha}{E_\rho(-\lambda_k \tau^\rho)} \right|^2 [|\Psi_k|^2 + |\omega_k(\tau)|^2] + |\omega_k(\xi)|^2 \right] \equiv \Phi_j^1 + \Phi_j^2 + \Phi_j^3.$$

Since $|E_\rho(-\lambda_k \xi^\rho) - \alpha| \leq C$, then by virtue of the asymptotic estimate (8) we obtain

$$\Phi_j^1 \leq \sum_{k=1}^{j} \frac{\lambda_k^2 \tau^{2\rho} \Gamma^2(1-\rho)}{\big(1 + O((-\lambda_k \tau^\rho)^{-1})\big)^2} |\Psi_k|^2 \leq C \sum_{k=1}^{j} \lambda_k^2 |\Psi_k|^2 \leq C ||\Psi||_1^2.$$

Similarly, by virtue of estimate (11) and the definition of ω_k, we have

$$\Phi_j^2 \leq \sum_{k=1}^{j} \frac{\lambda_k^2 \tau^{2\rho} \Gamma^2(1-\rho)}{\big(1 + O((-\lambda_k \tau^\rho)^{-1})\big)^2} \left| \int_0^\tau \eta^{\rho-1} E_{\rho,\rho}(-\lambda_k \eta^\rho) f_k(\tau - \eta) d\eta \right|^2 \leq$$

$$\leq \sum_{k=1}^{j} \frac{C_\varepsilon \lambda_k^2}{\big(1 + O((-\lambda_k \tau^\rho)^{-1})\big)^2} \left| \int_0^\tau \eta^{\varepsilon\rho-1} \lambda_k^{\varepsilon-1} |f_k(\tau - \eta)| d\eta \right|^2 \leq$$

$$\leq C_\varepsilon \left[\int_0^\tau \eta^{\varepsilon\rho-1} \left(\sum_{k=1}^{j} |\lambda_k^\varepsilon f_k(\tau - \eta)|^2 \right)^{\frac{1}{2}} d\eta \right]^2 \leq C_\varepsilon \max_{t\in[0,T]} ||f||_\varepsilon^2.$$

For the sum Φ_j^3 one has

$$\Phi_j^3 \leq \sum_{k=1}^{j} \left| \int_0^\xi \eta^{\rho-1} E_{\rho,\rho}(-\lambda_k \eta^\rho) f_k(\xi - \eta) d\eta \right|^2 \leq$$

$$\leq C \left[\int_0^\xi \eta^{\rho-1} \left(\sum_{k=1}^{j} |f_k(\xi - \eta)|^2 \right)^{\frac{1}{2}} d\eta \right]^2 \leq C \max_{t\in[0,T]} ||f||^2.$$

Thus, it is shown that $\varphi \in H$ (see (58)).

Fulfillment of the conditions of Definition 3 for function $u(t)$, defined by the series (57) is proved in exactly the same way as with Theorem 3.

Uniqueness. Let us prove that if $\{u(t), \varphi\}$ is a solution to the homogeneous problem:

$$D_t^\rho u(t) + Au(t) = 0, \qquad t > 0; \tag{59}$$

$$u(\xi) = u(0) + \alpha\varphi, \quad 0 < \xi \leq T, \tag{60}$$

$$u(\tau) = 0, \quad 0 < \tau \leq T, \; \tau \neq \xi, \tag{61}$$

then $u(t) \equiv 0$ and $\varphi = 0$.

Let $u(t)$ be a solution to this problem and let $u_k(t) = (u(t), v_k)$. Then

$$D_t^\rho u_k(t) + \lambda_k u_k(t) = 0, \quad t > 0; \quad u_k(\xi) = \alpha u_k(0) + \varphi_k.$$

The solution to this problem has the form (this is the same problem as (23))

$$u_k(t) = \frac{E_\rho(-\lambda_k t^\rho)}{E_\rho(-\lambda_k \xi^\rho) - \alpha}\, \varphi_k.$$

Condition (61) implies

$$u_k(\tau) = \frac{E_\rho(-\lambda_k \tau^\rho)}{E_\rho(-\lambda_k \xi^\rho) - \alpha}\, \varphi_k = 0.$$

Since $E_\rho(-\lambda_k \xi^\rho) \neq \alpha$ for all $k \geq 1$, then by virtue of the properties of the Mittag-Leffler functions (see Proposition 2) we have $\varphi_k = 0$ for all k. This, in turn, means $u_k(t) \equiv 0$ for all k. Therefore, due to the completeness of the system of eigenfunctions $\{v_k\}$, we have $\varphi = 0$ and $u(t) \equiv 0$, as required.

Theorem 5 is completely proved.

Remark 2. *If f does not depend on t, then the statement of Theorem 5 is true for all $f \in H$, and $\Psi \in D(A)$.*

Using Corollary 1, this statement can be proved in the same way as above.

6. Well-posedness of the problem (3)

In the case of fractional Riemann-Liouville derivatives, we consider only the forward problem for the homogeneous subdiffusion equation. The inhomogeneous equations and inverse problems considered above are studied in exactly the same way as in the case of the Caputo derivatives.

Consider the forward problem:

$$\begin{cases} \partial_t^\rho u(t) + Au(t) = 0, \quad 0 < t \leq T; \\ \partial_t^{\rho-1} u(t)\Big|_{t=\xi} = \alpha \lim\limits_{t\to 0} \partial_t^{\rho-1} u(t) + \phi, \quad 0 < \xi \leq T, \end{cases} \tag{62}$$

where $\phi \in H$ and the number α are given.

We will only consider the case $E_\rho(-\lambda_k \xi^\rho) \neq \alpha$ for all $k \geq 1$, other cases are treated similarly.

Theorem 6. *Let $\phi \in H$ and $E_\rho(-\lambda_k \xi^\rho) \neq \alpha$ for all $k \geq 1$. Then problem (62) has a unique solution and this solution has the form*

$$u(t) = \sum_{k=1}^{\infty} \frac{\phi_k}{E_\rho(-\lambda_k \xi^\rho) - \alpha}\, t^{\rho-1} E_{\rho,\rho}(-\lambda_k t^\rho) v_k. \tag{63}$$

Moreover, there is a constant $C_\xi > 0$ such that the following coercive type inequality holds:

$$||\partial_t^\rho u(t)||_1^2 + ||u(t)||_2^2 \leq C_\xi t^{-2\rho-2} ||\phi||^2, \quad 0 < t \leq T. \tag{64}$$

Proof. As in the case of problem (2), we will seek a solution to problem (62) in the form of a series: $\sum_{k=1}^{\infty} T_k(t)v_k$, where $T_k(t)$ is a solution of the problem:

$$\partial_t^\rho T_k(t) + \lambda_k T_k(t) = 0, \quad 0 < t \leq T; \tag{65}$$

$$\partial_t^{\rho-1} T_k(t)\Big|_{t=\xi} = \alpha \lim_{t\to 0} \partial_t^{\rho-1} T_k(t) + \phi_k, \quad 0 < \xi \leq T. \tag{66}$$

Let us denote $\lim_{t\to 0} \partial_t^{\rho-1} T_k(t) = b_k$. Then the unique solution to the differential equation (65) with this initial condition has the form $T_k(t) = b_k t^{\rho-1} E_{\rho,\rho}(-\lambda_k t^\rho)$ (see, for example, [2], p.173, [4], p. 16 and [33]).

Proposition 5 implies

$$\partial_t^{\rho-1} T_k(t)\Big|_{t=\xi} = b_k E_\rho(-\lambda_k \xi^\rho)$$

Then from the nonlocal conditions (66) we find the unknown numbers b_k:

$$b_k = \frac{\phi_k}{E_\rho(-\lambda_k \xi^\rho) - \alpha}.$$

Hence, function $u(t)$ defined by series (63) is a formal solution to problem (62).

Now let us show that series (63) is indeed a solution. To do this we denote by $S_j(t)$ the partial sum of series (63). Then

$$A^2 S_j(t) = \sum_{k=1}^{j} \lambda_k^2 \frac{\phi_k}{E_\rho(-\lambda_k \xi^\rho) - \alpha} t^{\rho-1} E_{\rho,\rho}(-\lambda_k t^\rho) v_k. \tag{67}$$

Due to the Parseval equality we may write

$$||A^2 S_j(t)||^2 = \sum_{k=1}^{j} \left| \lambda_k^2 \frac{\phi_k}{E_\rho(-\lambda_k \xi^\rho) - \alpha} t^{\rho-1} E_{\rho,\rho}(-\lambda_k t^\rho) \right|^2.$$

Estimate (10) of function $E_{\rho,\rho}(-t)$ implies (note, $E_\rho(-\lambda_k \xi^\rho) \neq \alpha$)

$$||A^2 S_j(t)||^2 \leq \frac{C t^{2\rho-2}}{|\alpha - E_\rho(-\lambda_1 \xi^\rho)|^2} \sum_{k=1}^{j} \lambda_k^4 \left| \frac{\phi_k}{1 + \lambda_k^2 t^{2\rho}} \right|^2 \leq C_\xi t^{-2\rho-2} \sum_{k=1}^{j} |\phi_k|^2 \leq C_\xi t^{-2\rho-2} ||\phi||^2.$$

Thus $A^2 u(t) \in C((0, T]; H)$ (and even more so $Au(t) \in C((0, T]; H)$).

Since $A\partial_t^\rho u(t) = -A^2 u(t)$, then from the above estimate we have

$$||\partial_t^\rho u(t)||_1^2 \leq C_\xi t^{-2\rho-2} ||\phi||^2.$$

Thus, (63) is a solution to problem (62). The coercivity inequality follows from the last two estimates. The uniqueness is proved in the same way as in the proof of Theorem 2. □

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1. Lizama, C., *Abstract linear fractional evolution equations*, Handbook of Fractional Calculus with Applications V. 2, J.A.T. Marchado Ed. DeGruyter, **2019**; 465-497.
2. Gorenflo, R.; Kilbas, A. A.; Mainardi, F.; Rogozin, S. V., *Mittag-Leffler functions, related topics and applications*, Springer **2014**.
3. Ashurov, R.; Cabada A.; Turmetov, B. *Operator method for construction of solutions of linear fractional differential equations with constant coefficients*, Fractional calculas and applied analysis, **2016** 1, 229-252.
4. Pskhu, A.V. *Fractional partial differential equations* (in Russian), M. NAUKA **2005**.
5. Liu, J.; Yamamoto, M. A backward problem for the time-fractional diffusion equation, *Appl. Anal.*, **2010** M. **89** 1769-1788.
6. Sakamoto, K.; Yamamoto, M. Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems, *J. Math. Anal. Appl.*, **2011**, **382**, 426-447.
7. Floridia, G.; Li, Z.; Yamamoto, M. Well-posedness for the backward problems in time for general time-fractional difussion equation, *Rend. Lincei Mat. Appl.*, **2020** **31**, 593-610.
8. Alimov, Sh.A.; Ashurov, R.R. On the backward problems in time for time-fractional subdiffusion equations, https://www.researchgate.net/publication/351575279.
9. Kabanikhin, S.I. *Inverse and Ill-Posed Problems. Theory and Applications*, De Gruyter **2011**.
10. Ashyralyev, A. O.; Sobolevskii, P. E. Coercive stability of a multidimensional difference elliptic equation of 2m-th order with variable coefficients, Investigations in the Theory of Differential Equations (Russian), Minvuz Turkmen. SSR, Ashkhabad, **1987**, pp. 31-43.
11. Ashyralyev, A. O.; Hanalyev, A.; Sobolevskii, P. E. Coercive solvability of nonlocal boundary value problem for parabolic equations, Abstract and Applied Analysis, **2001**, 6 (1), 53-61.
12. Krein, S. G. Linear Differential Equations in a Banach Space, Nauka, Moscow, **1966** (Russian).
13. Dzherbashian M. M. *Integral Transforms and Representation of Functions in the Complex Domain* (in Russian), M. NAUKA **1966**.
14. Zhang, Y.; Xu,X.; Inverse scource problem for a fractional differential equations, Inverse Prob. **2011**, V. 27. N 3. P. 31-42.
15. Furati,K. M.; Iyiola, O. S.; Kirane, M. An inverse problem for a generalized fractional diffusion. Applied Mathematics and Computation. **2014**, V. 249. P. 24-31.
16. Ismailov, M. I.; Cicek, M. Inverse source problem for a time-fractional diffusion equation with nonlocal boundary conditions. Applied Mathematical Modelling. **2016**, V. 40. P. 4891-4899.
17. Kirane, M.; Malik, A. S. Determination of an unknown source term and the temperature distribution for the linear heat equation involving fractional derivative in time. Applied Mathematics and Computation. **2011**, V. 218. P. 163-170.
18. Kirane, M.; Samet, B.; Torebek, B. T. Determination of an unknown source term and the temperature distribution for the subdiffusion equation at the initial and final data. Electronic Journal of Differential Equations. **2017**, V. 217. P. 1-13.
19. Nguyen, H. T.; Le. D. L.; Nguyen, V. T. Regularized solution of an inverse source problem for a time fractional diffusion equation. Applied Mathematical Modelling. **2016**, V. 40. P. 8244-8264.
20. Torebek, B. t.; Tapdigoglu, R. Some inverse problems for the nonlocal heat equation with Caputo fractional derivative. Mathematical Methods in Applied Sciences. **2017**, V. 40. P. 6468-6479.
21. Li, Z.; Liu,Y.; Yamamoto, M. Initial-boundary value problem for multi-term time-fractional diffusion equation with positive constant coefficients. Applied Mathematica and Computation. **2015**, V. 257. P. 381-397.
22. Rundell, W.; Zhang, Z. Recovering an unknown source in a fractional diffusion problem. Journal of Computational Physics. **2018**, V. 368. P. 299-314.
23. Asl, N. A.; Rostamy, D. Identifying an unknown time-dependent boundary source ib time-fractional diffusion equation with a non-local boundary condition. Journal of Computational and Applied Mathematics. **2019**, V. 335. P. 36-50.

24. Sun, L.; Zhang, Y.; Wei,T. Recovering the time-dependent potential function in a multi-term time-fractional diffusion equation. Applied Numerical Mathematics. **2019**, V. 135. P. 228-245.
25. Malik, S. A.; Aziz, S. An inverse source problem for a two parameter anomalous diffusion equation with nonlocal boundary conditions. Computers and Mathematics with applications. **2017**, V. 3. P. 7-19.
26. Ruzhansky, M.; Tokmagambetov, N.; Torebek, B. T. Inverse source problems for positive operators. I: Hypoelliptic diffusion and subdiffusion equations. J. Inverse Ill-Possed Probl. **2019**, V. 27. P. 891-911.
27. Ashurov, R.; Muhiddinova, O. Inverse problem of determining the heat source density for the subdiffusion equation. Differential equations, **2020**, Vol. 56, No. 12, pp. 1550-1563.
28. Ashurov, R.; Muhiddinova, O. Initial-boundary value problem for a time-fractional subdiffusion equation with an arbitrary elliptic differential operator. Lobachevskii Journal of Mathematics, **2021**, V 42, N 3.
29. Hilfer, R.; Luchko, Y.; Tomovski, Z. Operational method for solution of the fractional differential equations with the generalized Riemann-Liouville fractional derivatives. Frac.Calculus Appl. Anal. **2009**, V. 12. P. 299-318.
30. Ashurov, R.; Fayziev,Yu. Determination of fractional order and source term in a fractional subdiffusion equation. https://www.researchgate.net/publication/354997348
31. Ashurov, R.; Fayziev, Yu. Uniqueness and existence for inverse problem of determining an order of time-fractional derivative of subdiffusion equation. Lobachevskii journal of mathematics, **2021**, V 42, N 3, 508-516.
32. Ashurov, R.; Fayziev, Yu. Inverse problem for determining the order of the fractional derivative in the wave equation. Mathematical Notes. **2021**, 110:6, 824-836 (in Russian)
33. Ashurov, R.; Fayziev, Yu. On construction of solutions of linear fractional differentional equations with constant coefficients and the fractional derivatives, Uzb.Math.Journ. **2017**, 3, 3-21, (in Russian).