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DOUBLE HILBERT EXPONENTIAL SUMS ALONG POLYNOMIALS (PREPRINT VERSION)

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1. INTRODUCTION

Abstract : Double Hilbert Exponential Sums along polynomials is the Fourier multiplier of Discrete Double Hilbert Transform along polynomials. In this paper, we prove that Double Hilbert Exponential Sums along polynomials that satisfy a certain condition is uniformly bounded function except very small area.

Keywords: Discrete; Double Hilbert transform; Circle method; exponential sums; discrete double Hilbert transform; discrete double exponential sums.

The 1-parameter hilbert transform along polynomials has been studied (E. M. Stein and S. Wainger [7], F. Ricci, E. M. Stein [8], [9], [10]). For L^p theory of those singular integrals has also been studied quite well (M. Christ, A. Nagel, E. M. Stein and S. Wainger [18], M. Folch-Gabayet and James Wright [17]). The 2-parameter Hilbert transform along polynomials were introduced in [6],[16]. And the monomial case has been studied[10]. The necessary and sufficient condition for $L^p\left(\mathbb{R}^3\right)$ was obtained in Anthony Carbery, Stephen Wainger, James Wright [4]. Sanjay Patel [19] proved the necessary and sufficient condition for L^p boundedness of global case

$$Hf(x,y,z) = p\cdot v\cdot \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-s,y-t,z-P(s,t))\frac{\mathrm{d}s\ \mathrm{d}t}{st}.$$

The multiple parameter cases has been studied [15]. Similar question has been studied as discrete version. For this, the Circle method plays a great role(Hardy and Ramanujan [11]). The boundedness of the maximal operator

$$M(f)(n) = \sup_{N\in[1,\infty)} \frac{1}{N}\sum_{1\le m\le N} |f(n-P(m))|$$

($d_1 = d_2 = 1$, and P is a polynomial with integer coefficients) has been studied (J. Bourgain [14], [12], [13]). Alexandru D .lonescu and Stephen

Wainger([1]) proved the L^p boundedness of discrete singular radon transform

$$T(f)(x) = \sum_{n \in \mathbb{Z}^{d_1} \setminus \{0\}} f(x - P(n))K(n).$$

But there are no results about discrete double Hilbert transform before our paper. We will consider

$$H_{discrete} f(x, y, z) = \sum_{(s,t) \in \mathbb{Z}^2, s,t \neq 0} f(x - s, y - t, z - p(s,t)) \frac{1}{st}$$

and

$$\sum_{(s,t) \in \mathbb{Z}^2, s,t \neq 0} e^{-2\pi i \cdot (s \cdot \xi_{(1,0)} + t \cdot \xi_{(0,1)} + p(s,t) \cdot \gamma)} \cdot \frac{1}{st}.$$

Question: What is the necessary and sufficient condition of polynomials for

$$\sum_{(s,t) \in \mathbb{Z}^2, s,t \neq 0} e^{-2\pi i \cdot (s \cdot \xi_{(1,0)} + t \cdot \xi_{(0,1)} + p(s,t) \cdot \gamma)} \cdot \frac{1}{st} \leq C?$$

(The constant C may depend only on the polynomial and d).
Since $\frac{1}{s \cdot t}$ is not good funtion like Calderón-Zygmund kernel and s and t affect each other, we can't apply the circle method to 2-parameter in this case. So we should probably consider one variable fraction $\frac{1}{s}$ to apply the circle method. But if we do this, infinitely many changing variables related to s or t appear in the functions of the continuous version linked to the major arc. This is why the Question may be difficult. Our main result is double Hilbert exponential sums along polynomials that satisfy a certain condition is uniformly bounded function except very small area. We hope that one day we will know the necessary and sufficient conditions perfectly.

We first introduce continuous version of double Hilbert transform along polynomials. For $f \in \mathcal{S}$ (i.e., a Schwartz class function), we define

$$H_{loc} f(x, y, z) = p \cdot v \cdot \int_{-1}^{1} \int_{-1}^{1} f(x - s, y - t, z - P(s,t)) \frac{\mathrm{d}s \ \mathrm{d}t}{st}$$

where $P(s,t)$ is a real-valued polynomial in s and t. Carbery, Wainger and Wright determined the necessary and sufficient condition on the polynomial P so that H_{loc} is L^p bounded for $1 < p < \infty$. We state their result.
Let $P(s,t) = \sum_{(m,n) \in \Lambda} a_{m,n} s^m t^n$ be a polynomial with real coefficients such that $P(0,0) = 0, \nabla P(0,0) = 0$ and where Λ is indexing the set of lattice points $(m,n) \in \mathbf{N}^2$ such that $a_{m,n} \neq 0$. For each (m,n) in Λ, we let

$$Q_{m,n} = \left\{ (x,y) \in \mathbb{R}^2 \mid x \geq m \text{ and } y \geq n \right\},$$

Set $Q = \bigcup_{(m,n) \in \Lambda} Q_{m,n}$. Then the Newton diagram Π of P is the smallest (closed) convex set containing Q. Π is an unbounded polygon with a finite number of corners. We denote the set of corners by D. Then $D \subset \Lambda$.

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THEOREM 1.1. For any $p, 1 < p < \infty$,

$$\|H_{loc} f\|_{L^p} \leq A_p \|f\|_{L^p}$$

iff for each (m, n) that is a corner point of the Newton diagram corresponding to P, at least one of m and n is even.(Anthony Carbery, Stephen Wainger, James Wright [4].)

Sanjay Patel determined the necessary and sufficient condition on P so that the (global) double Hilbert transform defined by

$$Hf(x, y, z) = p \cdot v \cdot \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x - s, y - t, z - P(s, t)) \frac{\mathrm{d}s\mathrm{d}t}{st}$$

is bounded on $L^p, 1 < p < \infty$. The operator is defined for $f \in \mathcal{S}$ by integrating where $\epsilon' \leq |s| \leq R', \epsilon \leq |t| \leq R$, and then, taking the limits as $\epsilon, \epsilon' \to 0$ and $R, R' \to +\infty$. Let $P(s,t) = \sum_{(m,n)\in\Lambda} a_{m,n} s^m t^n$ be a polynomial with real coefficients such that $P(0,0) = 0, \nabla P(0,0) = 0$ and where Λ is indexing the set of lattice points $(m, n) \in \mathbf{Z}^2$ such that $a_{m,n} \neq 0$. Let $\mathcal{C}$ be the closed convex hull of Λ in $\mathbf{R}^2$ and $\mathcal{D} = \{(m,n) \in \Lambda : (m,n)$ is a corner point (vertex) of $\mathcal{C}\}$.

THEOREM 1.2. For any $p, 1 < p < \infty$,

$$\|Hf\|_{L^P} \leq A_p \|f\|_{L^P}$$

if and only if for each $(m,n) \in \mathcal{D}$, at least one of m and n is even and furthermore, if any (extended) edge of $\mathcal{C}$ passes through the origin (there are at most two such edges), then every point of Λ on that edge must have at least one even coordinate.(Sanjay Patel [19])

Assume that $d_1 \geq 1$ is an integer and $K \in C^1\left(\mathbb{R}^{d_1} \backslash \{0\}\right)$ satisfies the differential inequalities $|x|^{d_1}|K(x)| + |x|^{d_1+1}|\nabla K(x)| \leq 1$ for any $x \in \mathbb{R}^{d_1}, |x| \geq 1$, and the cancellation condition

$$\left| \int_{|x|\in[1,\lambda]} K(x) dx \right| \leq 1$$

for any $\lambda \geq 1$ (i.e., K is a Calderón-Zygmund kernel on $\mathbb{R}^{d_1}$ away from 0). Let $P = (P_1, \ldots, P_{d_2}) : \mathbb{R}^{d_1} \to \mathbb{R}^{d_2}$ denote a polynomial of degree $A \geq 1$ with real coefficients. We define the (translation invariant) discrete singular Radon transform operator T by the formula

$$T(f)(x) = \sum_{n \in \mathbb{Z}^{d_1} \backslash \{0\}} f(x - P(n)) K(n)$$

for any Schwartz function $f : \mathbb{R}^{d_2} \to \mathbb{C}$. Ionescu and Wainger proved following theorem.

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THEOREM 1.3. The operator T extends to a bounded operator on $L^p\left(\mathbb{R}^{d_2}\right), p \in (1,\infty)$, with $\|T(f)\|_{L^P\left(\mathbb{R}^{d_2}\right)} \leq C_p\|f\|_{L^p\left(\mathbb{R}^{d_2}\right)}$. The constant C_p may depend only on the exponent p, the dimension d_1, and the degree A. (Alexandru D .lonescu and Stephen Wainger [1])

We introduce some definitions. Assume $d \geq 1$ is an integer. For any $\mu \geq 1$, let $Z_\mu = \mathbb{Z} \cap [1,\mu]$. If $a = (a_1,\ldots,a_d) \in \mathbb{Z}^d$ is a vector and $q \geq 1$ is an integer, then we denote by (a,q) the greatest common divisor of a and q, i.e., the largest integer $q' \geq 1$ that divides q and all the components $a_1,\ldots,a_d$. Clearly, any vector in $\mathbb{Q}^d$ has a unique representation in the form a/q, with $q \in \{1,2,\ldots\}, a \in \mathbb{Z}^d$, and $(a,q) = 1$; such a vector a/q will be called an irreducible d -fraction. We also let $P_q = \{a \in \mathbb{Z}^d : (a,q) = 1, 0 \leq a_i \leq q\}$ and let $|(m,n)|$ denote $|(m,n)| = m+n$. We set $p(s,t) = \sum_{(m,n)\in\Lambda} a_{m,n}s^m t^n$ be a polynomial with real coefficients such that $p(0,0) = 0, \nabla p(0,0) = 0$. We let $\xi = (\xi_{(m,n)}) \in \mathbb{R}^d$ $((m,n) \in \{(0,1),(1,0)\} \cup \Lambda$, indeed $d = |\Lambda|+2)$. We also let $P : \mathbb{R}^2 \to \mathbb{R}^d, [P(x)]_\alpha = x^\alpha (x = (s,t), \alpha = (m,n) \in \{(0,1),(1,0)\} \cup \Lambda)$.
Now we define the discrete double Hilbert transform operator $H_{discrete}$ by following formula

$$H_{discrete}f(x,y,z) = \sum_{(s,t)\in\mathbb{Z}^2, s,t\neq 0} f(x-s,y-t,z-p(s,t))\frac{1}{s\cdot t}.$$

We also define the discrete double hilbert exponential sum :

$$\mu(\xi_{(1,0)},\xi_{(0,1)},\gamma) = \sum_{(s,t)\in\mathbb{Z}^2, s,t\neq 0} e^{-2\pi i\cdot P(s,t)\cdot\xi}\cdot\frac{1}{s}\cdot\frac{1}{t}, \xi \in \mathbb{R}^d$$

which is the Fourier multiplier of $H_{discrete}(f) : \hat{H}_{discrete}(\xi_{(1,0)},\xi_{(0,1)},\gamma)$ if we let $\xi_{(m,n)} = a_{m,n}\gamma$ for all $(m,n) \in \Lambda$. From now, we let $\xi_{(m,n)} = a_{m,n}\gamma$. Chronologically, there have been important results such that

$$\int_{-\infty}^{\infty} e^{-2\pi i\cdot(x\cdot\xi_1 + p(x)\xi_2)}\cdot\frac{1}{x} \leq C,$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi i\cdot(x\cdot\xi_1 + y\cdot\xi_2 + p(x,y)\cdot\xi_3)}\cdot\frac{1}{x}\cdot\frac{1}{y} \leq C$$

(with conditions mentioned in Theorem 1.2),

$$\sum_{s\in\mathbb{Z}\setminus\{0\}} e^{-2\pi i\cdot P(s)\cdot\xi}\cdot\frac{1}{s} \leq C, \quad \xi \in \mathbb{R}^d.$$

(by Theorem 1.3)

But followung inequality has not yet been solved.

$$\mu(\xi_{(1,0)},\xi_{(0,1)},\gamma) = \sum_{(s,t)\in\mathbb{Z}^2, s,t\neq 0} e^{-2\pi i\cdot P(s,t)\cdot\xi}\cdot\frac{1}{s}\cdot\frac{1}{t} \leq C, \quad \xi \in \mathbb{R}^d$$

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Our main result(Theorem 1.4) is about the last inequality.
We define funtion $\phi(\gamma)$: For $q \geq 100^{100d/\delta}$, if there is a/q ($(a,q)=1$) satisfying $|\gamma - a/q| \leq min\{1/a_{m,n}, 1\} \cdot 2^{-q^{\delta/2d}}$, then $\phi(\gamma)=0$. Otherwise, $\phi(\gamma)=1$. (δ will be mentioned in Lemma 1.7.)
Main Theorem If all $(m_{\geq 1}, n_{\geq 1}) \in \Lambda$ are not on one line passing through the origin, $\mu(\xi_{(1,0)}, \xi_{(0,1)}, \gamma)\phi(\gamma)$ is uniformly bounded by constant C. The constant C may depend only on the polynomial and the d.

Remark(1) If γ is not very close to a/q, $\mu(\xi_{(1,0)}, \xi_{(0,1)}, \gamma)$ is uniformly bounded by constant C. In $[0,1]^d$, the very small area may be less than $2^{-100^{49}}$.
(2) As we can see in the proof of the Main Theorem, if $\phi(\gamma)=0$, then $|\mu(\xi_{(1,0)}, \xi_{(0,1)}, \gamma)| \lessapprox \sum_{a,q} \log_2(|\gamma - a/q|)\chi(max\{a_{m,n}, 1\} \cdot 2^{q^{\delta/2d}}(\gamma - a/q))$. So it is easy to show $\left\|\mu(\xi_{(1,0)}, \xi_{(0,1)}, \gamma)\right\|_{L^p([0,1]^3)} \leq O(1)$.

We now introduce following Lemmas which are used many times in this paper.

Lemma 1.5. For fixed $\xi \in \mathbb{R}$ and increasing sequence q_n ($q_n \in \mathbb{N}$), If there are a_n satisfying $|\xi - a_n/q_n| \leq 1/(10q_n^2)$ for all n, then the sequence is lacunary series and $q_{n+1}/q_n \geq 2$.

Lemma 1.6. (Diriclet's principle) For any $\Lambda \geq 1$ and $\xi \in \mathbb{R}$, there are $q \in \mathbb{Z}_\Lambda$ and $a \in \mathbb{Z}$ with $(a,q)=1$, with the property that $|\xi - a/q| \leq 1/q\Lambda$.

Lemma 1.7. For any $R \geq 1$, let $B_R = \left\{x \in \mathbb{R}^{d_1} : |x_l| < R, l = 1, \ldots, d_1\right\}$. Assume that $k : B_R \to \mathbb{C}$ is a C^1 function with the property that

$$|k(x)| + R \cdot |\nabla k(x)| \leq 1$$

for any $x \in B_R$. Assume that $\epsilon \in (0, 1/10)$ is fixed and $\xi \in \mathbb{R}^d$ has the property that for some $\alpha, 1 \leq |\alpha| \leq A$, there are integers a and q, with $(a,q)=1, q \in \left[R^\epsilon, R^{|\alpha|-\epsilon}\right]$, and $|\xi_\alpha - a/q| \leq 1/q^2$. Then

$$\left|\sum_{n \in \Omega \cap \mathbb{Z}^{d_1}} e^{-2\pi i P(n)\cdot\xi} k(n)\right| \leq CR^{d_1 - \delta}, \delta > 0$$

for any open, convex set $\Omega \subset B_R$. The constants C and δ may depend only on d_1, A, and ϵ but not on R, ξ, or the irreducible fraction a/q. [6, Proposition 3]

We introduce Guass sum. For any $q \in \{1, 2, \ldots\}$ and $a \in \mathbb{Z}^d$ with $(a,q) = 1$, let

$$S(a/q) = \frac{1}{q^{d_1}} \sum_{n \in [Z_q]^{d_1}} e^{-2\pi i P(n)\cdot a/q}.$$

Lemma 1.8. If (a, q) = 1, $|S(a/q)| \leq C \cdot q^{-\delta}$ (for some constant $\delta = \delta(d) > 0$).

Proof of Lemma 1.8. Let $a = (a_\alpha)$, and assume that $a_\alpha/q = a'_\alpha/q'_\alpha$, where a'_α/q'_α is an irreducible. Since $(a, q) = 1$, there are d_α, q'_α satisfying $q = d_\alpha q'_\alpha, (d_1, \ldots, d_\alpha, \ldots) = 1$. Since q has d_α and $(d_1, \ldots, d_\alpha, \ldots) = 1, \Pi d_\alpha$ can't divide all d_α of q^d. So $\Pi_\alpha q'_\alpha = q^d/\Pi d_\alpha \geq q$. First, if $q'_\alpha \geq q^{1/10d^2}$ for some index α with $|\alpha| \geq 2$, we shall know

$$\left| \sum_{n \in [z_q]^{d_1}} e^{-2\pi i P(n) \cdot a/q} \right| \leq \mathrm{C} 2^{d_1} \cdot q^{d_1 - \delta}, \delta > 0$$

by lemma 1.7 with R $= 2q, k = 1$, and $\epsilon = 1/20d^2$ which means $q'_\alpha \in \left[R^\epsilon, R^{|\alpha| - \epsilon}\right]$. Second, if $q'_\alpha \leq q^{1/10d^2}$ for any α with $|\alpha| \geq 2$, then $q'_\alpha \geq 1/q^{2d}$ for some α_0 with $|\alpha_0| = 1$. In this case, by summing first the variable corresponding to the index α_0 with summation by parts, we have S(a/q) = 0. □

In chapter 2, we make asymtotic formula for the multiplier of 1-parameter discrete Hilbert transform which has variables related to t terms through the circle method. In chapter 3, we will show that double Hilbert exponential sum along polynomials($\mu(\xi)$) which is the Fourier multiplier of the operator is uniformly bounded by constants with the condition mentioned in Theorem 1.4. For this, we apply the circle method (Propsotion 2.1) to 1-parameter exponential sums and the other exponential sums related to the other variable. And we study the property of $a/(qt^n)$ in Lemma 3.1 and use this to reduce the complexity created by each variables. We also use combinatorial thinking related to $(j, k) \in \mathbb{N}^2$ plane. And we use geometric property of some inequalities about $(m, n) \in \Lambda$ for problem area in $(j, k) \in \mathbb{N}^2$ plane.

2. Asymtotic formula with Circle method.

In this section, We make asymtotic formula for the multiplier of 1-parameter discrete Hilbert transform(s valuable) which is according to ξ with t terms by circle method. Our method is similar to the method of Alexandru D. Ionescu, Stephen Wainger [1] and J. Bourgain [14].

Let $N(m, n)$ denote $N(m, n) = n$ and $M(m, n)$ denote $M(m, n) = m$. Set $\Lambda_{m \geq 1} = \{(m, n) \in \Lambda | m \geq 1\}$ and $\Lambda_{n \geq 1} = \{(m, n) \in \Lambda | n \geq 1\}$. Then we let $P_N(t) \otimes \xi$ denote $P_N(t) \otimes \xi = (t^{N(m,n)} \cdot \xi_{(m,n)}), (m, n) \in \{(1, 0)\} \cup \Lambda_{n \geq 1}$ and let $P_M(s) \otimes \xi$ denote $P_M(s) \otimes \xi = (s^{M(m,n)} \cdot \xi_{(m,n)}), (m, n) \in \{(0, 1)\} \cup \Lambda_{m \geq 1}$. For example, if $p(s, t) = s^{10} + t^{12} + s^2 t^2 + s^4 t^4 + s^6 t^8$, then $P_N(t) \otimes \xi = (\xi_{(1,0)}, t^2 \xi_{(2,2)}, t^4 \xi_{(4,4)}, t^8 \xi_{(6,8)})$ and $P_M(s) \otimes \xi = (\xi_{(0,1)}, s^2 \xi_{(2,2)}, s^4 \xi_{(4,4)}, s^6 \xi_{(6,8)})$.

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We will first consider the case when if (m_1, n_1) and $(m_2, n_2) \in \Lambda$, $M(m_1, n_1) \neq M(m_2, n_2)$ and $N(m_1, n_1) \neq N(m_2, n_2)$ in the chapter 2. The other cases is mentioned in the chapter 3.

We begin by choosing an odd C^∞-function $\psi(s)$, defined on the real line, nonnegative for $s \geq 0$, and supported in $1/2 \leq |s| \leq 2$ such that

$$\sum_{p=-\infty}^{\infty} 2^j \psi\left(2^j s\right) = \frac{1}{s}.$$

We also define

$$\mu_j(P_N(t) \otimes \xi) = \sum_{s \in \mathbb{Z}} 2^{-j} \psi(2^{-j} s) e^{-2\pi i (s\xi_{(1,0)} + \sum_{(m,n)\in\Lambda, n\geq 1} s^m (t^n \xi_{(m,n)}))}$$

for each t and j,
and

$$\mu_k(P_M(s) \otimes \xi) = \sum_{t \in \mathbb{Z}} 2^{-k} \psi(2^{-k} t) e^{-2\pi i (t\xi_{(0,1)} + \sum_{(m,n)\in\Lambda, m\geq 1} t^n (s^m \xi_{(m,n)}))}$$

for each s and k. And we will treat only $j, k \geq 0$.

Proposition 2.1 There is a large constant C_d with the property that for any $D_1 \geq 2$, we have

$$(1) \qquad \mu_j(P_N(t) \otimes \xi) = \sum_{q=1}^{q \leq (j+1)^{C_d D_1}} \sum_{a \in P_q} S(a/q) \cdot \varphi_j\left(P_N(t) \otimes \xi - a/q\right)$$
$$\cdot \chi\left(\left[2^{(m-1/4)j}\left(t^{N(m,n)} \xi_{(m,n)} - a_{(m,n)}/q\right)\right]_{(m,n)\in\{(1,0)\}\cup\Lambda_{n\geq 1}}\right)$$
$$+ \widetilde{E}_j(\xi)$$

The functions φ_j are defined in proof, and $|E_j(\xi)| \leq C_{D_1}(j+1)^{-D_1}$.

Proof of Proposition 2.1. For each fixed t, we fisrt consider this case(major arc): For all $\alpha = (m, n)$, there is q which satisfies

$$\left|t^{N(m,n)} \xi_{(m,n)} - a_{(m,n)}/q\right| \leq \left(2^{(m-1/2)j}\right)^{-1}, q \in \left[1, 2^{(1/10)j}\right].$$

If we let

$$L_j(s) = \eta_1\left(2^{-j} s\right) \frac{1}{s}, P_N(t) \otimes \xi = a/q + \beta, Q(s) = (s^{M(m,n)})$$

(a, q, β depend on t), then

$$\mu_j(P_N(t) \otimes \xi) = \sum_{n \in \mathbb{Z}^1} \sum_{l \in [\mathbb{Z}_q]^1} L_j(nq + l) \cdot e^{-2\pi i Q(l) \cdot a/q} \cdot e^{-2\pi i Q(nq+l) \cdot \beta}$$

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$$= \sum_{n\in\mathbb{Z}^1}\sum_{l\in[\mathbb{Z}_q]^1}\left(L_j(nq+l)\cdot e^{-2\pi iQ(nq+l)\cdot\beta} - L_j(nq)\cdot e^{-2\pi iQ(nq)\cdot\beta}\right)\cdot e^{-2\pi iQ(l)a/q}$$
$$+L_j(nq)\cdot e^{-2\pi iQ(nq)\cdot\beta}\cdot e^{-2\pi iQ(l)a/q}$$
$$\leq \sum_{n\in\mathbb{Z}^1}\sum_{l\in[\mathbb{Z}_q]^1}|(L_j(nq+l)\cdot e^{-2\pi iQ(nq+l)\cdot\beta} - L_j(nq)\cdot e^{-2\pi iQ(nq+l)\cdot\beta})$$
$$+(L_j(nq)\cdot e^{-2\pi iQ(nq+l)\cdot\beta} - L_j(nq)\cdot e^{-2\pi iQ(nq)\cdot\beta})|$$
$$+\sum_{n\in\mathbb{Z}^1}\sum_{l\in[\mathbb{Z}_q]^1} L_j(nq)\cdot e^{-2\pi iQ(nq)\cdot\beta}e^{-2\pi iQ(l)a/q}$$
$$= \left[\sum_{l\in[\mathbb{Z}_q]^1} e^{-2\pi iQ(l)\cdot\frac{a}{q}}\right]\cdot\left[\sum_{n\in\mathbb{Z}^1} L_j(nq)e^{-2\pi iQ(nq)\cdot\beta}\right] + \mathrm{O}\left(2^{-j/4}\right)$$

(by mean value theorem)

$$= \mathrm{S(a/q)}\int L_j(x)e^{-2\pi iQ(x)\cdot\beta}\mathrm{dx} + \mathrm{O}\left(2^{-j/4}\right).$$

(by Van Da Corput theorem)

Let $\varphi_j(\xi) = \int L_j(x)e^{-2\pi iQ(x)\cdot\xi}\mathrm{dx}$. Then, we know that

$$|\varphi_j(\xi)| \leq \mathrm{C}\ (d,A)(1+\sum_{1\leq|\alpha|\leq A} 2^{|\alpha|j}|\xi_\alpha|)^{-\frac{1}{d}} \tag{2}$$

which follows from [3, proposition 2.1].
Second, we consider this case(minor arc) : $(t^{N(m,n)}\xi_{(m,n)})$ does not belong to the union over $q\in\left[1,2^{j/10}\right]$ of the above arcs.
In this case, we can show

$$\left|\mu_{j(t)}(\xi)\right| = O\left(2^{-c_dj}\right), c_d > 0.$$

Let $\xi = \left(t^{N(m,n)}\xi_{(m,n)}\right)$ and, for each (m,n) consider a Farey dissection at level $\Lambda_{(m,n)} = 2^{(m-1/2)j}$. Thus

$$\left|t^{(m,n)}\xi_{(m,n)} - a_{(m,n)}/q_{(m,n)}\right| \leq \left(q_{(m,n)}\cdot 2^{(m-1/2)j}\right)^{-1}$$

for some integers $a_{(m,n)}$ and $q_{(m,n)}$, with $\left(a_{(m,n)},q_{(m,n)}\right) = 1$ and $q_{(m,n)}\in\left[1,2^{(m-1/2)j}\right]$. Since ξ does not belong to the union over $q\in\left[1,2^{j/10}\right]$ of the major arcs, at least one of the denominators $q_{(m,n)}$ is $\geq 2^{j/(10d)}$. The bound follows from Lemma 1.7 with $R = 2^{j+1}$, $K = 2^j\mathrm{Ł}_j$, and $\epsilon = 1/(20d)$.
Finally, we can insert the cutoff function χ :

$$\mu_j(P_N(t)\otimes\xi) = S(a/q)\cdot\varphi_j(P_N(t)\otimes\xi - a/q)$$
$$\cdot\chi\left(\left[2^{(m-1/2)j}\left(t^{N((m,n))}\xi_{(m,n)} - a_{(m,n)}/q\right)\right]_{1\leq|(m,n)|\leq A,(m,n)\in\{(1,0)\}\cup\Lambda_{n\geq 1}}\right)$$

$$+O\left(2^{-c_d j}\right).$$

Then it is easy to see (1) by Lemma 1.8 with the constant C_d equal to $1/\delta$ and (2) and disjointness of above arc.
Thus the proof of Proposition 2.1 is complete. □
Remark (1)In the last part of proof of proposition 2.1, we can replace $(m-1/2)j$ by $(m-1/4)j$ in χ because of (2). In this same manner, it is obvious that for $\epsilon \leq 1/4$, we can replace $(m-1/2)j$ by $(m-\epsilon)j$ in χ.
(2)We can't apply Lemma 1.7 to minor arc part if we take $k(x) = 1/(st), x = (s,t)$ because the K in the proof of Proposition 2.1 is not a function satisfying the condition required to use Lemma 1.7 and we can't get a good error term in major part because $k(x) = 1/(st)$ does not satisfy the ferential inequalities $|x|^{d_1}|K(x)| + |x|^{d_1+1}|\nabla K(x)| \leq 1$ for any $x \in \mathbb{R}^{d_1}, |x| \geq 1$.

3. Double Hilbert Exponential sums

This chapter is our main chapter and we will prove Main Theorem. For this, we apply the circle method (Propsotion 2.1) to 1-parameter exponential sums and the other exponential sums related to the other variable. And we will study the property of $a/(qt^n)$ in Lemma 3.1 and use this to reduce the complexity created by each variables. And we will study (i) of Lemma 3.2 for j sums and we will study (ii) of Lemma 3.2 for a sums. We also use combinatorial thinking related to $(j,k) \in \mathbb{N}^2$ plane. And we use geometric property of some inequalities about $(m,n) \in \Lambda$ for problem area in $(j,k) \in \mathbb{N}^2$ plane.

We first introduce some definitions which are used to prove Theorem 3.3. For each (m,n) in Λ, we define

$$Q_{m,n}^{Reverse} = \left\{(x,y) \in \mathbb{R}^2 \mid x \leq m, y \leq n\right\},$$

Set $Q^{Reverse} = \bigcup_{(m,n)\in\Lambda\cap\mathbb{N}^2} Q_{m,n}^{Reverse}$. Then we let the reverse Newton diagram $\Pi^{Reverse}$ of P denote the smallest (closed) convex set containing $Q^{Reverse}$. $\Pi^{Reverse}$ is an unbounded polygon with a finite number of corners. We denote the set of corners by D. Then $D \subset \Lambda$.

For $2^{j-1} \leq |s| \leq 2^{j+1}, 2^{k-1} \leq |t| \leq 2^{k+1}$, $q \leq (j+1)^{C_d D_1}, a \in P_q$ if $|t^n \xi_{(m,n)} - a/q| \leq 2^{-(m-1/4)j}$, then $|\xi_{(m,n)} - a/(q\cdot t^n)| \leq 2^{-(m-1/4)j}/|t|^n$. And there are q', a' satisfying $a/(q\cdot t^n) = a'/q'$, $q \leq q' \leq 2^{(k+1)n}\cdot(j+1)^{C_d D_1}$, $(a',q') = 1$.
In this manner, we define function $f_{t^n}(a/q)$

$$f_{t^n}(a/q) = a'/q',$$

$$(a/q \cdot t^n = a'/q', q \leq q' \leq 2^{(k+1)n}\cdot(j+1)^{C_d D_1}, (a',q') = 1).$$

Similarly, by exchaging t,s and m,n and j,k, we also define function $f_{s^m}(a/q)$

$$f_{s^m}(a/q) = a'/q',$$

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$$(a/q \cdot s^m = a'/q', q \leq q' \leq 2^{(j+1)m} \cdot (k+1)^{C_d D_1}, (a', q') = 1).$$

Lemma 3.1 For fixed (m, n), if j and k satisfies following inequality

(3) $$n(k+1) \leq (m - 1/4)j - 2C_d D_1 \log_2(j+1) - \log_2 10,$$

for fixed $a_{(m,n)}/q$, $q \leq (j+1)^{C_d D_1}$), there are at most $C_n(\log_2 q)$ t's in $2^{k-1} \leq |t| \leq 2^{k+1}$ satisfying

$$\chi\left(2^{(m-1/4)j}\left(t^{N((m,n))}\xi_{(m,n)} - a_{(m,n)}/q\right)\right)) \neq 0$$

(C_n depends only on n) and if (j, k) satisfies following inequality

(4) $$m(j+1) \leq (n - 1/4)k - 2C_d D_1 \log_2(k+1) - \log_2 10,$$

for fixed $a_{(m,n)}/q$, $q \leq (k+1)^{C_d D_1}$), there are at most $C_m(\log_2 q)$ s's in $2^{j-1} \leq |s| \leq 2^{j+1}$ satisfying

$$\chi\left(2^{(n-1/4)k}\left(s^{M((m,n))}\xi_{(m,n)} - a_{(m,n)}/q\right)\right)) \neq 0$$

(C_m depends only on m).

Proof of Lemma 3.1. It suffices to consider the case when $a_{(m,n)} \geq 0, t \geq 0$. If

$$n(k+1) \leq (m - 1/4)j - 2C_d D_1 \log_2(j+1) - \log_2 10,$$

then

$$2^{(k+1)n} \leq 2^{(m-1/4)j} 2^{-\log_2 (j+1)^{C_d D_1}} \cdot \frac{1}{10}$$

which means

$$2^{(k+1)n} \leq \frac{1}{10} \cdot 2^{(m-1/4)j} \cdot \frac{1}{(j+1)^{2C_d D_1}} \leq 2^{(m-1/4)j} \cdot \frac{1}{10q^2}.$$

Therefore if $2^{k-1} \leq t \leq 2^{k+1}$,

$$2^{-mj+1/4j}/t^n \leq C_n \frac{1}{10(qt^n)^2}.$$

So by above definition of q', q' satisfies

$$|\xi_{(m,n)} - a'/q'| \leq 2^{-(m-1/4)j} \cdot 1/t^n \leq C_n \frac{1}{10(q')^2}$$

which means q' runs over the integers in the dyadic interval $[2^y, 2^{y+1} - 1]$ by Lemma 1.5 (q' is from $f_{t^n}(a_{(m,n)}/q) = a'/q'$). And since $l \leq q$ and the fact that if $(a_{(m,n)}, t^n) = l$, $q' = qt^n/l$, we know that $2^{n(k-1)}/q \leq q' \leq q2^{n(k+1)}$. So there are at most $C_n(\log_2 q)$ of t's in $2^{k-1} \leq t \leq 2^{k+1}$ satisfying

$$\cdot\chi\left(2^{(m-1/4)j}\left(t^{N((m,n))}\xi_{(m,n)} - a_{(m,n)}/q\right)\right)) \neq 0$$

(C_n depends only on n). Counting s is exactly same.
Thus Lemma 3.1 is complete. □
Remark (1)If we replace above χ by

$$\chi\left(2^{(m-1/4)j}\left((\sum_{n_r} t^{n_r})\gamma - a/q\right)\right)),$$

the biggest number of n_r dominates in the proof. So for large j and k, if j and k satisfies following inequality $(\max_r n_r)(k+1) \leq (m-1/4)j - 2C_dD_1\log_2(j+1) - \log_2 10$, for fixed a/q $((a,q)=1, q \leq (j+1)^{C_dD_1})$, there are at most $C_n(\log_2 q)$ t's in $2^{k-1} \leq |t| \leq 2^{k+1}$ satisfying

$$\chi\left(2^{(m-1/4)j}\left((\sum_{n_r} t^{n_r})\gamma - a/q\right)\right)) \neq 0$$

(C_n depends only on n).

We let

$$(5)\quad \mu_{j,k}(\xi) = \sum_t\sum_s 2^{-j}\psi(2^{-j}s)2^{-k}\psi(2^{-k}t)e^{-2\pi i(P(s,t)\cdot\xi)}\phi(\gamma)$$
$$= (\sum_t 2^{-k}\psi(2^{-k}t)e^{-2\pi i(t\xi_{(0,1)}+\sum_{(0,n)\in\Lambda}t^n\xi_{(0,n)})})\cdot\mu_j(P_N(t)\otimes\xi)\phi(\gamma)$$
$$= (\sum_s 2^{-j}\psi(2^{-j}s)e^{-2\pi i(s\xi_{(1,0)}+\sum_{(m,0)\in\Lambda}s^m\xi_{(m,0)})})\cdot\mu_k(P_M(s)\otimes\xi)\phi(\gamma).$$

In order to prove Theorem 3.3, we define following function which are used in major parts when we apply Proposition 2.1 to $\mu_j(P_N(t)\otimes\xi)$.

$$\varphi_j^1(\xi) = \int 2^{-j}\psi(2^{-j}x)e^{-2\pi i(x\xi_{(1,0)}+\sum_{(m,n)\in\Lambda,n\geq 1}x^m\cdot\xi_{(m,n)})}dx.$$

And we let $k_{q,\Lambda}$ denote a subset of P_q which has all a such that there exist $t \in [2^{k-1}, 2^{k+1}]$ satisfying

$$\chi\left(\left[2^{(m-1/4)nk/m}\left(t^{N((m,n))}\xi_{(m,n)} - a_{(m,n)}/q\right)\right]_{(m,n)\in\{(1,0)\}\cup\Lambda,n\geq 1}\right) \neq 0.$$

Lemma 3.2 (i) For fixed t, ξ and a/q,

$$\sum_j |\varphi_j^1(P_N(t)\otimes\xi - a/q)| \leq O(1).$$

(ii) If $\phi(\gamma)=1$ and $q \geq 100^{100/\delta}$,

$$\sum_k 2^{-k}|k_{q,\Lambda}| \leq O(q^{\delta/2})$$

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Proof of Lemma 3.2. (i)By the mean value theorem and [3, proposition 2.1], it is easy to show. (ii) □

Now we are ready to prove our main Theorem.

Proof of Main Theorem If $|D| = 0$, $\mu(\xi)$ is bounded function by Theorem 1.3. Otherwise, we first consider the case where if (m_1, n_1) and $(m_2, n_2) \in \Lambda$, $M(m_1, n_1) \neq M(m_2, n_2)$ and $N(m_1, n_1) \neq N(m_2, n_2)$.
By proposition 2.1,

$$\mu_j(P_N(t) \otimes \xi) = \sum_{q=1}^{q \le (j+1)^{C_d D_1}} \sum_{a \in P_q} S(a/q) \cdot \varphi_j^1(P_N(t) \otimes \xi - a/q)$$

$$\cdot \chi \left(\left[2^{(m-1/4)j} \left(t^{N((m,n))} \xi_{(m,n)} - a_{(m,n)}/q \right) \right]_{(m,n) \in \{(1,0)\} \cup \Lambda_{n \ge 1}} \right) + \widetilde{E}_j(\xi).$$

($|\widetilde{E}_j(\xi)| \le C_{D_1}(j+1)^{-D_1}$ and we choos $D_1 \ge 3$ for proof.)
So by (5), we see

$$(6)\quad \mu_{j,k} = (\sum_t 2^{-k} \psi(2^{-k}t) e^{-2\pi i (t\xi_{(0,1)} + \sum_{(0,n)\in\Lambda} t^n \xi_{(0,n)})}) \cdot \mu_j(P_N(t) \otimes \xi)\phi(\gamma)$$

$$= ((\sum_t 2^{-k} \psi(2^{-k}t) e^{-2\pi i (t\xi_{(0,1)} + \sum_{(0,n)\in\Lambda} t^n \xi_{(0,n)})})$$

$$\cdot (\sum_{q=1}^{q \le (j+1)^{C_d D_1}} \sum_{a \in P_q} S(a/q) \cdot \varphi_j^1(P_N(t) \otimes \xi - a/q))$$

$$\cdot \chi \left(\left[2^{(m-1/4)j} \left(t^{N((m,n))} \xi_{(m,n)} - a_{(m,n)}/q \right) \right]_{(m,n) \in \{(1,0)\} \cup \Lambda, n \ge 1} \right))$$

$$+ (\sum_t 2^{-k} \psi(2^{-k}t) e^{-2\pi i (t\xi_{(0,1)} + \sum_{(0,n)\in\Lambda} t^n \xi_{(0,n)})}) \cdot \widetilde{E}_j(\xi))\phi(\gamma).$$

On the other hand, since Lemma 3.1, for fixed (m, n), a/q, if (j, k) satisfies (3), there are at most $C_n(\log_2 q)$ t's in $2^{k-1} \le |t| \le 2^{k+1}$ which make

$$S(a/q) \cdot \varphi_j^1(P_t(s,t) \otimes \xi - a/q)$$

$$\cdot \chi \left(\left[2^{(m-1/4)j} \left(t^{N((m,n))} \xi_{(m,n)} - a_{(m,n)}/q \right) \right]_{(m,n) \in \{(1,0)\} \cup \Lambda, n \ge 1} \right) \neq 0.$$

Therefore, by applying the fact that $|2^{-k}\psi(2^{-k}t)e^{-2\pi i (t\xi_{(0,1)} + \sum_{(0,n)\in\Lambda} t^n \xi_{(0,n)})}| \le 2^{-k}$ and (i) of Lemma 3.2 to (6) (when we sum j terms first), it suffices to consider following (7) for boundedness if (j, k) satisfies (3).

$$(7)\quad \sum_k (2^{-k} \sum_{q=1}^{\infty} \sum_{a \in k_\Lambda} \phi(\gamma) C_n (\log_2 q) |S(a/q)| \chi \left(\left[q^2 \left(\xi_{(1,0)} - a_{(1,0)}/q \right) \right] \right)$$

$$+ C_{D_1}(nk/m+1)^{-D_1+1}).$$

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Thus, since Lemma 1.5, Lemma 1.8 and (ii) of Lemma 3.2, if we let
$$E_{m,n} = \{(j,k)|(j,k) \quad \text{satisfy} \quad (3)\},$$
we see
$$\sum_{(j,k)\in E_{m,n}} |\mu_{j,k}(\xi)| \le O(1).$$
And by exactly same method, if we let
$$F_{m,n} = \{(j,k)|(j,k) \quad \text{satisfy} \quad (4)\},$$
we know
$$\sum_{(j,k)\in F_{m,n}} |\mu_{j,k}(\xi)| \le O(1).$$
Next we will conside the cases where there exist m_1 and n_2 which satisfy that there are many n_r satisfying $(m_1, n_r) \in \Lambda$ and there are many $m_{r'}$ satisfying $(m_{r'}, n_2) \in \Lambda$. For this cases, we define $P'_N(t)\otimes\xi$, $P'_M(s)\otimes\xi$
$$P'_N(t)\otimes\xi = (\cdots, \sum_r (t^{n_r})\xi_(m_1, n_r), \cdots) = (\cdots, \gamma\sum_r (t^{n_r})a_{m_1,n_r}, \cdots),$$
$$P'_M(s)\otimes\xi = (\cdots, \sum_r (s^{m_{r'}})\xi_(m_{r'}, n_2), \cdots) = (\cdots, \gamma\sum_r (s^{m_{r'}})a_{m_{r'},n_2}, \cdots).$$
We will do the same things above exept that we replace $P_N(t)\otimes\xi$, $P_M(s)\otimes\xi$, by $P'_N(t)\otimes\xi$, $P'_M(s)\otimes\xi$ in the Proposition 2.1. Since the biggest number of n_r and m_r dominate when we apply the Lemma 3.1(see remark (1) of Lemma 3.1), we shall consider the D(the corners of $\Pi^{Reverse}$).
If $|D| \ge 2$, there are two points $(m_1, n_1), (m_2, n_2) \in D$. If we do above same things, we know that
$$\sum_{(j,k)\in E_{m_1,n_1}} |\mu_{j,k}(\xi)| \le O(1), \quad \sum_{(j,k)\in F_{m_1,n_1}} |\mu_{j,k}(\xi)| \le O(1),$$
$$\sum_{(j,k)\in E_{m_1,n_1}} |\mu_{j,k}(\xi)| \le O(1), \quad \sum_{(j,k)\in F_{m_1,n_1}} |\mu_{j,k}(\xi)| \le O(1).$$
Then, $E^c_{m_1,n_1} \cap F^c_{m_1,n_1}$ and $E^c_{m_2,n_2} \cap F^c_{m_2,n_2}$ are the remaining area that we need to treat. Since for large j, k, lines like $n_1(k+1) = (m_1 - 1/4)j$, $m_1(j+1) = (n_1 - 1/4)k$ or $n_2(k+1) = (m_2 - 1/4)j$, $m_2(j+1) = (n_2 - 1/4)k$ dominate the inequalities (3),(4) with m_1, n_1 or m_2, n_2 , it suffices to consider those four lines for the remaining area. However, we can replace $1/4$ in the proof of the Proposition 2.1 by any small positive number $\epsilon \le 1/4$ (see remark(1) of Proposition 2.1) and there is a $\epsilon \le 1/4$ which satisfies
$$m_1/(n_1 - \epsilon) \le (m_2 - \epsilon)/n_2$$
or
$$(m_1 - \epsilon)/(n_1) \ge m_2/(n_2 - \epsilon).$$
So $E_{m_1,n_1} \cup F_{m_1,n_1}$ and $E_{m_2,n_2} \cup F_{m_2,n_2}$ cover the remaining area each other which means $\sum_{(j,k)} |\mu_{j,k}(\xi)| \le O(1)$.
The proof of the case when $|D| = 1$ and all (m,n) are not on one line passing

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through the origin is same. Thus Theorem 3.3 is complete.□

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