
Article

Architectural Significance of the Seokguram Buddhist Grotto in Gyeongju (Korea)

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Abstract: The purpose of this article is to disclose the architectural proportions and nature of the Korean national treasure in Seokguram Grotto, Gyeongju. The authors compare its features with those of other ancient hypogeal or ashlar constructions and intend to rediscover its relevant hidden configuration and latent structural properties to show its uniqueness. The methods employed in the research belong initially to architectural design and composition to advance at a later stage, into the nuances of stone masonry, lighting effects or cohesive construction. In this discussion and thorough analysis, different philosophical and scientific subtleties come afloat. The results demonstrate a significant potential capable to be applied in part to recent architectural developments like Tadao Ando's Buddha Hill in Hokkaido (2017) and the authors' own project for a Buddhist monument.

Keywords: Korean Heritage; Asian Architecture; Tadao Ando; Hypogeal Chambers; Architectural Proportions; Ashlar Construction; Innovative Architectural Projects.

1. Introduction. The Seokguram syndrome

It seems that in Korean language there is no appropriate word to describe what one feels when contemplating the Seokguram temple. Indeed, in the book of Joon- Sik Choi [1] we can read:

The moment you see it you will be stunned by the breathtaking sight. It radiates a dignified yet compassionate aura. The cave in which the Buddha statue is enshrined does not feel like being from this world. It is a wondrous sight that cannot be properly expressed in words. You cannot help but wonder how on earth it was possible to carve such a soft and flowing appearance out of granite, a rock that is so hard that it is considered the most difficult type of stone to sculpt anything out of. Various thoughts will come to mind, but I think it is best to just observe without trying to think of anything at all. Isn't it enough for the best works of art simply to exist?

In the same way, the celebrated work of Yu Hong-June [2] cites the manuscript of Ko Yusoeop "Our Art and Handicrafts":

This great statue, beginning as a bloodless, passionless lump of granite, has been endowed with a strong pulse, breath, divinity, gentleness and dignity. When unveiled, the joy was surely not limited to the sculptor, but extended throughout Silla and echoed towards the cosmos.

Most people who visit the Seokguram, feel as if they lacked the words to express what they have adumbrated in it. Some have compared this situation to the impressions experienced in seeing for the first time the sculpture of the Pietà by Michelangelo Buonarroti (unfortunately also through a glass pane), today located in the entrance chapel to the right side of the Basilica of St. Peter in Rome (Figure 1).

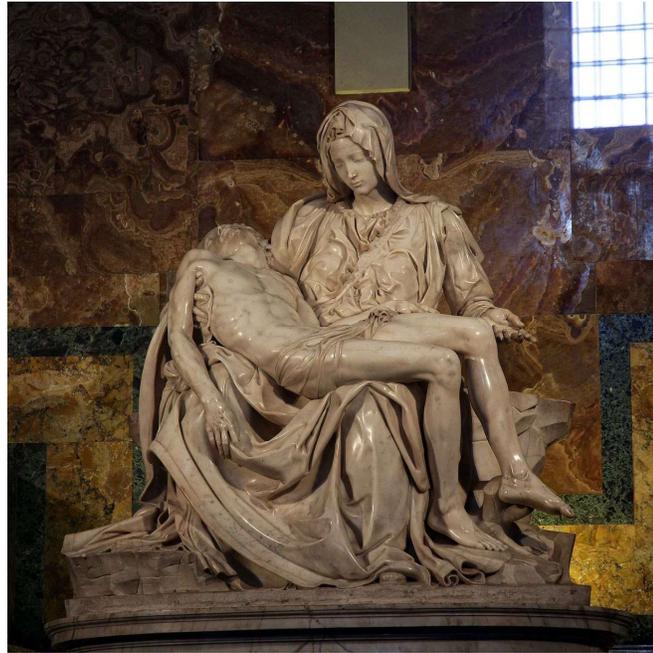


Figure 1. Pietà by Michelangelo Buonarroti. Source https://commons.m.wikimedia.org/wiki/Category:Piet%C3%A0_in_Saint_Peter%27s_Basilica#/media/File%3A'Pietà'_by_Michelangelo_JBU21.jpg

However, history of art comes to our aid when we intend to find words to express what we may recollect in such quaint occasions. The feeling stems from the 19th-century French author Stendhal (pseudonym of Marie-Henri Beyle), who described his own impressions during a stint in 1817 to Florence in his book *Naples and Florence: A Journey from Milan to Reggio*.

When he visited the Basilica of *Santa Croce*, where *Niccolò Machiavelli*, *Michelangelo Buonarroti* and Galileo Galilei are interred, he saw Giotto's frescoes for the first time and was overwhelmed with emotion. He wrote:

I was in a sort of ecstasy, from the idea of being in Florence, close to the great men whose tombs I had seen. Absorbed in the contemplation of sublime beauty... I reached the point where one encounters celestial sensations... Everything spoke so vividly to my soul [...] [3].

The syndrome was only named in 1979, as it was described by the Italian psychiatrist Graziella Magherini, who reported more than 100 similar cases among tourists and visitors in Florence. There is no scientific proof to define the Stendhal syndrome as a specific psychiatric disorder; on the other hand, there is evidence that the same cerebral areas involved in emotional reactions are activated during the exposure to artworks. Thus, Salguero and Cabeza-Láinez propose to coin anew the term “Seokguram Syndrome” to describe what happens in our interior when we contemplate this Korean masterpiece of universal art.

Consequently, to disclose the main features of Seokguram, our research will be presented in the following way:

In chapter 2, we will compare the main similarities of the constructive characteristics of the temple with other buildings in different latitudes and epochs. Section 3, enumerates diverse constructions based on the employ of cupolas. Section 4, describes the evolution that took place in Korea from the ceremonial tumulus to the Buddhist chaitya or ritual cave. On chapter 5 we would identify the mathematic fundamentals of the dimensions and disposition of the Seokguram temple.

In the structural analysis of Chapter 6, we will focus in the possibility of building the Seokguram dome of 'dry' granite buried without 'forearms', what could have been the structural function of these 'forearms' and more intriguingly if structural stability would be compromised by the existence of a vaulted opening in the front of the dome, used to bring forth introspective lighting effects.

Finally, in section 7 we would present and explore future architectural projects that are inspired and derive from the findings attained in the previous chapters.

2. Some Characteristics of the Tectonic Art of Silla

There are specific points in the architecture of the Unified Silla Period in Korea (668-935 A. D.) to which the Seokguram enclave belongs, that are designated as exclusive of its epoch and culture. One of them is the stone carving method known as 'grangie' which consists in adapting the straight contours of rectangular blocks to dovetail within the natural stones located underneath in a manner of foundation (Figure 2).



Figure 2. The 'Grangie' ('Kŭraeng'í) technique.

In wooden architecture, this is known as the 'kŭraeng'í technique and, according to several authors, it only appears in Korea. However, we have found some examples of this 'technique' in other places and epochs like Machu Picchu (literally 'Old Mountain') in *Peru* built between 1438 and 1470 A. D. (Figure 3), and at a stone wall of the church of Santa

María de Melque which dates from the 10th century) in Toledo (Spain), to name just a few examples (Fig. 4).



Figure 3. Cyclopean wall in Machu Picchu (*Peru*). Joseph Cabeza.



Figure 4. Window of the church of *Santa María de Melque* 10th century, Toledo (Spain).

Another typical feature of the original Seokguram temple is the placement of granite ashlar without mortar. In Spain, the most significant construction built using this technique is the Roman aqueduct of Segovia (Figure 5)



Figure 5. Aqueduct of Segovia (Spain). Granite ashlars without mortar.

Although we have to outline a recently detected phenomenon by which the vibrations due to traffic and the musical concerts that are celebrated in its vicinity are literally pulverising the granite ashlars. This circumstance should also be taken into account in the conservation of the Seokguram temple through precise measurements of the vibrations to which it is subject daily.

3. The Dome of the Seokguram temple. A singular case in universal Architecture

It is difficult to find examples of real domes built using ashlar directly cut from the rock. One of the most cited examples is the Treasury of Atreus or *Tomb of Agamemnon*. It is a large "tholos" tomb on the Panagitsa Hill at Mycenae, Greece, which was constructed during the Bronze Age (around 1250 B. C.). The lintel stone above the doorway weighs 120 tons, with approximate dimensions 8.3x5.2x1.2 m, it constitutes the largest in the world.

However, as figures 5 and 6 show, it is not a true dome, but instead a corbelled or false dome made by minute displacements of the stones to the inside and towards the apex.



Figure 5. Interior of the Treasury of Atreus, near the entrance.

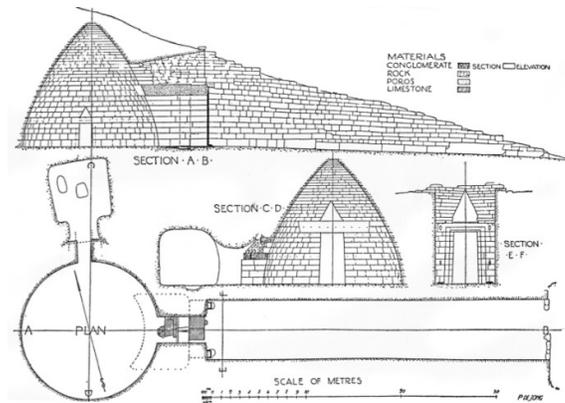


Figure 6. Schematic plan and sections of the Treasury of Atreus.

Another difficulty encountered in the finding of these types of buildings is that, on a number of occasions, the ceiling is no longer extant because it was built with friable materials, whereby it is very difficult to know if these roofs were really domes, as is the case of Ġgantija, one of the main Megalithic Complexes of Malta. Their makers erected the two Ġgantija structures during the Neolithic (3600–2500 B. C.), which assigns to these sanctuaries more than 5500 years of age and thus they would constitute the world's second oldest religious structures known to man.



Figure 6. Ġgantija, the Megalithic Temples of Malta. Source: Joseph Cabeza-Lainez

One of the problems raised by the construction of domes is that, however light they may be, they tend to create a very strong horizontal thrust force. In Rome, the simplest formula was chosen to bear the loads induced by them: placing the support of the dome directly over circular or octagonal walls. This is what happens in the Pantheon of Agrippa (that we will explain later on). In Byzantium a wise equilibrium was devised by setting the walls free of their sustaining role, that is, conceiving the dome over spandrels.



Figure 7. Dome on spandrels at the Hagia Sophia church in Istanbul, Turkey.

In the case of the Roman Pantheon, the dome is sorted out by means of brickwork discharge arches, ribs made of the same material and hydraulic cement mortar (a kind of roman concrete), but it is not built by the simple stacking of granite blocks.



Figure 8. Concentric nerves used for the construction of the Pantheon in Rome.

A rather original formula to balance the forces induced by the cupola, is the one created in 751 A.D. at the Seokguram temple, where a kind of counterweights or 'forearms' are used, acting at the same time as levers and wedges. As far as we can report, it is the only dome possessing such artifacts among its features (Figure 9).



Figure 9. 'Forearms' that balance the horizontal and vertical forces at Seokguram's cupola

4. Evolution of the Korean tumuli

The dome made with masses of stones had been used in Korea since ancient times to mark a sort of ceremonial mound known as tumulus. It was covered with grass as we can see in figure 10. Unlike the Atrius' Treasury these tumuli possessed no internal space but they enshrined a geomantic significance as ritual landmarks. At a later stage such landscape's Feng Shui (Pung Sui in Korean) was made apparent by the orderly carving of statues of the twelve celestial branches or oriental horoscope which measured the seasons and the times.



Figure 10. Tumulus of Hwangnam-ri. Gyeongju (Korea).

These mythical figures provided with animal heads superposed over a human warmongering appearance, were carved in the cylindrical drum of the tumulus at regular intervals with respect to the solar orientation [4] which begins, as we know, with the horse 午 (*Wu* in Chinese). The celestial horse is aligned with due South as it indicates noon (Fig. 11). The correct time and sequence order of the figures is as follows, Rat (24h) Ox (2h), Tiger (4h), Hare (6h), Dragon (8h) Serpent (10h), Horse (12h) Ram (14h), Ape (16h), Bird (18h), Dog (20h) and Boar (22h). It is interesting to outline that although the animals face exactly their cardinal orientation, the entrance and ceremonial way to the tumulus are usually deviated between 5 and 10 degrees from the astronomical north-south axis for reasons that we are still researching.

	Ox 丑	Rat 子
	Hare 卯	Tiger 寅
	Serpent 巳	Dragon 辰
	Ram 未	Horse 午
	Bird 酉	Ape 申
	Boar 亥	Dog 戌

Fig. 11. The cosmological disposition of the Twelve animals found at diverse tumuli in Korea. The hour of the rat is midnight (north) and the hour of the horse is noon (south), the order segues from right to left and top to bottom in each inclined column. Source Joseph Cabeza-Lainez



Fig. 12. The relief of the horse at a tumulus and its 3D computer reconstruction.

By a parallel to what happened in India, the tumulus that formed the stupa began to be carved on the inside to become a hypogeal temple devoted to the Buddha, usually known as chaitya. The main reason for that was a climatic one, the absolute necessity of building a protected space to perform religious ceremonies and liturgy when the excesses of the monsoon weather impeded wandering or any kind of outside practice for both monks and believers. Also to prevent the incidental killing of any insects that surged in the heat and humidity of the rainy season. In this way, the Seokguram grotto as a renowned Buddhist space, derived and evolved from the previous ritual tumuli that we have discussed beforehand [5].

5. Some mathematic and cosmic fundamentals of the architecture of Seokguram

DIN is widely known as the acronym for *Deutsches Institut für Normung* (German Institute for Standardization). This institute, based in Berlin, constitutes the national standardization body in Germany. A commonly used DIN standard is DIN 476 (equivalent to ISO 216), which defines paper sizes and has been adopted by most national institutions in the world. It can be assumed that this was developed by the Berliner engineer Dr. Walter Porstmann.

However, we could verify in our research that the universally accepted system of paper sizes might have been invented before 751 in the Unified Silla Period, but a detailed explanation of its scientific value had to wait for the Japanese civil engineer Yoneda Miyoji. In fact, in the trace of the Seokguram temple (Figures 13, 14) we find a demonstration and the answer to the origin of the paper size DIN A4, whose greater side is exactly the size of the *cha* (a unit of measurement used during the Unified Silla Period). Scholars had previously assumed that the *cha* in use at that time, was equivalent to a *kokch'ök* (303 mm), but in his survey of the Pulguksha and Seokpulsha temples, Yoneda discovered that the assumption was incorrect. Surveying each part of the temple's stone stupas, he arrived to measurements equal to 0.98 *kokch'ök*, a unit of measure that he dubbed a *tangch'ök* (Tang foot).

Indeed, if we establish:

$$\begin{aligned} a \cdot b &= 1 \\ b &= a\sqrt{2} \end{aligned} \tag{1}$$

We obtain

$$b = \sqrt{\sqrt{2}} = 1.1892 \quad (2)$$

Dividing this amount twice between two, we obtain a *tangch'ök* (Tang foot), i. e.: 297.3 mm, which DIN 476 rounded to 297 mm.

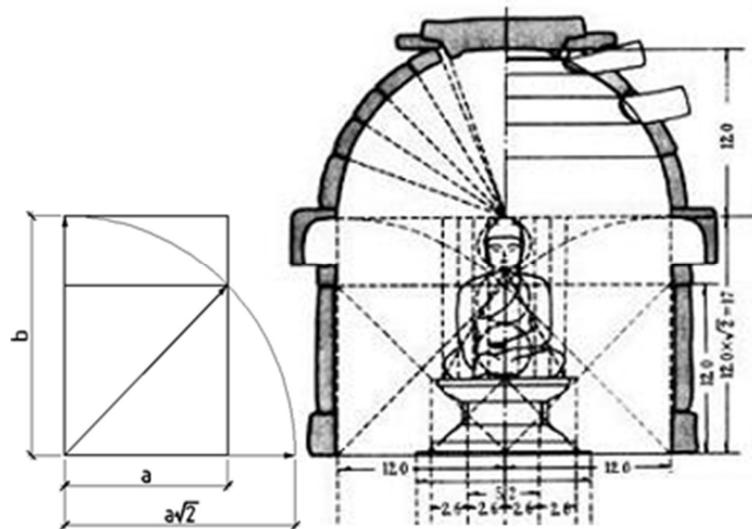


Figure 13. Use of proportions DIN 476 (1922 A. D.) before 751 A. D. by Korean architects and masons.

Yoneda's investigation showed that the grotto's rotunda has a radius of 12 *cha*. If a hexagon were inserted into the circular chamber, the entranceway, also of 12 *cha* in length would have a width equivalent to one side of the hexagon. The Buddha statue actually stands slightly back from the center of the chamber. If one imagines the entranceway, 12 *cha* across, as one side of an equilateral triangle, the front of the Buddha pedestal would touch the tip of the triangle (Figure 14).

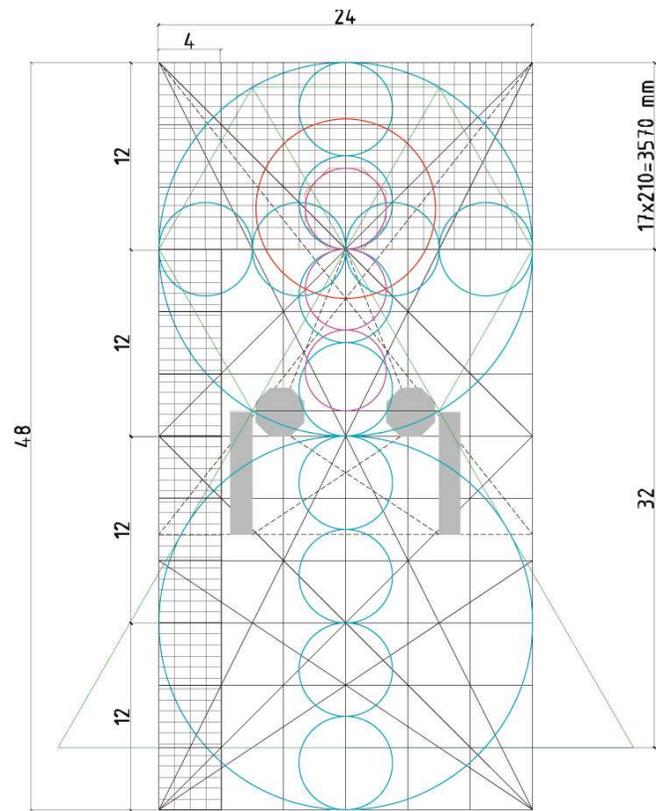


Figure 14. The unusual proportions of the Seokguram temple. Its floor plan can be exactly tessellated with 1632 DIN A4 units.

As can be clearly seen in Figure 14, the Seokguram temple surface in plan, can be tiled exactly with 1632 DIN A4 units, i. e.: $2^5 \times 3 \times 17 = 1632$. This is arguably not a coincidence.

In Western culture it is assumed that the most beautiful proportion between the sides of a rectangle is precisely $\Phi = (1 + \sqrt{5})/2 = 1.618$, although this could be rapidly changing for Korea, after the study of the Seokguram's rectangle relationship. (Figs. 15 and 16)

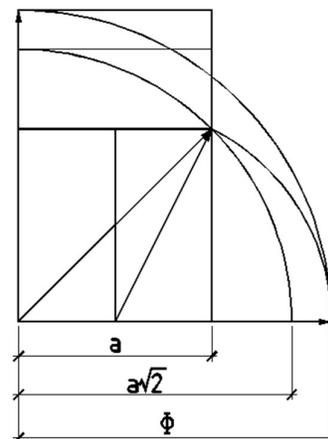


Figure 15. Difference between Seokguram temple proportion and golden ratio: one uses the diagonal of the square instead of the diagonal of half square.

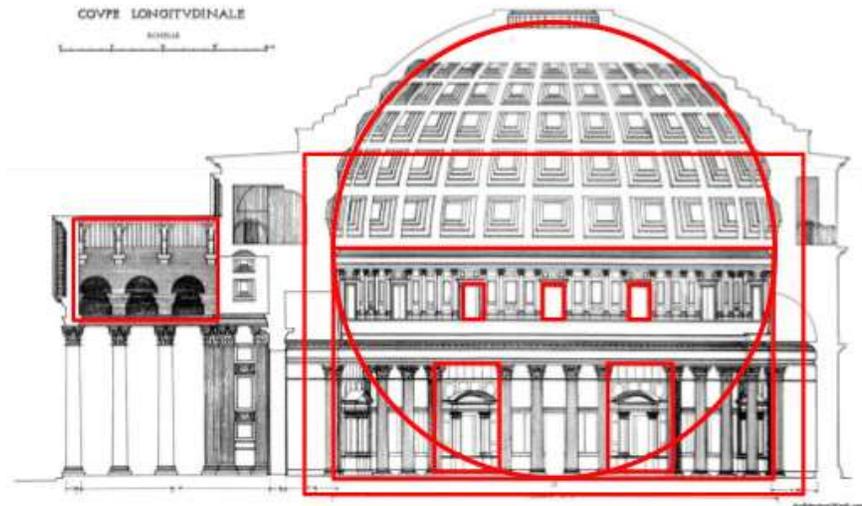


Figure 16. Roman Pantheon. Golden rectangles.

As previously discussed, the inner drum of the temple is composed of reliefs that present twelve arahats or disciples that accompany the Buddha and have clearly evolved from the twelve animals of the Daoist credence but this time to the inside of the sanctuary. A fact that reveals a clear tendency towards esoterism in Korean religion. Also the entrance is guarded by the four celestial Kings 四天王, which have dedicated temples elsewhere in Korea and Japan. Other apotropaic figures like Light and Darkness, Dragons and Giants complete the awe-inspiring scene (Fig. 17).

A whitish granite statue so polished as to resemble pure marble, inspired the pilgrims in all likelihood, to contemplate the stone become spiritual life. In this sense Seokguram becomes one of these quintessential rarefied spaces, like Ryoanji or Ajanta, in which the combined effects of shimmering materials, subdued light and tenuous shades, beckon and manifest the radiance of the supernatural. (Fig. 19).

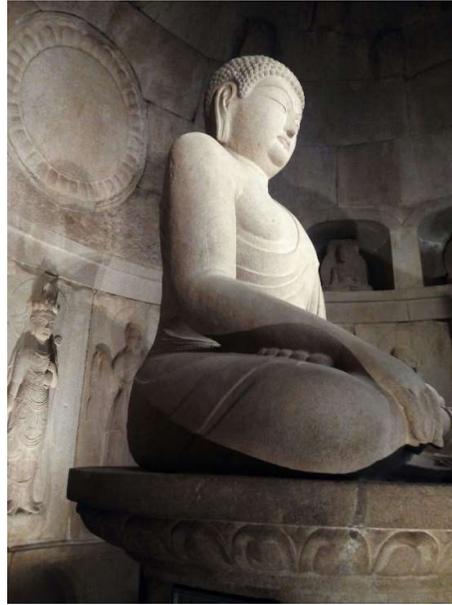


Figure 19. A view of inner tranquility at the central chamber of Seokguram

6. Structural Behavior Analysis

6.1. Models

At this point in the study, the authors faced a relevant decision. On receiving precise information on the current interior dimensions of the grotto (obtained with high-tech equipment from the company Wipco, 3D Scan & Solutions), that is to say, after the dismantling and subsequent piece-by-piece mounting of the temple during Japanese occupation, including later reforms of the 1960's and beyond. These data, although precise, lacked any kind of coincidence with the supposed original plan of the project. In other words, they were incoherent data in which almost no element was perfectly circular or straight or spherical and there was no available module to reconstruct the whole temple with such meagre figures.

The final agreement was to return to the assumed plan of the original architect of the temple and respect the trace shown in Figure 14, with which, a model can practically be constructed recalling the numbers 2, 6, 8, 10, 12, 17, 20 and 297.

We have performed three approximations to the problem, namely: to study the structural behavior of the dome without considering the 'forearms' to begin with. Then we will place the 'forearms' and compare the results in efforts and displacements and finally, we would analyze a third hypothesis with 'forearms' and considering the existence of a hypothetical window in the front of the dome to determine if its presence compromises the structural security of the building.

Different views of the initial model are shown in Figures 20-26 corresponding to the common theories on the early proposal for the original shrine.

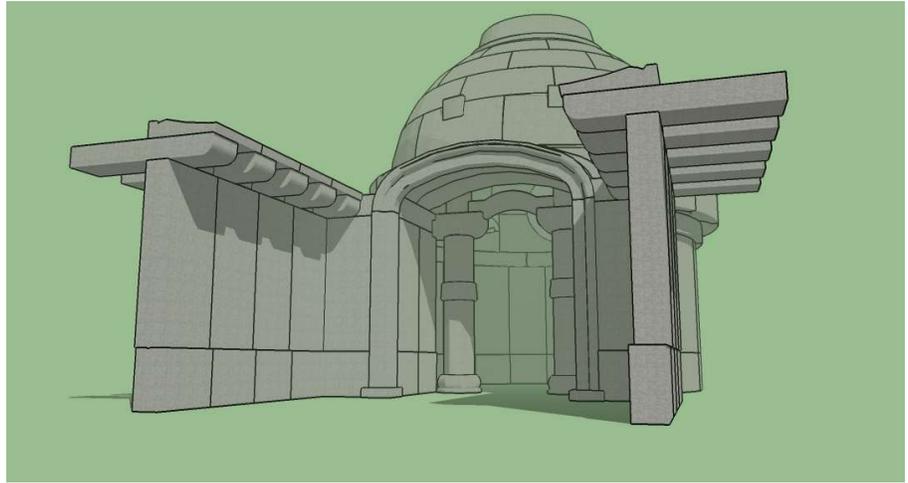


Figure 20. Model for the first structural study. It is assumed the absence of 'forearms'.

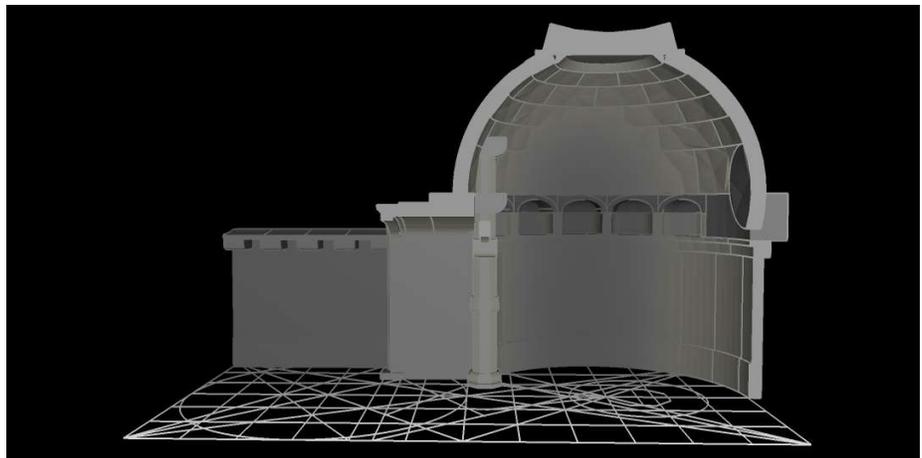


Figure 21. Section of the initial model.

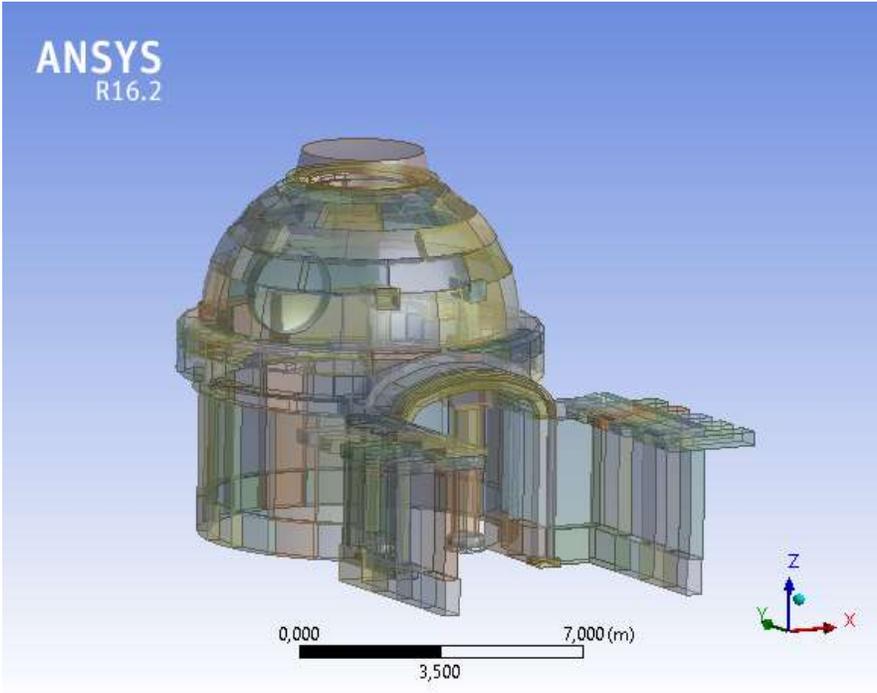


Figure 22. Process of importing the model to ANSYS.

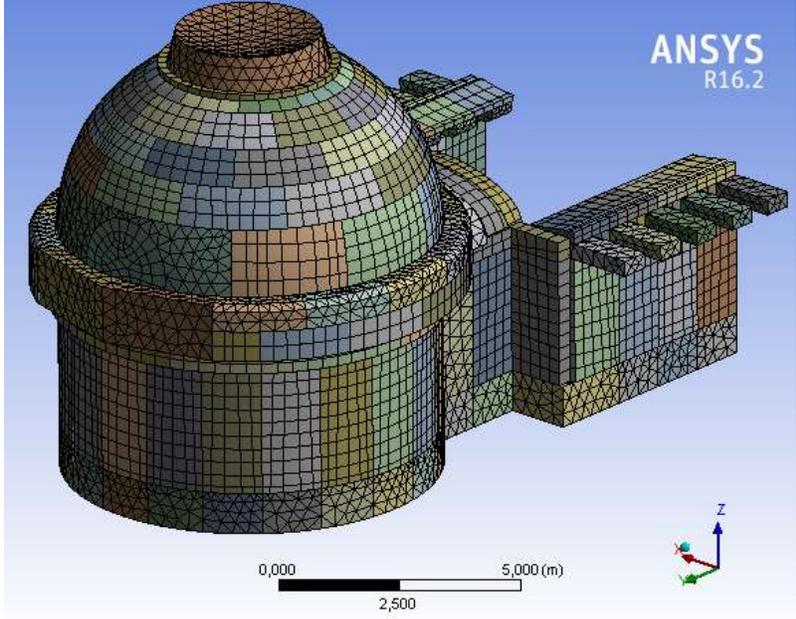


Figure 23. First test of meshing the structure.

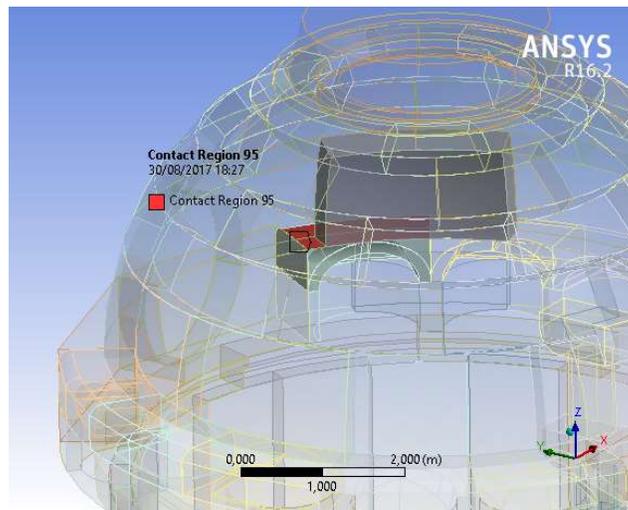


Figure 24. Rendering of one of the granite-granite contact regions. Coefficient of static friction $\mu_s = 0.6$.

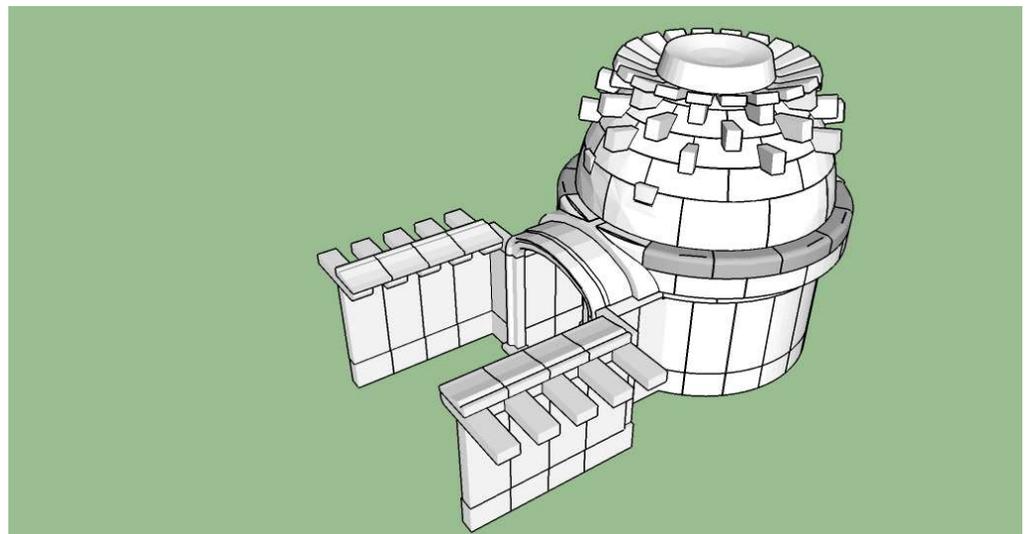


Figure 25. Visualization of the model as it currently is

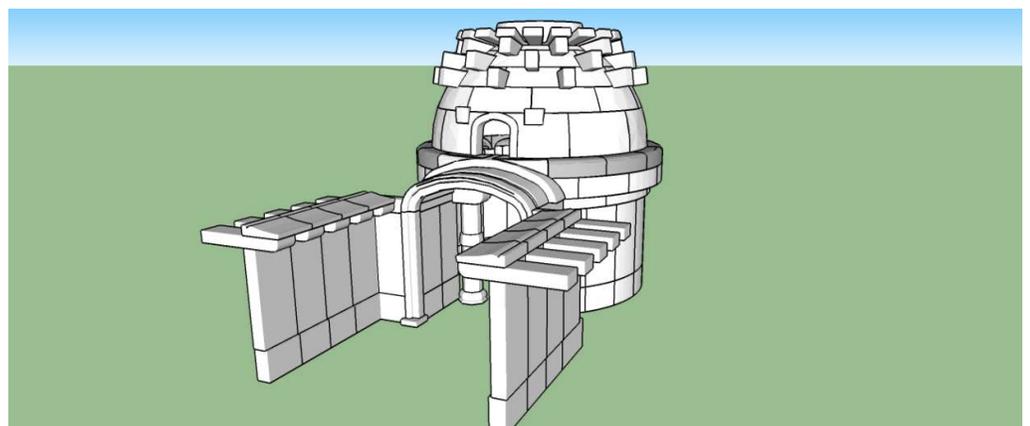


Figure 26. The model with a supposed opening in the front side to induce light effects

The results, although not totally conclusive, due to some uncertainties in the nature of the material which was clearly different 1200 years ago to what we now possess and can analyze with modern techniques, determine the extreme importance of the forearms to sustain the dome in place and without damage under the thrust of the mountain above and other unexpected forces that surely came to pass with the course of the centuries. At the same time, it has been proven that a reduced eastern aperture in the base of the dome would have been compatible with the overall stability of the construction. Thus, the authors have further investigated with their own specific software [5] the daylight distribution inside the grotto and additional effects as depicted in figure 27. It is convenient to remark that direct linkage of the statue and its chamber with the sun and the moon should have been demanded by the ritual cosmology prevailing on the epoch.

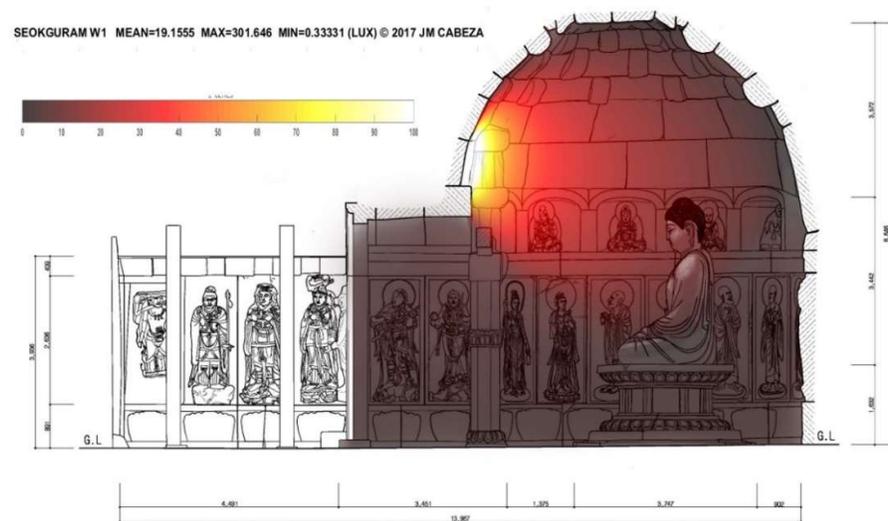


Figure 27. Sectional view of the grotto including simulation of lighting levels by a surmised circular opening.

7. Recent and future architectural proposals in the wake of Buddhist chaitya

We have surfaced in the previous discussion how the ancient builders of Korea, possessed unforeseen knowledge on complicated techniques for the 8th century A.D. In particular, the evolution from the ceremonial tumuli to the Buddhist chaitya must have been a strenuous process entailing decades of planning and craftsmanship.

At the same time, thinking in retrospect, the lessons that we have learned from Seokguram and other derived enclaves are ones to recognize the path of creation for a more serene architecture, which brings a much desired between buildings and nature. In moments of turmoil like the ones we are facing, we should carefully design spaces for hope in the modelling of our common future where, sustained by unflinching love and kindness, we could hone our insight.

There is a demand in the architectural profession to identify the entangled and sometimes misleading, forking threads to accrue the significance of our buildings.

We feel that it is imperative to ignite the signals that lead us into a future, possible and humanized, for once.

Two examples in direct relationship to this research are given, the first one is Tadao Ando's recent project for renovation of a landscape cemetery in the outskirts of Sapporo (Japan).

To conceal an excessively imposing Buddha statue that the visitors apparently disfavored, Ando, inspired by the tradition of Buddhist grottoes, executed an artificial earthen shell that generates an open dome shrine. The renovated complex is provided with ceremonial access and other facilities intended for religious services (Figs. 28 and 29). The outer and gentle slopes of the artificial mound are immersed in a mantle of lavender plants from whence the hallowed head arises like "the sun on the waves of a purple sea".



Figure 28. Tadao Ando, Hill of the Buddha (2017) with Buddha's Head emerging from a sea of lavender shrubs at Makomanai Takino Cemetery (Sapporo).



Figure 29. Tadao Ando's Buddha Hill at Makomanai Takino Cemetery (Sapporo).

The second example is introduced by the authors themselves, it consists in a proposal of monument to the Holy Buddha at Gyeongju the same city of the Seokguram.

The authors were inspired by another stone construction without mortar from the oriental roman outpost of Hierapolis (Turkey) (Fig. 30). The semicircular apse is a typical feature of many ancient roman buildings. We conceived that it was possible to complete the circle by means of conoid-like shapes, whose resistance and simplicity has been previously demonstrated in other articles [8]



Figure 30. Three-lobed apse of stone at the Roman Baths in Hierapolis (Turkey)

The monument presents a curved frame organized by tiers in which every arch finds its support naturally on the keystone of the following arch underneath (Fig. 31). Such structure in the symmetric shape of a lotus, which is the flower favored by Buddhism, would be constructed mainly with masonry and special reinforcements at the joints. (Fig. 32) The concept of architecture as an ever-opening flower as exposed in the Lotus Sutra (*saddharma pundarika sutra*) is the recognizable inspiration for our design.

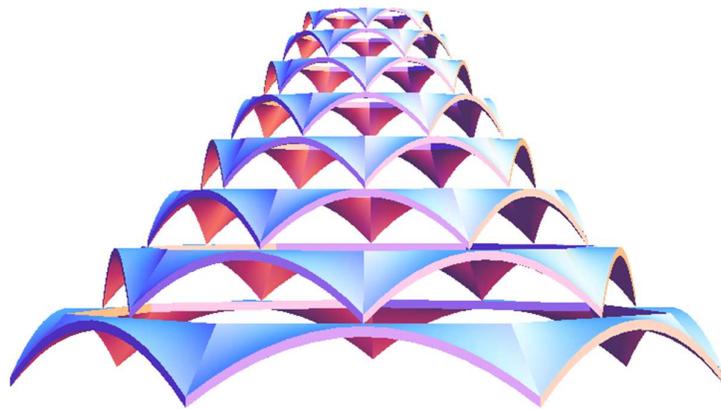


Figure 31. Proposal of an eight tiered Buddhist monument. Elevation. Drawing courtesy of Encarnación Abajo.

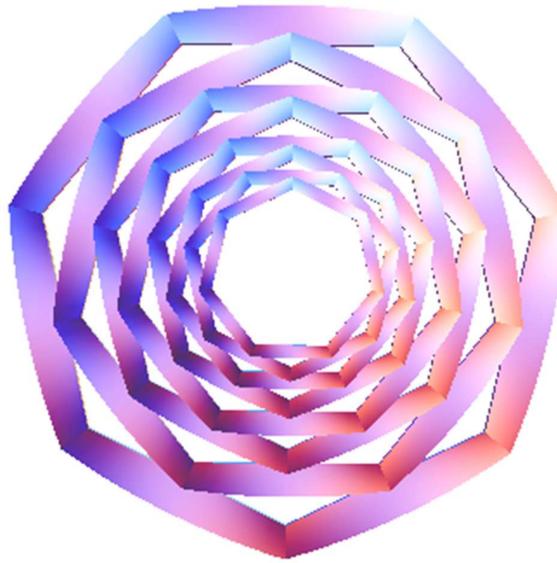


Figure 32. Plan of the proposed Buddhist monument, notice the opening lotus shape. By courtesy of Encarnación Abajo.

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Appendix A

Fractional Hooke's law, Continuum Damage Mechanics and the Weibull's distribution for granite's behaviour

In the ensuing section, we would describe how we obtained a universal equation to model the stress-strain behavior of any material and apply it to the granite. These values are necessary in order to introduce the parameters required by the software ANSYS to proceed with an analysis in accordance to such an important building.

Fractional Hooke's law

Beginning with Cauchy's formula to calculate the iterated integral:

$${}_0D_x^{-n}f(x) = \iint \dots \int f(x) dx^n = \frac{1}{(n-1)!} \int_0^x f(t)(x-t)^{n-1} dt \quad (3)$$

and from the generalization of the $n!$ notion introduced by Euler through the Gamma function:

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt \Rightarrow \Gamma(z) = (z-1)! \quad (4)$$

where z represents any complex number with $\text{Re}(z) > 0$, Liouville [9] obtained the integral that bears his name, generalizing the natural number n to any complex number with real part greater than zero, thus yielding the first formal expression for a fractional order integral –noninteger–:

$${}_0D_x^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t) dt}{(x-t)^{1-\alpha}} \quad (5)$$

This approach offers great possibilities of generalization with respect to the classical laws of Hooke and Newton about the behavior of materials. In fact, in 1947, Blair [10,11], highlighting that stress is proportional to the zero ordered derivative of strain in solids and also related to the first derivative in fluids, proposed intermediate materials, provided with quasi-properties, for which stress would be proportional to the intermediate ordered derivative –non-integer– of strain, that is:

Hooke:

$$\sigma(t) = E \frac{d^0 \varepsilon(t)}{dt^0} \quad (6)$$

Blair:

$$\sigma(t) = K_0 D_t^\alpha \varepsilon(t) \quad (7)$$

Newton:

$$\sigma(t) = \eta \frac{d\varepsilon(t)}{dt} \quad (8)$$

where K and α would be dependent constants of the material ($0 < \alpha < 1$).

Therefore, since the Hooke's law can be expressed as:

$$\frac{d\sigma}{d\varepsilon} = E \quad (9)$$

In view of the approaches of the differintegral calculus, we could generalize the derivation order of this differential equation and propose a Hooke's law extended to non integer derivation orders:

$${}_0D_\varepsilon^\alpha \sigma = K \quad (10)$$

Eq. 10 gives a fractional ordered differential equation of immediate solution:

$${}_0D_\varepsilon^{-\alpha} {}_0D_\varepsilon^\alpha \sigma = {}_0D_\varepsilon^{-\alpha} K \Rightarrow \sigma = {}_0D_\varepsilon^{-\alpha} K \quad (11)$$

which, citing the definition of Liouville's fractional integral –Eq. (5)– it results in:

$$\sigma = \frac{1}{\Gamma(\alpha)} \int_0^\varepsilon \frac{K dt}{(\varepsilon-t)^{1-\alpha}} \quad (12)$$

from which, integrating, we obtain,

$$\sigma = \frac{K}{\alpha \Gamma(\alpha)} \varepsilon^\alpha \quad (13)$$

Therefore, we will use Eq. (13) in order to represent the behavior of the material points when the damage process, inherent in deformation, has not yet occurred.

Continuum Damage Mechanics

Originally proposed by Kachanov [12] and later modified by Rabotnov [13], the Continuum Damage Mechanics –CDM– has been widely accepted to simulate the complex constitutive behavior of many materials used in engineering. Specially, the models based on an internal variable of damage represented by a scalar function, are characterized by their implementation simplicity and versatility. This variable of damage reflects the deterioration level of the material as it is strained and it turns real stresses into effective stresses, so a general equation that relates stresses with strains may be expressed as:

$$\sigma = \Psi(\varepsilon)[1 - \omega(\varepsilon)] \quad (14)$$

where $\Psi(\varepsilon)$ represents the response of the undamaged material and $\omega(\varepsilon)$ is a scalar function of damage that ranges between 0 –when the material has not yet been stressed– and 1 –when such material collapses–. The proposal we present in this study assumes that function corresponds to the Weibull's probability distribution function.

Weibull's distribution

The Weibull's theory of resistance of materials [14,15] may be related to the hypotheses of Griffith [16] and Smekal [17]. In fact, assuming the existence of numerous weak zones within a material, either as microcracks –according to Griffith–, as dislocations in the atomic mesh –following Taylor [18,19,20]– or as Stone-Wales defects on Carbon NanoTubes –SWCNT–according to Ebbensen and Takada [21] and Miyamoto et al. [22], resistance would decrease from theoretical values. Assuming also that all those weak zones are of such nature that they will break as soon as they reach a

volume subjected to σ stress and supposing that there are n weak zones per volume unit and that the σ stress is concentrated within a small volume dv , then the probability of breakage for the element of volume will be $dP = ndv$. If, instead of one, there are N elements of volume dv and the probability of breakage is P , then, the probability of survival will be $S=1-P=(1-dP)^N=(1-ndv)^N$, thus:

$$P = 1 - (ndv)^N \quad (15)$$

The total volume subjected to stress will then be $V=Ndv$ and we will receive

$$P = 1 - \left(1 - \frac{nV}{N}\right)^N = 1 - \left(1 - \frac{nV}{N}\right)^{\frac{N}{nV}nV} \quad (16)$$

However, if N increases as dv decreases indefinitely until V becomes constant, we obtain

$$P = 1 - \lim_{\frac{N}{nV} \rightarrow \infty} \left(1 - \frac{nV}{N}\right)^{\frac{N}{nV}nV} = 1 - e^{-n} \quad (17)$$

At this point, Weibull introduces his most famous proposal: if we choose $nV=f\omega(\varepsilon)dv$ as

$$\omega(\varepsilon) = \left(\frac{\varepsilon - \varepsilon_a}{\varepsilon_0}\right)^m \quad (18)$$

then, as Weibull heralded, this function allows the data to be the ones that give us the distribution function. [23, 24]

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