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L^2 BOUNDEDNESS OF DISCRETE DOUBLE HILBERT TRANSFORM ALONG POLYNOMIALS

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Abstract : We will show L^2 boundedness of Discrete Double Hilbert Transform along polynomials satisfying some conditions. Double Hilbert exponential sum along polynomials:$\mu(\xi)$ is Fourier multiplier of the operator. In chapter 1, we introduce definitions and main Theorem and we also define the reverse Newton diagram. In chapter 2, We make approximation formula for the multiplier of one valuable discrete Hilbert transform by circle method as [1]. In chapter 3, We obtain the result that $\mu(\xi)$ is bounded by constants if $|D| \geq 2$ or all (m, n) are not on one line passing through the origin ($\Lambda \subseteq \{(m, n)|m, n$ are even natural number$\}$). We will study property of $1/(qt^n)$ and use circle method (Propsotion 2.1) to calculate sums. We also envision combinatoric thinking about $\mathbb{N}^2$ lattice points in j-k plane for some estimates. Finally, we use geometric property of some inequalities about $(m, n) \in \Lambda$ to prove Theorem 3.3. In chapter 4, We obtain the result that $\mu(\xi)$ is bounded by sums which are related to $\log_2(\xi_1 - a_1/q)$ and $\log_2(\xi_2 - a_2/q)$ and the boundedness of double Hilbert exponential sum for even polynomials with torsion without conditions in Theorem 3.3. We also use $\mathbb{N}^2$ lattice points in j-k plane and Proposition 2.1 which are shown in chapter 2 and some estimates to show that the Fourier multiplier of discrete double Hilbert transform is bounded by terms about log.

Keyword : Discrete Double Hilbert transform, Circle method, Exponential sums, Double Hilbert transform, Discrete Hilbert transform.

1. INTRODUCTION

There are good results of continuous version of L^p boundedness of double moreover multiple hilbert tranform along the polynomials with special conditions and L^p boudedness of discrete singular trasnform along polynomials. For $f \in \mathcal{S}$ (i.e., a Schwartz class function), we define

$$H_{loc}f(x, y, z) = p \cdot v \cdot \int_{-1}^{1} \int_{-1}^{1} f(x - s, y - t, z - P(s, t)) \frac{\mathrm{d}s\ \mathrm{d}t}{st}$$

where $P(s, t)$ is a real-valued polynomial in s and t. Carbery, Wainger and Wright determined the necessary and sufficient condition on the polynomial P so that H_{loc} is L^p bounded for $1 < p < \infty$. We state their result.
Let $P(s, t) = \sum_{(m,n)\in\Lambda} a_{m,n}s^m t^n$ be a polynomial with real coefficients such

that $P(0,0)=0, \nabla P(0,0)=0$ and where Λ is indexing the set of lattice points $(m,n)\in \mathbf{N}^2$ such that $a_{m,n}\neq 0$. For each (m,n) in Λ, we let

$$Q_{m,n}=\left\{(x,y)\in\mathbb{R}^2 \mid x\geq m \text{ and } y\geq n\right\},$$

Set $Q=\bigcup_{(m,n)\in\Lambda} Q_{m,n}$. Then the Newton diagram Π of P is the smallest (closed) convex set containing Q. Π is an unbounded polygon with a finite number of corners. We denote the set of corners by D. Then $D\subset\Lambda$.

THEOREM 1.1. For any $p, 1<p<\infty$,

$$\|H_{loc}f\|_{L^p}\leq A_p\|f\|_{L^p}$$

iff for each (m, n) that is a corner point of the Newton diagram corresponding to P, at least one of m and n is even.(Anthony Carbery, Stephen Wainger, James Wright [2].)

Sanjay Patel determined the necessary and sufficient condition on P so that the (global) double Hilbert transform defined by

$$Hf(x,y,z)=p\cdot v\cdot\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-s,y-t,z-P(s,t))\frac{\mathrm{d}s\mathrm{d}t}{st}$$

is bounded on $L^p, 1<p<\infty$. The operator is defined for $f\in\mathcal{S}$ by integrating where $\epsilon'\leq|s|\leq R', \epsilon\leq|t|\leq R$, and then, taking the limits as $\epsilon,\epsilon'\to 0$ and $R,R'\to+\infty$. Let $P(s,t)=\sum_{(m,n)\in\Lambda} a_{m,n}s^m t^n$ be a polynomial with real coefficients such that $P(0,0)=0, \nabla P(0,0)=0$ and where Λ is indexing the set of lattice points $(m,n)\in\mathbf{Z}^2$ such that $a_{m,n}\neq 0$. Let $\mathcal{C}$ be the closed convex hull of Λ in $\mathbf{R}^2$ and $\mathcal{D}=\{(m,n)\in\Lambda : (m,n)$ is a corner point (vertex) of $\mathcal{C}\}$.

THEOREM 1.2. For any $p, 1<p<\infty$,

$$\|Hf\|_{L^P}\leq A_p\|f\|_{L^P}$$

if and only if for each $(m,n)\in\mathcal{D}$, at least one of m and n is even and furthermore, if any (extended) edge of $\mathcal{C}$ passes through the origin (there are at most two such edges), then every point of Λ on that edge must have at least one even coordinate.(Sanjay Patel [6])

Assume that $d_1\geq 1$ is an integer and $K\in C^1\left(\mathbb{R}^{d_1}\backslash\{0\}\right)$ satisfies the differential inequalities $|x|^{d_1}|K(x)|+|x|^{d_1+1}|\nabla K(x)|\leq 1$ for any $x\in\mathbb{R}^{d_1}, |x|\geq 1$, and the cancellation condition

$$\left|\int_{|x|\in[1,\lambda]} K(x)dx\right|\leq 1$$

for any $\lambda\geq 1$ (i.e., K is a Calderón-Zygmund kernel on $\mathbb{R}^{d_1}$ away from 0). Let $P=(P_1,\ldots,P_{d_2}):\mathbb{R}^{d_1}\to\mathbb{R}^{d_2}$ denote a polynomial of degree $A\geq 1$

with real coefficients. We define the (translation invariant) discrete singular Radon transform operator T by the formula

$$T(f)(x) = \sum_{n\in\mathbb{Z}^{d_1}\setminus\{0\}} f(x - P(n))K(n)$$

for any Schwartz function $f : \mathbb{R}^{d_2} \to \mathbb{C}$. Ionescu and Wainger proved following theorem.

THEOREM 1.3. The operator T extends to a bounded operator on $L^p\left(\mathbb{R}^{d_2}\right), p \in (1,\infty)$, with $\|T(f)\|_{L^p\left(\mathbb{R}^{d_2}\right)} \le C_p\|f\|_{L^p\left(\mathbb{R}^{d_2}\right)}$. The constant C_p may depend only on the exponent p, the dimension d_1, and the degree A. (Alexandru D .lonescu and Stephen Wainger [1])

We introduce some definitions. Assume $d \ge 1$ is an integer. For any $\mu \ge 1$, let $Z_\mu = \mathbb{Z} \cap [1,\mu]$ and $P_q = \{a \in \mathbb{Z} : (a,q) = 1\}$. If $a = (a_1,\ldots,a_d) \in \mathbb{Z}^d$ is a vector and $q \ge 1$ is an integer, then we denote by (a,q) the greatest common divisor of a and q, i.e., the largest integer $q' \ge 1$ that divides q and all the components $a_1,\ldots,a_d$. Clearly, any vector in $\mathbb{Q}^d$ has a unique representation in the form a/q, with $q \in \{1,2,\ldots\}, a \in \mathbb{Z}^d$, and $(a,q) = 1$; such a vector a/q will be called an irreducible d -fraction. We let $|(m,n)|$ denote $|(m,n)| = m+n$. We set $p(s,t) = \sum_{(m,n)\in\Lambda} a_{m,n}s^m t^n$ be a polynomial with real coefficients such that $p(0,0) = 0, \nabla p(0,0) = 0.(\Lambda \subseteq \{(m,n)|m,n$ are even natural number, $|(m,n)| \le A$ $\})$.
We let $\xi = (\xi_{(m,n)}) \in \mathbb{R}^d$ $((m,n) \in \{(0,1),(1,0)\}\cup\Lambda$, indeed $d = |\Lambda|+2)$. We also let $P : \mathbb{R}^2 \to \mathbb{R}^d, [P(x)]_\alpha = x^\alpha(x = (s,t), \alpha = (m,n) \in \{(0,1),(1,0)\} \cup \Lambda)$.
For each (m,n) in Λ, we let

$$Q^{Reverse}_{m,n} = \left\{(x,y) \in \mathbb{R}^2 \mid x \le m, y \le n\right\},$$

Set $Q^{Reverse} = \bigcup_{(m,n)\in\Lambda} Q^{Reverse}_{m,n}$. Then we let the reverse Newton diagram $\Pi^{Reverse}$ of P denote the smallest (closed) convex set containing $Q^{Reverse}$. Π is an bounded polygon with a finite number of corners. We denote the set of corners by D. Then $D \subset \Lambda$.
Now we define the discrete double Hilbert transform operator $H_{discrete}$ by following formula

$$H_{discrete}f(x,y,z) = \sum_{(s,t)\in\mathbb{Z}^2\setminus\{0\}} f(x-s, y-t, z-p(s,t))\frac{1}{s\cdot t}.$$

We also define the discrete double hilbert exponential sum :

$$\mu(\xi) = \sum_{(s,t)\in\mathbb{Z}^2\setminus\{0\}} e^{-2\pi i\cdot P(s,t)\cdot\xi}\cdot\frac{1}{s}\cdot\frac{1}{t}, \xi \in \mathbb{R}^d$$

which is the Fourier multiplier of $H_{discrete}(f)$: $\hat{H}_{discrete}(\xi_{(1,0)}, \xi_{(0,1)}, \gamma)$ if we let $\xi_{(m,n)} = a_{m,n}\gamma$ for all $(m,n) \in \Lambda$. It suffices to show that $\mu(\xi)$ is bounded by constants to show Theorem 1.4.
Chronologically, there have been important results such that

$$\int_{-\infty}^{\infty} e^{-2\pi i\cdot(x\cdot\xi_1+p(x)\xi_2)}\cdot\frac{1}{x}\leq C,$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2\pi i\cdot(x\cdot\xi_1+y\cdot\xi_2+p(x,y)\cdot\xi_3)}\cdot\frac{1}{x}\cdot\frac{1}{y}\leq C$$

(with conditions mentioned in Theorem 1.2),

$$\sum_{s\in\mathbb{Z}\setminus\{0\}} e^{-2\pi i\cdot P(s)\cdot\xi}\cdot\frac{1}{s}\leq C, \quad \xi\in\mathbb{R}^d.$$

But following inequality is unsolved.

$$\sum_{(s,t)\in\mathbb{Z}^2\setminus\{0\}} e^{-2\pi i\cdot P(s,t)\cdot\xi}\cdot\frac{1}{s}\cdot\frac{1}{t}\leq C, \quad \xi\in\mathbb{R}^d \quad (?)$$

Our main result(Theorem 3.3) is last inequality along polynomials with some conditions.

Theorem 1.4 $\|H_{discrete}(f)\|_{L^2(\mathbb{R}^3)} \leq C|f\|_{L^2(\mathbb{R}^3)}$ if $|D| \geq 2$ or all (m,n) are not on one line passing through the origin ($\Lambda \subseteq \{(m,n)|m,n$ are even natural number$\}$). The constant C may depend only on the polynomial and d.

Proof of Theorem 1.4. It follows from Theorem 3.3. □

Remarks (1) Since $\frac{1}{s\cdot t}$ does not have good property of Calderón-Zygmund kernel, we can't apply Theorem 1.3 to Theorem 1.4 but when we consider one valuable fraction $\frac{1}{s}$, we can apply the same methods used to prove Theorem 1.3. in [1].
(2) Since $\mu(\xi)$ is periodic, we concentrate only on $[0,1]^d$.

Question 1.5 What are the necessary and sufficient conditions of Λ for $\|H_{discrete}(f)\|_{L^p(\mathbb{R}^3)} \leq C|f\|_{L^p(\mathbb{R}^3)}$? (The constant C may depend only on the polynomial and d)

We introduce following Lemmas which are used many times in this paper.

Lemma 1.6. (Diriclet's principle) For any $\Lambda \geq 1$ and $\xi \in \mathbb{R}$, there are $q \in \mathbb{Z}_\Lambda$ and $a \in \mathbb{Z}$ with $(a,q) = 1$, with the property that $|\xi - a/q| \leq 1/q\Lambda$.

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Lemma 1.7. For any $R \geq 1$, let $B_R = \left\{x \in \mathbb{R}^{d_1} : |x_l| < R, l = 1, \ldots, d_1\right\}$. Assume that $k : B_R \to \mathbb{C}$ is a C^1 function with the property that

$$|k(x)| + R \cdot |\nabla k(x)| \leq 1$$

for any $x \in B_R$. Assume that $\epsilon \in (0, 1/10)$ is fixed and $\xi \in \mathbb{R}^d$ has the property that for some $\alpha, 1 \leq |\alpha| \leq A$, there are integers a and q, with $(a, q) = 1, q \in \left[R^\epsilon, R^{|\alpha|-\epsilon}\right]$, and $|\xi_\alpha - a/q| \leq 1/q^2$. Then

$$\left|\sum_{n \in \Omega \cap \mathbb{Z}^{d_1}} e^{-2\pi i P(n)\cdot\xi} k(n)\right| \leq C R^{d_1 - \delta}, \delta > 0$$

for any open, convex set $\Omega \subset B_R$. The constants C and δ may depend only on d_1, A, and ϵ but not on R, ξ, or the irreducible fraction a/q. [22, Proposition 3]

Remark (1)We can't apply Lemma 1.7 to $k(x) = 1/(st)$ because the $k(x)$ is not a function with the property in 1.7. This is reason why it is difficult to prove L^p boundedness of double Hilbert transform along general polynomials.

We introduce Guass sum. For any $q \in \{1, 2, \ldots\}$ and $a \in \mathbb{Z}^d$ with $(a, q) = 1$, let

$$S(a/q) = \frac{1}{q^{d_1}} \sum_{n \in [Z_q]^{d_1}} e^{-2\pi i P(n)\cdot a/q}.$$

Lemma 1.8. If $(\mathrm{a}, \mathrm{q}) = 1$, $|S(a/q)| \leq C \cdot q^{-\delta}$ (for some constant $\delta = \delta(d) > 0$).

Proof of Lemma 1.8. Let $a = (a_\alpha)$, and assume that $a_\alpha/q = a'_\alpha/q'_\alpha$, where a'_α/q'_α is an irreducible. Since $(a, q) = 1$, there are d_α, q'_α satisfying $q = d_\alpha q'_\alpha, (d_1, \ldots, d_\alpha, \ldots) = 1$. Since q has d_α and $(d_1, \ldots, d_\alpha, \ldots) = 1, \Pi d_\alpha$ can't divide all d_α of q^d. So $\Pi_\alpha q'_\alpha = q^d/\Pi d_\alpha \geq q$. First, if $q'_\alpha \geq q^{1/10d^2}$ for some index α with $|\alpha| \geq 2$, we shall know

$$\left|\sum_{n \in [z_q]^{d_1}} e^{-2\pi i P(n)\cdot a/q}\right| \leq \mathrm{C}2^{d_1} \cdot q^{d_1 - \delta}, \delta > 0$$

by lemma 1.7 with $\mathrm{R} = 2q, k = 1$, and $\epsilon = 1/20d^2$ which means $q'_\alpha \in \left[R^\epsilon, R^{|\alpha|-\epsilon}\right]$. Second, if $q'_\alpha \leq q^{1/10d^2}$ for any α with $|\alpha| \geq 2$, then $q'_\alpha \geq 1/q^{2d}$ for some α_0 with $|\alpha_0| = 1$. In this case, by summing first the variable corresponding to the index α_0 with summation by parts, we have $\mathrm{S(a/q)} = 0$. □

In chapter 2, We make approximation formula for the multiplier of one valuable discrete Hilbert transform by circle method as [1]. In chapter 3, We obtain the result that $\mu(\xi)$ is bounded by constants if $|D| \geq 2$ or all (m, n) are not on one line passing through the origin ($\Lambda \subseteq \{(m, n) | m, n$ are even

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natural number}). We will study property of $1/(qt^n)$ and use circle method (Propsotion 2.1) to calculate sums. We also envision combinatoric thinking about $\mathbb{N}^2$ lattice points in j-k plane for some estimates. Finally, we use geometric property of some inequalities about $(m, n) \in \Lambda$ to prove Theorem 3.3. In chapter 4, We obtain the result that $\mu(\xi)$ is bounded by sums which are related to $\log_2(\xi_1 - a_1/q)$ and $\log_2(\xi_2 - a_2/q)$ and the boundedness of double Hilbert exponential sum for even polynomials with torsion without conditions in Theorem 3.3. We also use $\mathbb{N}^2$ lattice points in j-k plane and Proposition 2.1 which are shown in chapter 2 and some estimates to show that the Fourier multiplier of discrete double Hilbert transform is bounded by terms about log.

2. Some estimates with circle method.

In this section, We make approximation formula for the multiplier of one valuable discrete Hilbert transform by study circle method. Our method is similar to the method of Alexandru D. Ionescu, Stephen Wainger [1] and J. Bourgain [4].

Let $N(m, n)$ denote $N(m, n) = n$ and $M(m, n)$ denote $M(m, n) = m$. Let $P_t(s, t) \otimes \xi$ denote $P_t(s, t) \otimes \xi = (t^{N(m,n)} \cdot \xi_{(m,n)}), (m, n) \in \{(1, 0)\} \cup \Lambda$ and let $P_s(s, t) \otimes \xi$ denote $P_s(s, t) \otimes \xi = (s^{M(m,n)} \cdot \xi_{(m,n)}), (m, n) \in \{(0, 1)\} \cup \Lambda$. For example, if $p(s, t) = s^2t^2 + s^4t^4 + s^6t^8$, then $P_t(s, t) \otimes \xi = (\xi_{(1,0)}, t^2 \cdot \xi_{(2,2)}, t^4 \cdot \xi_{(4,4)}, t^8 \cdot \xi_{(6,8)})$ and $P_s(s, t) \otimes \xi = (\xi_{(0,1)}, s^2 \cdot \xi_{(2,2)}, s^4 \cdot \xi_{(4,4)}, s^6 \cdot \xi_{(6,8)})$. We also let $P_t^{\Lambda}(s, t) \otimes \xi$ denote $P_t(s, t) \otimes \xi = (t^{N(m,n)} \cdot \xi_{(m,n)}), (m, n) \in \Lambda$ and let $P_s^{\Lambda}(s, t) \otimes \xi$ denote $P_s(s, t) \otimes \xi = (s^{M(m,n)} \cdot \xi_{(m,n)}), (m, n) \in \Lambda$.

We will first consider the case when if (m_1, n_1) and $(m_2, n_2) \in \Lambda$, $M(m_1, n_1) \neq M(m_2, n_2)$ and $N(m_1, n_1) \neq N(m_2, n_2)$ in the chapter 2. The proof for other cases is similar but we should consider reverse Newton diagram. We will mention this in the last part of Theorem 3.3.

Let $\eta : \mathbb{R}^{d_1} \to [0, 1]$ denote a smooth function supported in $\{x : |x| \in [1/2, 2]\}$ with the property that $\sum_{j=0}^{\infty} \eta\left(2^{-j}x\right) = 1$ for any $x \in \mathbb{R}$ with $|x| \geq 1$.

We also define $\mu_{j(t)}(\xi)$ as

$$\mu_{j(t)}(\xi) = \sum_{s \in \mathbb{Z}} \eta\left(2^{-j}s\right) e^{-2\pi i \cdot (s^{M(m,n)}) \cdot P_t(s,t) \otimes \xi} \cdot \frac{1}{s}$$

for each t and j.

Proposition 2.1 There is a large constant C_d with the property that for any $D_1 \geq 2$, we have (2.1)

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$$\mu_{j(t)}(\xi) = \sum_{q=1}^{q\leq (j+1)^{C_d D_1}} \sum_{a\in P_q} S(a/q)\cdot \varphi_j\left(P_t(s,t)\otimes\xi - a/q\right)$$
$$\cdot\chi\left(\left[2^{(m-1/4)j}\left(t^{N(m,n)}\xi_{(m,n)} - a_{(m,n)}/q\right)\right]_{(m,n)\in\{(1,0)\}\cup\Lambda}\right)$$
$$+\widetilde{E}_j(\xi)$$

The functions φ_j are defined in proof, and $|E_j(\xi)| \leq C_{D_1}(j+1)^{-D_1}$.

Proof of Proposition 2.1. For Each fixed t, Fisrt we consider this case: For all $\alpha = (m,n)$, there is q which satisfies

$$\left|t^{N(m,n)}\xi_{(m,n)} - a_{(m,n)}/q\right| \leq \left(2^{(m-1/2)j}\right)^{-1}, q\in\left[1, 2^{(1/10)j}\right].$$

If we let

$$L_j(s) = \eta_1\left(2^{-j}s\right)\frac{1}{s}, P_t(s,t)\otimes\xi = a/q+\beta, Q(s) = (s^{M(m,n)}).$$

(a, q, β depend on t)

Then

$$\mu_{j(t)}(\xi) = \sum_{n\in\mathbb{Z}^1}\sum_{l\in[\mathbb{Z}_q]^1} L_j(nq+l)\cdot e^{-2\pi i Q(l)\cdot a/q}\cdot e^{-2\pi i Q(nq+l)\cdot\beta}$$

$$= \sum_{n\in\mathbb{Z}^1}\sum_{l\in[\mathbb{Z}_q]^1}\left(L_j(nq+l)\cdot e^{-2\pi i Q(nq+l)\cdot\beta} - L_j(nq)\cdot e^{-2\pi i Q(nq)\cdot\beta}\right)\cdot e^{-2\pi i Q(l)a/q}$$
$$+L_j(nq)\cdot e^{-2\pi i Q(nq)\cdot\beta}\cdot e^{-2\pi i Q(l)a/q}$$
$$\leq \sum_{n\in\mathbb{Z}^1}\sum_{l\in[\mathbb{Z}_q]^1}|(L_j(nq+l)\cdot e^{-2\pi i Q(nq+l)\cdot\beta} - L_j(nq)\cdot e^{-2\pi i Q(nq+l)\cdot\beta})$$
$$+(L_j(nq)\cdot e^{-2\pi i Q(nq+l)\cdot\beta} - L_j(nq)\cdot e^{-2\pi i Q(nq)\cdot\beta})|$$
$$+\sum_{n\in\mathbb{Z}^1}\sum_{l\in[\mathbb{Z}_q]^1} L_j(nq)\cdot e^{-2\pi i Q(nq)\cdot\beta}e^{-2\pi i Q(l)a/q}$$
$$= \left[\sum_{l\in[\mathbb{Z}_q]^1} e^{-2\pi i Q(l)\cdot\frac{a}{q}}\right]\cdot\left[\sum_{n\in\mathbb{Z}^1} L_j(nq)e^{-2\pi i Q(nq)\cdot\beta}\right] + \mathrm{O}\left(2^{-j/4}\right)$$

(by mean value theorem)

$$= \mathrm{S(a/q)}\int L_j(x)e^{-2\pi i Q(x)\cdot\beta}\mathrm{dx} + \mathrm{O}\left(2^{-j/4}\right).$$

(by Van Da Corput theorem)

Let $\varphi_j(\xi) = \int L_j(x)e^{-2\pi i Q(x)\cdot\xi}\mathrm{dx}$. In addition, we know that $|\varphi_j(\xi)| \leq$

C $(d, A)(1+\sum_{1\leq|\alpha|\leq A} 2^{|\alpha|j}|\xi_\alpha)^{-\frac{1}{d}} \ldots (2.2)$ which follows from [3, proposition 2.1] and the fact that

$$|\varphi_j(\xi)| = |\int L_j(x)e^{-2\pi i Q(x)\cdot\xi}\mathrm{dx}| \leq C|2^j\xi_1|$$

which follows from mean-value theorem.
Second we consider this case : $(t^{N(m,n)}\xi_{(m,n)})$ does not belong to the union over $q \in \left[1, 2^{j/10}\right]$ of the above arcs.
In this case, we can show

$$\left|\mu_{j(t)}(\xi)\right| = O\left(2^{-c_d j}\right), c_d > 0$$

Let $\xi = \left(t^{N(m,n)}\xi_{(m,n)}\right)$ and, for each (m,n) consider a Farey dissection at level $\Lambda_{(m,n)} = 2^{(|(m,n)|-1/2)j}$. Thus

$$\left|t^{(m,n)}\xi_{(m,n)} - a_{(m,n)}/q_{(m,n)}\right| \leq \left(q_{(m,n)}\cdot 2^{(m-1/2)j}\right)^{-1}$$

for some integers $a_{(m,n)}$ and $q_{(m,n)}$, with $\left(a_{(m,n)}, q_{(m,n)}\right) = 1$ and $q_{(m,n)} \in \left[1, 2^{(m-1/2)j}\right]$. Since ξ does not belong to the union over $q \in \left[1, 2^{j/10}\right]$ of the major arcs, at least one of the denominators $q_{(m,n)}$ is $\geq 2^{j/(10d)}$. The bound follows from Lemma 1.12 with $R = 2^{j+1}, K = 2^j\mathrm{L}_j$, and $\epsilon = 1/(20d)$.
Finally we can insert the cutoff function χ :

$$\mu_{j(t)}(\xi) = S(a/q)\cdot\varphi_j(P_t(s,t)\otimes\xi - a/q)$$

$$\cdot\chi\left(\left[2^{(m-1/2)j}\left(t^{N((m,n))}\xi_{(m,n)} - a_{(m,n)}/q\right)\right]_{1\leq|(m,n)|\leq A,(m,n)\in\{(1,0)\}\cup\Lambda}\right)$$

$$+O\left(2^{-c_d j}\right).$$

Then it is easy to see (2.1) by Lemma 1.8 with the constant C_d equal to $1/\delta$ and (2.2) and disjointness of above arc (socalled t-major arc). □

Before proposition 2.2, we define

$$m_k(\beta) = \chi(\beta/2)\sum_{j=k+1}^{\infty}\varphi_j(\beta), \quad \Lambda^t = (1,0)\cup\Lambda.$$

Proposition 2.2 Given C_d and $D_1 \geq 2$ as is Proposition 2.1, we have (2.3)

$$\sum_{j=k+1}^{\infty} \mu_{j(t)}(\xi) = \sum_{q=1}^{q \leq k^{C_d D_1}} \sum_{a \in P_q} S(a/q) \cdot m_k(P_t(s,t) \otimes \xi - a/q)$$
$$\cdot \chi\left(\left[2^{(m-1/4)k}\left(t^{N(m,n)}\xi_{(m,n)} - a_{(m,n)}/q\right)\right]_{(m,n)\in\Lambda^t}\right)$$
$$+ \widetilde{E}_k(\xi)$$

for any integer $k \geq 1$, where $\left|\widetilde{E}_k(\xi)\right| \leq C_{d,D_1} k^{-(D_1-1)}$.

Proof of Proposition 2.2 . This proof is almost same with [1. Lemma [6.4]]. But we rewrite for reader. We use formula (2.1). The sum of the error terms $E_j(\xi), j \geq k+1$, can be incorporated into the error term $\widetilde{E}_k$. Let χ_+ denote the characteristic function of the set $[0,\infty)$. By (2.1) , we can get following folmula (2.4)

$$\sum_{j\geq k+1}^{\infty} \mu_{j(t)}(\xi) = \sum_{q=1}^{\infty} \sum_{a\in P_q} S(a/q) \cdot \sum_{j \geq k+1} \chi_+\left((j+1)^{C_d D_1} - q\right) \cdot \varphi_j(P_t(s,t) \otimes \xi - a/q)$$
$$\cdot \chi\left(\left[2^{(m-1/4)j}\left(t^{N(m,n)}\xi_{(m,n)} - a_{(m,n)}/q\right)\right]_{(m,n)\in\Lambda^t}\right) + \widetilde{E}_k(\xi).$$

Assume first that $q \leq k^{C_d D_1}$. Then $\chi_+\left((j+1)^{C_d D_1} - q\right) = 1$. In addition,

$$\varphi_j(P_t(s,t) \otimes \xi - a/q) \cdot \chi\left(\left[2^{(m-1/4)j}\left(t^{N(m,n)}\xi_{(m,n)} - a_{(m,n)}/q\right)\right]_{(m,n)\in\Lambda^t}\right)$$
$$= \varphi_j(P_t(s,t) \otimes \xi - a/q) \cdot \chi\left(\left[2^{(m-1/4)k}\left(t^{N(m,n)}\xi_{(m,n)} - a_{(m,n)}/q\right)\right]_{(m,n)\in\Lambda^t}\right)$$
$$+ O\left(2^{-j/(4d)}\right)$$

by (2.2). Thus, the sum over $q \leq k^{C_d D_1}$ in (2.4) coincides with the main term in (2.3), modulo acceptable errors.

We break up the sum over $q > k^{C_d D_1}$ in (2.4) into dyadic pieces, $q \in \left[2^s, 2^{s+1}\right) \cap \left(k^{C_d D_1}, \infty\right) \cap \mathbb{Z}$. Since $C_d = 1/\delta$, it suffices to prove that (2.5)

$$\sum_{q=2^s}^{2^{s+1}-1} \sum_{a \in P_q} S(a/q) \sum_{j \geq k+1} \chi_+\left((j+1)^{C_d D_1} - q\right) \cdot$$
$$\varphi_j(P_t(s,t) \otimes \xi - a/q) \cdot \chi\left(\left[2^{(m-1/4)j}\left(t^{N(m,n)}\xi_{(m,n)} - a_{(m,n)}/q\right)\right]_{(m,n)\in\Lambda^t}\right)$$
$$= O\left(2^{-\delta s}\right).$$

whenever $2^{s+1} \geq k^{C_d D_1}$. We may assume $j \geq C_{D_1}$. Then the support of the sum over j in (2.5) is contained in the set $\left\{\xi : |P_t(s,t) \otimes \xi - a/q| \leq \left(10q^2\right)^{-1}\right\}$. These sets are disjoint when q runs over the integers in the dyadic interval $\left[2^s, 2^{s+1}-1\right]$. By Lemma 2.1 , $S(a/q) = O\left(2^{-\delta s}\right)$. Therefore it remains to prove that

$$\sum_{j \geq k+1} \chi_+ \left((j+1)^{C_d D_1} - q\right) \varphi_j(P_t(s,t) \otimes \xi - a/q)$$
$$\cdot \chi\left(\left[2^{(m-1/2)j}\left(t^{N(m,n)}\xi_{(m,n)} - a_{(m,n)}/q\right)\right]_{(m,n)\in\Lambda^t}\right)$$
$$= O(1)$$

for any irreducible d -fraction a/q. This follows easily from

$$|\varphi_j(\xi)| = |\int L_j(x) e^{-2\pi i Q(x)\cdot\xi} \mathrm{dx}| \leq \mathrm{C}(d,A)\left(1 + \sum_{1\leq|\alpha|\leq A} 2^{|\alpha| j}\left|\xi_\alpha\right|\right)^{-\frac{1}{d}}$$

and

$$|\varphi_j(\xi)| = |\int L_j(x) e^{-2\pi i Q(x)\cdot\xi} \mathrm{dx}| \leq C\left|2^j \xi_1\right| \cdots (2.6).$$

□

3. Uniformly boundedness of Discrete Double Hilber Exponential Sum

This chapter is our main chapter and our own results are in chapter 3 and chapter 4. In this chapter, We will show that for $p(s,t) = \sum_{(m,n)\in\Lambda} a_{m,n} s^m t^n$,

$$\|H_{discrete}(f)\|_{L^2(\mathbb{R}^3)} \leq C\|f\|_{L^2(\mathbb{R}^3)}$$

if all $(m,n) \in \Lambda \subseteq \{(m,n)|\text{m,n are even natural number}\}$ are not on one line passing through the origin. It suffices to prove the Fourier multiplier of the operator $\mu(\xi)$ is bounded function.
In this chpater, We study property of $1/(qt^n)$ and use circle method (Propsotion 2.1) to calculate sums. We also use combinatoric thinking in j-k $\mathbb{N}^2$ plane. Finally, we use geometric property of some inequalities about $(m,n) \in \Lambda$ to prove Theorem 3.3.

For $2^{j-1} \leq |s| \leq 2^{j+1}, 2^{k-1} \leq |t| \leq 2^{k+1}$, $q \leq (j+1)^{C_d D_1}, a \in P_q$ if $|t^n \xi_{(m,n)} - a/q| \leq 2^{-(m-1/4)j}$, then $|\xi_{(m,n)} - a/(q\cdot t^n)| \leq 2^{(j-1)m)+1/4j}/|t|^n$. And there are q', a' satisfying $a/(q\cdot t^n) = a'/q', q \leq q' \leq 2^{(k+1)n} \cdot (j+1)^{C_d D_1}$ and $(a',q') = 1$.
In this manner, we define function $f_{t^n}(a/q)$

$$f_{t^n}(a/q) = a'/q',$$
$$(a/q \cdot t^n = a'/q', q \leq q' \leq 2^{(k+1)n} \cdot (j+1)^{C_d D_1}, (a',q') = 1).$$

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and we let $\beta^{(m,n)}_{a/q,t}(j,k)$

$$\beta^{(m,n)}_{a/q,t}(j,k) = |\xi_{(m,n)} - f_{t^n}(a/q)| \leq 2^{-((k-1)n+(j-1)m)+1/4j}.$$

With same manner by exchaging t,s and m,n and j,k, we also define function $f_{s^m}(a/q)$

$$f_{s^m}(a/q) = a'/q',$$

$$(a/q \cdot s^m = a'/q', q \leq q' \leq 2^{(j+1)m} \cdot (k+1)^{C_d D_1}, (a', q') = 1).$$

and we also let $\beta^{(m,n)}_{a/q,s}(j,k)$

$$\beta^{(m,n)}_{a/q,s}(j,k) = |\xi_{(m,n)} - f_{s^m}(a/q)| \leq 2^{-((k-1)n+(j-1)m)+1/4k}.$$

Lemma 3.1 If $nk \leq (m-1/4)j - C_d D_1 \log_2(j+1) - \log_2 10$, for fixed $a_{(m,n)}/q$ $((a_{(m,n)}, q) = 1)$, there are at most $C_n(\log_2 q)$ t's in $2^{k-1} \leq t \leq 2^{k+1}$ satisfying

$$\chi\left(2^{(m-1/4)j}\left(t^{N((m,n))}\xi_{(m,n)} - a_{(m,n)}/q\right)\right)) \neq 0$$

(C_n depends only on n) and if $mj \leq (n-1/4)k - C_d D_1 \log_2(k+1) - \log_2 10$, for fixed $a_{(m,n)}/q$ $((a_{(m,n)}, q) = 1)$, there are at most $C_m(\log_2 q)$ s's in $2^{j-1} \leq s \leq 2^{j+1}$ satisfying

$$\chi\left(2^{(n-1/4)k}\left(s^{M((m,n))}\xi_{(m,n)} - a_{(m,n)}/q\right)\right)) \neq 0$$

(C_m depends only on m).

Proof of Lemma 3.1. It suffices to think when $a_{(m,n)} \geq 0, t \geq 0$. If $nk \leq (m-1/4)j - C_d D_1 \log_2(j+1) - \log_2 10$, then $2^{-mj+1/4j}/t^n \leq 1/(qt^n)^2$. So q' satisfies $|\xi_{(m,n)} - a'/q'| \leq 2^{-(m-1/4)j} \cdot 1/t^n \leq 1/(10q')^2$ which means q' runs over the integers in the dyadic interval $[2^y, 2^{y+1}-1]$ (q' is from $f_{t^n}(a_{(m,n)}/q) = a'/q'$). Since $l \leq q$ and the fact that if $(a_{(m,n)}, t^n) = l$, $q' = qt^n/l$, we know that $q2^{n(k-1)}/q \leq q' \leq 2^{n(k+1)}$. So there are at most $C_n(\log_2 q)$ of t's in $2^{k-1} \leq t \leq 2^{k+1}$ satisfying

$$\cdot\chi\left(2^{(m-1/4)j}\left(t^{N((m,n))}\xi_{(m,n)} - a_{(m,n)}/q\right)\right)) \neq 0$$

(C_n depends only on n). Counting $\beta^{(m,n)}_{a/q,s}(j,k)$ is exactly same. $\square$

Remark(1) For fixed a/q, We can choose any $(m,n) \in \Lambda$ so that there are at most $C_n(\log_2 q)^2$ t's in $2^{k-1} \leq t \leq 2^{k+1}$ satisfying

$$\cdot\chi\left(\left[2^{(m-1/4)j}\left(t^{N((m,n))}\xi_{(m,n)} - a_{(m,n)}/q\right)\right]_{(m,n)\in\{(1,0)\}\cup\Lambda}\right)) \neq 0.$$

For Theorem 3.3, we begin by choosing an odd C^∞-function $\psi(s)$, defined on the real line, nonnegative for $s \geq 0$, and supported in $1/2 \leq |s| \leq 2$ such that

$$\sum_{p=-\infty}^{\infty} 2^j \psi\left(2^j s\right) = \frac{1}{s}.$$

We let (3.1)

$$\mu_{j,k}(\xi) = \sum_t \sum_s 2^{-j}\psi(2^{-j}s) \cdot 2^{-k}\psi(2^{-k}t) \cdot e^{-2\pi i (P(s,t)\cdot\xi)}.$$

and set

$$\varphi_j^1(\xi) = \int 2^{-j}\psi(2^{-j}x) e^{-2\pi i (x\xi_1 + \sum_{(m,n)\in\Lambda} x^m \cdot \xi_{(m,n)})} dx,$$

$$\varphi_j^2(\xi) = \int 2^{-j}\psi(2^{-j}x) e^{-2\pi i (x\xi_1)} dx,$$

$$\varphi_j^3(\xi) = \int 2^{-j}\psi(2^{-j}x) e^{-2\pi i (\sum_{(m,n)\in\Lambda} x^m \cdot \xi_{(m,n)})} dx.$$

Lemma 3.2
(1)

$$|\varphi_j^1(\xi) - \varphi_j^3(\xi)| \leq C \cdot 2^j \xi_{(1,0)}$$

(2) If $s \leq 2^j$,

$$\int_{|x|\leq 2^{j-1}+s} 2^{-j}\psi(2^{-j}x) e^{-2\pi i (x\xi_1)} dx \leq C \cdot (s/2^j) \cdot 2^{j+1} \xi_{(1,0)}.$$

Proof of Lemma 3.2. By mean value theorem, It is easy to show. □

Theorem 3.3 $\mu(\xi)$ is uniformly bounded according to $\xi_{(1,0)}, \xi_{(0,1)}, \gamma$ by constant C if $|D| \geq 2$ or all (m,n) are not on one line passing through the origin ($\Lambda \subseteq \{(m,n)|$m,n are even natural number$\}$). The constant C may depend only on polynomial and d..

Proof of Theorem 3.3. We first consider the case when if (m_1, n_1) and $(m_2, n_2) \in \Lambda, M(m_1, n_1) \neq M(m_2, n_2)$ and $N(m_1, n_1) \neq N(m_2, n_2)$. Since m,n are both even,

$$|\mu_{j,k}(\xi)| = |\sum_t \sum_s 2^{-j}\psi(2^{-j}s) \cdot 2^{-k}\psi(2^{-k}t) e^{-2\pi i (P(s,t)\xi_{(m,n)})}|$$

$$= |\sum_t \sum_s 2^{-k}\psi(2^{-k}t) \cdot e^{-2\pi i (t\xi_{(0,1)})} \cdot (2^{-j}\psi(2^{-j}s) \cdot e^{-2\pi i (s\xi_{(1,0)} + \sum_{(m,n)\in\Lambda} a_{m,n} s^m t^n \xi_{(m,n)})}$$

$$-2^{-j}\psi(2^{-j}s) \cdot e^{-2\pi i (\sum_{(m,n)\in\Lambda} a_{m,n} s^m t^n \xi_{(m,n)})})| \cdots (3.2)$$

If we apply Proposition2.1, then for any $D_1 \geq 2$,

$$\sum_s 2^{-j}\psi(2^{-j}s) e^{-2\pi i (s\xi_{(1,0)} + \sum_{(m,n)\in\Lambda} a_{m,n} s^m t^n \xi_{(m,n)})}$$

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$$= \sum_{q=1}^{q\le(j+1)^{C_dD_1}} \sum_{a\in P_q} S(a/q)\cdot \varphi_j^1(P_t(s,t)\otimes\xi - a/q)$$
$$\cdot\chi\left(\left[2^{(m-1/4)j}\left(t^{N((m,n))}\xi_{(m,n)} - a_{(m,n)}/q\right)\right]_{(m,n)\in\{(1,0)\}\cup\Lambda}\right) + \widetilde{E}_j^1(\xi),$$
$$\sum_t e^{-2\pi i(t\xi_{(0,1)})}\cdot 2^{-k}\psi(2^{-k}t)$$
$$= \sum_{q=1}^{q\le(k+1)^{C_dD_1}} \sum_{a\in P_q} S(a/q)\cdot \varphi_k^2(\xi_{(0,1)} - a/q)$$
$$\cdot\chi\left(\left[2^{(n-1/4)k}\left(s^{M((m,n))}\xi_{(m,n)} - a_{(m,n/q)}\right)\right]_{(m,n)\in\{(0,1)\}}\right) + \widetilde{E}_k^2(\xi),$$
$$\sum_s e^{-2\pi i\sum_{(m,n)\in\Lambda} a_{m,n}s^mt^n\xi_{(m,n)}}\cdot 2^{-j}\psi(2^{-j}s)$$
$$= \sum_{q=1}^{q\le(j+1)^{C_dD_1}} \sum_{a\in P_q} S(a/q)\cdot \varphi_j^3(P_t^\Lambda(s,t)\otimes\xi - a/q)$$
$$\cdot\chi\left(\left[2^{(m-1/4)j}\left(t^{N((m,n))}\xi_{(m,n)} - a_{(m,n)}/q\right)\right]_{(m,n)\in\{(m,n)\}}\right) + \widetilde{E}_j^3(\xi)$$

(For all i, $|\widetilde{E}_j^i(\xi)| \le C_{D_1}(j+1)^{-D_1}$, for any $D_1 \ge 2$).

(1) We first consider the case when $nk \le (m-1/4)j - C_dD_1\log_2(j+1) - \log_2 10$. We let $E_{m,n} = \{(j,k)|nk \le (m-1/4)j - C_dD_1\log_2(j+1) - \log_2 10\}$). By applying summation by part with Lemma 3.2 (2),(2.2) and applying Lemma 3.1, (2.2) to χ part and applying the fact that $\sum_t |2^{-k}\psi(-2^kt)|\cdot C_{D_1}(j+1)^{-D_1} \le 2\cdot C_{D_1}(j+1)^{-D_1}$ to $\widetilde{E}_j^1(\xi)$, (3.1) is bounded by

$$(|\sum_{q=1}^{q\le(k+1)^{C_dD_1}} \sum_{a\in P_q} S(a/q)\cdot\min\{C\cdot 2^k|\xi_{(0,1)} - a/q|, C_{1,1}/(2^k|\xi_{(0,1)} - a/q|)\}$$
$$\cdot\chi\left(\left[2^{(n-1/4)k}\left(s^{M((m,n))}\xi_{(m,n)} - a_{(m,n)}/q\right)\right]_{(m,n)\in\{(0,1)\}}\right) + C_{D_1}(k+1)^{-D_1}|)$$
$$\cdot(4C_n\log_2 q\cdot \sum_{q=1}^{q\le(j+1)^{C_dD_1}} \sum_{a\in P_q} |S(a/q)|\cdot\frac{C_{d,A}}{(1+2^j\cdot\beta_{a/q,t}^{(1,0)} + \sum_{(m,n)\in\Lambda}(2^{mj}\cdot\beta_{a/q,t}^{(m,n)}))^{1/d-1}}$$
$$\cdot\chi(\left[2^{(m-1/4)j}\left(t^{N((m,n))}\xi_{(m,n)} - a_{(m,n)}/q\right)\right]_{(m,n)\in\{(1,0)\}})) + 2\cdot C_{D_1}(j+1)^{-D_1},$$

and by (3.2) and Lemma 3.2, (3.1) is also bounded by

$$(|\sum_{q=1}^{q\le(k+1)^{C_dD_1}} \sum_{a\in P_q} S(a/q)\cdot\min\{C\cdot 2^k|\xi_{(0,1)} - a/q|, C_{1,1}/(2^k|\xi_{(0,1)} - a/q|)\}$$

$$\cdot\chi\left(\left[2^{(n-1/4)k}\left(s^{M((m,n))}\xi_{(m,n)}-a_{(m,n)}/q\right)\right]_{(m,n)\in\{(0,1)\}}\right)+C_{D_1}(k+1)^{-D_1}|)$$

$$\cdot(4C_n\log_2 q\cdot\sum_{q=1}^{q\leq(j+1)^{C_dD_1}}\sum_{a\in P_q}|S(a/q)|\cdot(C2^j\beta^{(1,0)}_{a/q,t,0})$$

$$\cdot\chi(\left[2^{(m-1/4)j}\left(t^{N((m,n))}\xi_{(m,n)}-a_{(m,n)}/q\right)\right]_{(m,n)\in\{(1,0)\}}))+2\cdot C_{D_1}(j+1)^{-D_1}.$$

So (3.1) is bounded by (3.3)

$$(|\sum_{q=1}^{q\leq(k+1)^{C_dD_1}}\sum_{a\in P_q}S(a/q)\cdot\min\{C\cdot 2^k|\xi_{(0,1)}-a/q|,C_{1,1}/(2^k|\xi_{(0,1)}-a/q|)\}$$

$$\cdot\chi\left(\left[2^{(n-1/4)k}\left(s^{M((m,n))}\xi_{(m,n)}-a_{(m,n)}/q\right)\right]_{(m,n)\in\{(0,1)\}}\right)+C_{D_1}(k+1)^{-D_1}|)$$

$$\cdot(4C_n\log_2 q\cdot\sum_{q=1}^{q\leq(j+1)^{C_dD_1}}\sum_{a\in P_q}|S(a/q)|\cdot\min(C2^j\beta^{(1,0)}_{a/q,t},\frac{C_{d,A}}{(2^j\cdot\beta^{(1,9)}_{a/q,t})^{1/d-1}})$$

$$\cdot\chi\left(\left[2^{(m-1/4)j}\left(t^{N((m,n))}\xi_{(m,n)}-a_{(m,n)}/q\right)\right]_{(m,n)\in\{(1,0)\}}\right))+2\cdot C_{D_1}(j+1)^{-D_1}.$$

So by Lemma 1.8 with the fact that q runs dydically and summing j first and choosing $D_1=3$, it is easy to show that

$$\sum_{(j,k)\in E_{m,n}}|\mu_{j,k}(\xi)|\leq O(1).$$

(2) We will consider the case when $mj\leq(n-1/4)k-C_dD_1\log_2(k+1)-\log_2 10$. We set $F_{m,n}=\{(j,k)|mj\leq(n-1/4)k-C_dD_1\log_2(k+1)-\log_2 10\}$. Since m and n are both even, we can do the exactly same thing with exchanging s and t. So (3.1) is also bounded by

$$(|\sum_{q=1}^{q\leq(j+1)^{C_dD_1}}\sum_{a\in P_q}S(a/q)\cdot\min\{C\cdot 2^j|\xi_{(1,0)}-a/q|,C_{1,1}/(2^j|\xi_{(1,0)}-a/q|)\}$$

$$\cdot\chi\left(\left[2^{(m-1/4)j}\left(t^{N((m,n))}\xi_{(m,n)}-a_{(m,n)}/q\right)\right]_{(m,n)\in\{(1,0)\}}\right)+C_{D_1}(k+1)^{-D_1}|)$$

$$\cdot(4C_m\log_2 q\cdot\sum_{q=1}^{q\leq(k+1)^{C_dD_1}}\sum_{a\in P_q}|S(a/q)|\cdot\min(C2^k\beta^{(0,1)}_{a/q,s},\frac{C_{d,A}}{(2^k\cdot\beta^{(0,1)}_{a/q,s})^{1/d-1}})$$

$$\cdot\chi\left(\left[2^{(n-1/4)k}\left(s^{M((m,n))}\xi_{(m,n)}-a_{(m,n)}/q\right)\right]_{(m,n)\in\{(0,1)\}}\right))+2\cdot C_{D_1}(j+1)^{-D_1}.$$

So by Lemma 1.8 with the fact that q runs dydically and summing k first and choosing $D_1 = 3$, it is easy to show that

$$\sum_{(j,k)\in F_{m,n}} |\mu_{j,k}(\xi)| \leq O(1).$$

(3) Since our the assumption($|\Lambda| \neq 1$), we can do the same things to another $(m,n) \in \Lambda$. For $(m_1,n_1),(m_2,n_2)$, since we only need to condsider $E^c_{m_1,n_1} \cap F^c_{m_1,n_1}$ and $E^c_{m_2,n_2} \cap F^c_{m_2,n_2}$ by (1),(2), it suffices to prove that $(m_1,n_1),(m_2,n_2) \in \Lambda$ satisfy

$$m_1 n_2 \leq (n_1 - 1/4)(m_2 - 1/4)$$

or

$$m_2 n_1 \leq (n_2 - 1/4)(m_1 - 1/4).$$

Since above can be changed as

$$n_2/(m_2 - 1/4) \leq (n_1 - 1/4)/m_1,$$

$$n_1/(m_1 - 1/4) \leq (n_2 - 1/4)/m_2,$$

we can understand this as the absolute values of the slopes of the line joining $(1/4,0)$ to (m_2,n_2), $(0,1/4)$ to (m_1,n_1) and $(1/4,0)$ to (m_1,n_1), $(0,1/4)$ to (m_2,n_2). On the other hand, if all (m,n) are not on one line passing through the origin, there are small number ϵ which make $(m_1,n_1),(m_2,n_2) \in \Lambda$ satisfy

$$m_1 n_2 \leq (n_1 - \epsilon)(m_2 - \epsilon)$$

or

$$m_1 n_2 \leq (n_1 - \epsilon)(m_2 - \epsilon).$$

However, We can replace $1/4$ in the proof of the Proposition 2.1 with ϵ and the constant depends only on $(m_1,n_1),(m_2,n_2) \in \Lambda$ which means the constant depends only on Λ.

(4)We will consider the cases when there exist m_1 and n_2 which satisfy that there are many n_r satisfying $(m_1,n_r) \in \Lambda$ and there are many $m_{r'}$ satisfying $(m_{r'},n_2) \in \Lambda$. In this cases, We will do the same things above exept that we replace $P_t(s,t)\otimes\xi$, $P_s(s,t)\otimes\xi$, with $P'_t(s,t)\otimes\xi$, $P'_s(s,t)\otimes\xi$ in the Proposition 2.1. We define $P'_t(s,t)\otimes\xi$, $P'_s(s,t)\otimes\xi$

$$P'_t(s,t)\otimes\xi = (\cdots, \sum_r (t^{n_r})\xi_(m_1,n_r),\cdots),$$

$$P'_s(s,t)\otimes\xi = (\cdots, \sum_r (s^{m_{r'}})\xi_(m_{r'},n_2),\cdots).$$

Since when we apply the Lemma 3.1, the biggest n_r and m_r dominate, we only need to consider the D(the corners of $\Pi^{Reverse}$). So in this cases, if $|D| \geq 2$, the Theorem 3.3 is true because any two corners of $\Pi^{Reverse}$ can not be on line passing through the origin. And if $|D| = 1$ and all (m,n) are not on one line passing through the origin, Theorem 3.3 is also true as we can see above. $\square$

4. Double Hilbert Exponetial sums with torsion.

In this chapter, we will remove all conditions of Λ of Theorem 1.4 and 3.3 except $\Lambda \subseteq \{(m,n)|$m,n are even natural number$\}$, but we will consider only the case when if (m_1,n_1) and $(m_2,n_2) \in \Lambda, M(m_1,n_1) \neq M(m_2,n_2)$ and $N(m_1,n_1) \neq N(m_2,n_2)$. We will use $\mathbb{N}^2$ lattice points in j-k plane and Proposition 2.1 which are shown in chapter 3 and integralable function $\log x$ in local area to show that Fourier multiplier of discrete double Hilbert transform is bounded by terms about log and integral of this with torsion is bounded by constants.

For each integers $p \geq 1$ and $n, N \geq 2$, we denote $J_{p,n}(N)$ by the number of integral solutions for the following system

$$X_1^i + \ldots + X_p^i = X_{p+1}^i + \ldots + X_{2p}^i, \quad 1 \leq i \leq n$$

with $1 \leq X_1, \ldots, X_{2p} \leq N$. The number $J_{p,n}(N)$ has the following analytic representation

$$J_{p,n}(N) = \int_{[0,1]^n} \left| \sum_{j=1}^{N} e^{-2\pi i \left(x_1 j + x_2 j^2 + \ldots + x_n j^n\right)} \right|^{2p} dx_1 \ldots dx_n.$$

We introduce main conjecture in Vinogradov's Mean Value Theorem. Apart from the N^ϵ loss, this bound has been known to be sharp. The case $n = 2$ follows easily from elementary estimates for the divisor function.

THEOREM 4.1. For each $p \geq 1$ and $n, N \geq 2$ we have the upper bound

$$J_{p,n}(N) \leq_\epsilon N^{p+\epsilon} + N^{2p - \frac{n(n+1)}{2} + \epsilon}.$$

This is proved by Jean Bourgain, Ciprian Demeter, Larry Guth [7] and a second proof was later provided by Wooley in [9].

We also know following results.

THEOREM 4.2. For $p \geq 2$ and $a_t \in \mathbb{C}$, we have

$$\left\| \sum_{t=-N}^{N} a_t e^{-2\pi i \left(x_1 t^{\beta_1} + \cdots + x_n t^{\beta_n}\right)} \right\|_{L^p([0,1]^n)} \leq_\epsilon N^\epsilon \left(1 + N^{\frac{1}{2} - \frac{n(n+1)}{2p}}\right) \|a_t\|_{l^2}.$$

But following conjectures have not been proved yet.

Conjecture 4.3. For each $p \geq 2$ and $a_t \in \mathbb{C}$

$$\left\| \sum_{t=-N}^{N} a_t e^{-2\pi i \left(x_1 t^{\beta_1} + \cdots + x_n t^{\beta_n}\right)} \right\|_{L^p([0,1]^n)} \leq_\epsilon N^\epsilon \left(1 + N^{\frac{1}{2} - \frac{\beta_1 + \cdots + \beta_n}{p}}\right) \|a_t\|_{\ell^2}.$$

Conjecture 4.4 is double case of Conjecture 4.3

Conjecture 4.4. For each $p \geq 2$ and $a_t, b_s \in \mathbb{C}$

$$\left\| \sum_{s=-M}^{M} \sum_{t=-N}^{N} a_t b_s e^{-2\pi i \left(x_1 s^{\alpha_1} t^{\beta_1} + \cdots + x_n s^{\alpha_n} t^{\beta_n}\right)} \right\|_{L^p([0,1]^n)}$$

$$\leq_\epsilon N^\epsilon \left(1 + N^{\frac{1}{2} - \frac{\beta_1 + \cdots + \beta_n}{p}}\right) M^\epsilon \left(1 + M^{\frac{1}{2} - \frac{\alpha_1 + \cdots + \alpha_n}{p}}\right) \|a_t\|_{\ell^2} \|b_s\|_{\ell^2}.$$

We define Double Hilbert Exponetial sums for even polynomials with torsion.

$$\left\| \sum_{s=-\infty}^{\infty} \sum_{t=-\infty}^{\infty} \frac{1}{s} \frac{1}{t} e^{-2\pi i \left(x_1 s^1 + x_2 t^1 + x_3 s^{\alpha_3} t^{\beta_3} \cdots + x_n s^{\alpha_n} t^{\beta_n}\right)} \right\|_{L^p([0,1]^n)}$$

$(\alpha_n, \beta_n$ are even).

This is special case of 4.4 when $a_t = 1/t, b_s = 1/s$ and $\alpha_1 = 1, \beta_1 = 0, \alpha_2 = 0, \beta_2 = 1, \alpha_n, \beta_n$ are evens and $N \to \infty$, $M \to \infty$. We get better boundedness of this without N^ϵ, M^ϵ in conjecture 4.4 by Theorem 3.4 which is our main results.

Theorem 4.5

$$\left\| \sum_{s=-\infty}^{\infty} \sum_{t=-\infty}^{\infty} \frac{1}{s} \frac{1}{t} e^{-2\pi i \left(x_1 s^1 + x_2 t^1 + x_3 s^{\alpha_3} t^{\beta_3} \cdots + x_n s^{\alpha_n} t^{\beta_n}\right)} \right\|_{L^p([0,1]^n)} \leq C_{p,n}.$$

$(\alpha_n, \beta_n$ are even.)

Proof of Thoerem 4.5. The following Theorem 4.9 means exactly Theorem 4.5 □

Let $\xi_1 = \xi_{(1,0)}$ and $\xi_2 = \xi_{(0,1)}$ for comfort.

Lemma 4.6. For any positive small number ε and $\beta \in \mathbb{R}^d$ and $k_0, k \in \mathbb{N}$ satisfying $\beta_1 \approx 2^{-k_0}$, if let $\mathrm{A}_{k,k_0} = ([1, k_0 - (1+\varepsilon)\log_2 k] \cup [k_0 + d(1+\varepsilon)\log_2 k, \infty]) \cap \mathbb{Z} \cap [k, \infty]$, *then* $\sum_{j \in A_{k,k_0}} \varphi_j(\beta) \leq k^{-(1+\varepsilon)}$.

Proof of Lemma 4.6. $2^{j-k_0} \leq k^{-(1+\epsilon)}$ means $j - k_0 \leq -(1+\epsilon)\log_2 k$ and $2^{j-k_0} \geq k^{+d(1+\epsilon)}$ means $j - k_0 \geq +d(1+\epsilon)\log_2 k$. If we sum $\varphi_j(\beta)$ only on j satisfying $|2^j \beta_1| \leq k^{-(1+\epsilon)}$ or $|2^j \beta_1| \geq k^{+d(1+\epsilon)}$, We can get the boundedness in Lemma 4.6 by the two facts that

$$\varphi_j(\xi) = \int L_j(x) e^{-2\pi i Q(x)\cdot\xi} \mathrm{dx} \leq C \left|2^j \xi_1\right|,$$

$$\varphi_j(\xi) = \int L_j(s)e^{-2\pi i P(s)\cdot\xi}\mathrm{ds} \leq \mathrm{C}(d,A)\left(1 + \sum_{1\leq|\alpha|\leq A} 2^{|\alpha|j}\,|\xi_\alpha|\right)^{-\frac{1}{d}}. \quad \square$$

Lemma 4.7 If we let $K_{k_0} = \{j \mid j \in \bigcup_k (A^C_{k,k_0} \cap [k,\infty])\}$, K_{k_0} has at most $2d(1+\epsilon)\log_2 M(d)k_0$ elements.

Proof of Lemma 4.7. We let $k^{\#} \leq k_0$ a solution of equation $d(1+\epsilon)\log_2 k = k - k_0$. By definition of A_{k,k_0}, smallest integer j in $\bigcup_k (A^C_{k,k_0} \cap [k,\infty])\}$ is bigger than $k_0 - d(1+\epsilon)\log_2 k^{\#}$ and largest integer j in A_{k,k_0} is smaller than $k_0 + d(1+\epsilon)\log_2 k^{\#}$. So there are at most $2d(1+\epsilon)\log_2 k^{\#}$ elements in K_{k_0}. Meanwhile it is easy to check that there is $M(d)$ which satisfies $d(1+\epsilon)\log_2 M(d)k_0 \leq M(d)k_0 - k_0$.($M(d)$ depends only on dimension d.) So $k^{\#} \leq M(d)k_0$. And this means K_{k_0} has at most $2d(1+\epsilon)\log_2 M(d)k_0$ elements. $\square$

To prove proposition 4.8, we shall apply the above results for $k \leq j$ case, and for the other case ($j \leq k$ case), we can apply the same above results. For this we define $J_{j_0} = \{j \mid (j,k) \in \bigcup_j (A^C_{j,j_0} \cap [j,\infty])\}$ with the same manner in Lemma 3.1 and Lemma 3.2. We also define $\sum_{k\geq j+1}^{\infty} \mu_{k(s)}(\xi)$ with the same manner in Proposition 2.2, Proposition 2.3.

Propostion 4.8. There are constants C,D,E which depend only on $|\Lambda|$ satisfying

$$|\mu(\xi)| = O\Big(\sum_q^{\infty} \sum_{a_1\in P_q} (\chi\left(\left[2^{(1/4)q^{1/C_d D_1}}(\xi_1 - a_1/q)\right]\right) \cdot |\log_2(\xi_1 - a_1/q)|)$$

$$+ \sum_q^{\infty} \sum_{a_2\in P_q} (\chi\left(\left[2^{(1/4)q^{1/C_d D_1}}(\xi_2 - a_2/q)\right]\right) \cdot |\log_2(\xi_2 - a_2/q)|) + C \quad \Big).$$

Proof of Proposition 4.8. For any q, there are $k_{a_1,q}$ and $j_{a_2,q}$ satisfying

$$|\xi_1 - a_1/q| \approx 2^{-k_{a_1,q}},$$

$$|\xi_2 - a_2/q| \approx 2^{-k_{a_1,q}}.$$

Then

$$|\mu(\xi)| = |\sum_{(k,j)\in\mathbb{N}^2} \mu_{k,j}(\xi)| \leq \sum_q \sum_{j\in K_{k_{a_1,q}}} \sum_{k\in[1,j]} |\mu_{j,k}(\xi)|$$

$$+ \sum_q \sum_{k\in J_{j_{a_2,q}}} \sum_{j\in[1,k]} |\mu_{k,j}(\xi)| + |\sum_{(k,j)\in remainder} \mu_{k,j}(\xi)|.$$

By remark (1) in chapter 1, for any k, there is D satisfying $\sum_j |\mu_{k,j}(\xi)| \le D$ and for any j, there is E satisfying $\sum_k |\mu_{k,j}(\xi)| \le E$. So

$$|\mu(\xi)| = |\sum_{(k,j)\in\mathbb{N}^2} \mu_{k,j}(\xi)|$$

$$\le \ D \cdot \sum_q |K_{k_{a_1,q}}| + E \cdot \sum_q |K_{k_{a_1,q}}| + | \sum_{(k,j)\in remainder} \mu_{k,j}(\xi)|.$$

By Lemma 4.7,

$$D \cdot \sum_q |K_{k_{a_1,q}}| + E \cdot \sum_q |J_{j_{a_2,q}}| + | \sum_{(k,j)\in remainder} \mu_{k,j}(\xi)|$$

$$\le D \cdot \sum_q \log_2 M(d) k_{a_1,q} + E \cdot \sum_q \log_2 M(d) j_{a_2,q} + | \sum_{(k,j)\in remainder} \mu_{k,j}(\xi)|.$$

For $|\sum_{(k,j)\in remainder} \mu_{k,j}(\xi)|$, first we will sum (k,j) of case $k \le j$. Let $\beta_{1,q} = \xi_1 - a_1/q$,$\beta_{2,q} = \xi_2 - a_2/q$ and we now use fomula (2.3) with Lemma 4.6. Then we get

$$| \sum_{(k,j)\in remainder, k\le j} \mu_{k,j}(\xi)|$$

$$\le \sum_k^\infty \sum_{t\approx 2^k} |1/t \cdot (\sum_{q=1}^{q\le k^{C_d D_1}} \sum_{a\in P_q} S(a/q) \cdot \sum_{j\in remainder} \varphi_j(P_t(s,t)\otimes \xi - a_1/q)$$

$$\cdot \chi\left(\left[2^{(|(m,n)|-1/4)k}\left(t^{N(m,n)}\xi_{(m,n)} - a_{1,(m,n)}/q\right)\right]_{(m,n)\in\Lambda^t}\right)| + \sum_k^\infty \widetilde{E}_k(\xi)$$

$$\le \sum_k^\infty |(\sum_{q=1}^{q\le k^{C_d D_1}} \sum_{a\in P_q} S(a/q) \sum_{j\in remainder} \min\{\mathrm{C}(d,A)\left(1+2^j|\beta_{1,q}|\right)^{-\frac{1}{d}}, |2^j\beta_{1,q}|\}$$

$$\cdot \chi\left(\left[2^{(3/4)q^{1/C_d D_1}}(\xi_1 - a_1/q)\right]\right)| + \sum_k^\infty \widetilde{E}_k(\xi)$$

$$\le |(\sum_{q=1}^\infty \sum_{a\in P_q} S(a/q) \cdot \chi\left(\left[2^{(3/4)q^{1/C_d D_1}}(\xi_1 - a_1/q)\right]\right)| \cdot | \sum_k C(d) k^{-(1+\epsilon)}| + \sum_k^\infty \widetilde{E}_k(\xi).$$

So it is easy to show that there is C satisfying

$$| \sum_{(k,j)\in remainder, k\le j} \mu_{k,j}(\xi)| \le C$$

by Lemma 1.8 with the fact that there is constant $C(C_d D_1)$ satisfying $|\xi_1 - a_1/q_1| \le C(C_d D_1)\cdot 1/q^2$ which follows from $\chi\left(\left[2^{(1/4)q^{1/C_d D_1}}(\xi_1 - a_1/q)\right]\right)$.

For the other case: $j \leq k$, we can apply same idea.
So it is clear that

$$|\mu(\xi)| = O(\quad \sum_{q}^{\infty} \sum_{a_1 \in P_q} (\chi\left(\left[2^{(1/4)q^{1/C_d D_1}}(\xi_1 - a_1/q)\right]\right) \cdot |\log_2(\xi_1 - a_1/q)|)$$

$$+ \sum_{q}^{\infty} \sum_{a_2 \in P_q} (\chi\left(\left[2^{(1/4)q^{1/C_d D_1}}(\xi_2 - a_2/q)\right]\right) \cdot |\log_2(\xi_2 - a_2/q)|) + C \quad).\square$$

Theorem 4.9 There are C_p satisfying $\int_{[0,1]^d} |\mu(\xi)|^p d\xi \leq C_p$ for all $p \in [1, \infty)$.

Proof of Theorem 4.9. By the fact that $\int \log x dx = xlogx - x + c$ and $|P_q| \leq q - 1$ and Proposition 4.8,

$$\int_{[0,1]^d} \mu(\xi) d\xi = O(\sum_q (q-1) \cdot q \cdot 2^{-(1/4)q^{1/C_d D_1}} + C).$$

So there is C_1 satisfying $\int_{[0,1]^d} |\mu(\xi)| d\xi \leq C_1$. With the same above methods, we can show that there are C_p satisfying $\int_{[0,1]^d} |\mu(\xi)|^p d\xi \leq C_p$ for all $p \in [1, \infty)$ easily. $\square$

5. Reference

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