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When the Riemann Hypothesis might be false

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Abstract Robin criterion states that the Riemann Hypothesis is true if and only if the inequality $\sigma(n) < e^{\gamma} \times n \times \log\log n$ holds for all natural numbers $n > 5040$, where $\sigma(n)$ is the sum-of-divisors function and $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. Let $q_1 = 2, q_2 = 3, \ldots, q_m$ denote the first m consecutive primes, then an integer of the form $\prod_{i=1}^{m} q_i^{a_i}$ with $a_1 \geq a_2 \geq \cdots \geq a_m \geq 0$ is called an Hardy-Ramanujan integer. If the Riemann Hypothesis is false, then there are infinitely many Hardy-Ramanujan integers $n > 5040$ such that Robin inequality does not hold and $n < (4.48311)^m \times N_m$, where $N_m = \prod_{i=1}^{m} q_i$ is the primorial number of order m.

Keywords Riemann hypothesis · Robin inequality · sum-of-divisors function · prime numbers

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1 Introduction

In mathematics, the Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$ [4]. As usual $\sigma(n)$ is the sum-of-divisors function of n [2]:

$$\sum_{d|n} d$$

where $d \mid n$ means the integer d divides to n and $d \nmid n$ means the integer d does not divide to n. Define $f(n)$ to be $\frac{\sigma(n)}{n}$. Say $\mathsf{Robins}(n)$ holds provided

$$f(n) < e^{\gamma} \times \log\log n.$$

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The constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant, and log is the natural logarithm. The importance of this property is:

Theorem 1.1 *If the Riemann Hypothesis is false, then there are infinitely many natural numbers $n > 5040$ such that $\mathsf{Robins}(n)$ does not hold [4].*

We recall that an integer n is said to be square free if for every prime divisor q of n we have $q^2 \nmid n$ [2]. $\mathsf{Robins}(n)$ holds for all natural numbers $n > 5040$ that are square free [2]. In addition, we show that $\mathsf{Robins}(n)$ holds for some $n > 5040$ when $\frac{\pi^2}{6} \times \log\log n' \leq \log\log n$ such that n' is the square free kernel of the natural number n. Let $q_1 = 2, q_2 = 3, \ldots, q_m$ denote the first m consecutive primes, then an integer of the form $\prod_{i=1}^{m} q_i^{a_i}$ with $a_1 \geq a_2 \geq \cdots \geq a_m \geq 0$ is called an Hardy-Ramanujan integer [2]. Based on the theorem 1.1, we know this result:

Theorem 1.2 *If the Riemann Hypothesis is false, then there are infinitely many natural numbers $n > 5040$ which are an Hardy-Ramanujan integer and $\mathsf{Robins}(n)$ does not hold [2].*

We prove if the Riemann Hypothesis is false, then there are infinitely many Hardy-Ramanujan integers $n > 5040$ such that $\mathsf{Robins}(n)$ does not hold and $n < (4.48311)^m \times N_m$, where $N_m = \prod_{i=1}^{m} q_i$ is the primorial number of order m.

2 A Central Lemma

These are known results:

Lemma 2.1 *[2]. For $n > 1$:*

$$f(n) < \prod_{q \mid n} \frac{q}{q-1}. \tag{2.1}$$

Lemma 2.2 *[3].*

$$\prod_{k=1}^{\infty} \frac{1}{1 - \frac{1}{q_k^2}} = \zeta(2) = \frac{\pi^2}{6}. \tag{2.2}$$

The following is a key lemma. It gives an upper bound on $f(n)$ that holds for all natural numbers n. The bound is too weak to prove $\mathsf{Robins}(n)$ directly, but is critical because it holds for all natural numbers n. Further the bound only uses the primes that divide n and not how many times they divide n.

Lemma 2.3 *Let $n > 1$ and let all its prime divisors be $q_1 < \cdots < q_m$. Then,*

$$f(n) < \frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i}.$$

Proof We use that lemma 2.1:

$$f(n) < \prod_{i=1}^{m} \frac{q_i}{q_i - 1}.$$

Now for $q > 1$,

$$\frac{1}{1-\frac{1}{q^2}} = \frac{q^2}{q^2-1}.$$

So

$$\frac{1}{1-\frac{1}{q^2}} \times \frac{q+1}{q} = \frac{q^2}{q^2-1} \times \frac{q+1}{q}$$
$$= \frac{q}{q-1}.$$

Then by lemma 2.2,

$$\prod_{i=1}^{m} \frac{1}{1-\frac{1}{q_i^2}} < \zeta(2) = \frac{\pi^2}{6}.$$

Putting this together yields the proof:

$$f(n) < \prod_{i=1}^{m} \frac{q_i}{q_i-1}$$
$$\leq \prod_{i=1}^{m} \frac{1}{1-\frac{1}{q_i^2}} \times \frac{q_i+1}{q_i}$$
$$< \frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i+1}{q_i}.$$

3 A Particular Case

We can easily prove that $\mathsf{Robins}(n)$ is true for certain kind of numbers:

Lemma 3.1 $\mathsf{Robins}(n)$ *holds for* $n > 5040$ *when* $q \leq 5$*, where* q *is the largest prime divisor of* n*.*

Proof Let $n > 5040$ and let all its prime divisors be $q_1 < \cdots < q_m \leq 5$, then we need to prove

$$f(n) < e^{\gamma} \times \log\log n$$

that is true when

$$\prod_{i=1}^{m} \frac{q_i}{q_i-1} \leq e^{\gamma} \times \log\log n$$

according to the lemma 2.1. For $q_1 < \cdots < q_m \leq 5$,

$$\prod_{i=1}^{m} \frac{q_i}{q_i-1} \leq \frac{2\times 3\times 5}{1\times 2\times 4} = 3.75 < e^{\gamma} \times \log\log(5040) \approx 3.81.$$

However, we know for $n > 5040$

$$e^{\gamma} \times \log\log(5040) < e^{\gamma} \times \log\log n$$

and therefore, the proof is complete when $q_1 < \cdots < q_m \leq 5$.

4 Helpful Lemmas

For every prime number $p_n > 2$, we define the sequence $Y_n = \frac{e^{\frac{1}{2\times\log(p_n)}}}{(1-\frac{1}{\log(p_n)})}$.

Lemma 4.1 *For every prime number $p_n > 2$, the sequence Y_n is strictly decreasing.*

Proof For every real value $x \geq 3$, we state the function

$$f(x) = \frac{e^{\frac{1}{2\times\log(x)}}}{(1 - \frac{1}{\log(x)})}$$

which is equivalent to

$$f(x) = g(x) \times h(u)$$

where $g(x) = e^{\frac{1}{2\times\log(x)}}$ and $h(u) = \frac{u}{u-1}$ for $u = \log(x)$. We know that $g(x)$ decreases as $x \geq 3$ increases, Moreover, we note that $h(u)$ decreases as $u > 1$ increases where $u = \log(x) > 1$ for $x \geq 3$. In conclusion, we can see that the function $f(x)$ is monotonically decreasing for every real value $x \geq 3$ and therefore, the sequence Y_n is monotonically decreasing as well. In addition, Y_n is essentially a strictly decreasing sequence, since there is not any natural number $n > 1$ such that $Y_n = Y_{n+1}$.

In mathematics, the Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{p\leq x} \log p$$

where $p \leq x$ means all the prime numbers p that are less than or equal to x.

Lemma 4.2 *[5]. For $x \geq 41$:*

$$\theta(x) > (1 - \frac{1}{\log(x)}) \times x.$$

Besides, we know that

Lemma 4.3 *[5]. For $x \geq 286$:*

$$\prod_{q\leq x} \frac{q}{q-1} < e^{\gamma} \times (\log x + \frac{1}{2\times\log(x)}).$$

We will prove another important inequality:

Lemma 4.4 *Let $q_1, q_2, \ldots, q_m$ denote the first m consecutive primes such that $q_1 < q_2 < \cdots < q_m$ and $q_m > 286$. Then*

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} < e^{\gamma} \times \log\left(Y_m \times \theta(q_m)\right).$$

Proof From the theorem 4.2, we know that

$$\theta(q_m) > (1 - \frac{1}{\log(q_m)}) \times q_m.$$

In this way, we can show that

$$\begin{aligned}\log(Y_m \times \theta(q_m)) &> \log\left(Y_m \times (1 - \frac{1}{\log(q_m)}) \times q_m\right) \\ &= \log q_m + \log\left(Y_m \times (1 - \frac{1}{\log(q_m)})\right).\end{aligned}$$

We know that

$$\begin{aligned}\log\left(Y_m \times (1 - \frac{1}{\log(q_m)})\right) &= \log\left(\frac{e^{\frac{1}{2\times\log(q_m)}}}{(1 - \frac{1}{\log(q_m)})} \times (1 - \frac{1}{\log(q_m)})\right) \\ &= \log\left(e^{\frac{1}{2\times\log(q_m)}}\right) \\ &= \frac{1}{2\times\log(q_m)}.\end{aligned}$$

Consequently, we obtain that

$$\log q_m + \log\left(Y_m \times (1 - \frac{1}{\log(q_m)})\right) \geq (\log q_m + \frac{1}{2\times\log(q_m)}).$$

Due to the theorem 4.3, we prove that

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} < e^{\gamma} \times (\log q_m + \frac{1}{2\times\log(q_m)}) < e^{\gamma} \times \log(Y_m \times \theta(q_m))$$

when $q_m > 286$.

5 Proof of Main Theorems

The next theorem implies that $\mathsf{Robins}(n)$ holds for a wide range of natural numbers $n > 5040$.

Theorem 5.1 *Let* $\frac{\pi^2}{6} \times \log\log n' \leq \log\log n$ *for some* $n > 5040$ *such that* n' *is the square free kernel of the natural number* n*. Then* $\mathsf{Robins}(n)$ *holds.*

Proof Let n' be the square free kernel of the natural number n. Let n' be the product of the distinct primes $q_1, \ldots, q_m$. By assumption we have that

$$\frac{\pi^2}{6} \times \log\log n' \leq \log\log n.$$

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For all square free $n' \leq 5040$, $\mathsf{Robins}(n')$ holds if and only if $n' \notin \{2,3,5,6,10,30\}$ [2]. However, $\mathsf{Robins}(n)$ holds for all natural numbers $n > 5040$ when $n' \in \{2,3,5,6,10,15,30\}$ due to the lemma 3.1. When $n' > 5040$, we know that $\mathsf{Robins}(n')$ holds and so

$$f(n') < e^{\gamma} \times \log\log n'.$$

By the previous lemma 2.3:

$$f(n) < \frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i}.$$

Suppose by way of contradiction that $\mathsf{Robins}(n)$ fails. Then

$$f(n) \geq e^{\gamma} \times \log\log n.$$

We claim that

$$\frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i} > e^{\gamma} \times \log\log n.$$

Since otherwise we would have a contradiction. This shows that

$$\frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i} > \frac{\pi^2}{6} \times e^{\gamma} \times \log\log n'.$$

Thus

$$\prod_{i=1}^{m} \frac{q_i + 1}{q_i} > e^{\gamma} \times \log\log n',$$

and

$$\prod_{i=1}^{m} \frac{q_i + 1}{q_i} > f(n'),$$

This is a contradiction since $f(n')$ is equal to

$$\frac{(q_1 + 1) \times \cdots \times (q_m + 1)}{q_1 \times \cdots \times q_m}.$$

Theorem 5.2 *If the Riemann Hypothesis is false, then there are infinitely many Hardy-Ramanujan integers $n > 5040$ such that $\mathsf{Robins}(n)$ does not hold and $n < (4.48311)^m \times N_m$, where $N_m = \prod_{i=1}^{m} q_i$ is the primorial number of order m.*

Proof Let $\prod_{i=1}^{m} q_i^{a_i}$ be the representation of some natural number $n > 5040$ as a product of primes $q_1 < \cdots < q_m$ with natural numbers as exponents $a_1, \ldots, a_m$. The primes $q_1 < \cdots < q_m$ must be the first m consecutive primes and $a_1 \geq a_2 \geq \cdots \geq a_m \geq 0$ since the natural number $n > 5040$ could be an Hardy-Ramanujan integer. We assume that $\mathsf{Robins}(n)$ does not hold. Indeed, we know there are infinitely many Hardy-Ramanujan integers such as $n > 5040$ when the Riemann Hypothesis is false according to the theorem 1.2. From the lemma 4.4, we know that

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} < e^{\gamma} \times \log(Y_m \times \theta(q_m)) = e^{\gamma} \times \log\log(N_m^{Y_m})$$

when $q_m > 286$. In this way, if $\mathsf{Robins}(n)$ does not hold, then $n < N_m^{Y_m}$ since by the lemma 2.1 we have that

$$f(n) < \prod_{i=1}^{m} \frac{q_i}{q_i - 1}.$$

That is the same as $n < N_m^{Y_m - 1} \times N_m$. We can check that $q_m^{Y_m - 1}$ is monotonically decreasing for all primes $q_m > 286$ due to the lemma 4.1. Certainly, the function

$$g(x) = x^{\left(\frac{e^{\frac{1}{2\times\log(x)}}}{(1-\frac{1}{\log(x)})} - 1\right)}$$

complies that its derivative is lesser than zero for all real numbers $x > 286$. Indeed, a function $g(x)$ of a real variable x is monotonically decreasing in some interval if the derivative of $g(x)$ is lesser than zero and the function $g(x)$ is continuous over that interval [1]. We know that q_m could comply with $q_m \geq 1000000!$ for infinitely many Hardy-Ramanujan integers $n > 5040$ such that $\mathsf{Robins}(n)$ does not hold, where $(\ldots)!$ is the factorial function. Certainly, if q_m would have an upper bound by some positive value, then there would not be infinitely many natural numbers $n > 5040$ which are an Hardy-Ramanujan integer and $\mathsf{Robins}(n)$ does not hold because of the theorem 5.1. Consequently, it is enough to show that

$$q_m^{Y_m - 1} \leq g(1000000!) < 4.48311$$

for all primes $q_m \geq 1000000!$. Moreover, we would obtain that

$$q_m^{Y_m - 1} > q_j^{Y_m - 1}$$

for every integer $1 \leq j < m$. Finally, we can state that $n < (4.48311)^m \times N_m$ since $N_m^{Y_m - 1} < (4.48311)^m$ when $n > 5040$ could be any of the infinitely many Hardy-Ramanujan integers such that $\mathsf{Robins}(n)$ does not hold and $q_m \geq 1000000!$.

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