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Article

# A Note on the Beal Conjecture

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## Abstract

Around 1637, Pierre de Fermat famously wrote in the margin of a book that he had a proof showing the equation  $a^n + b^n = c^n$  has no positive integer solutions for exponents  $n$  greater than 2. This statement, known as Fermat's Last Theorem, remained unproven for over three centuries despite efforts by countless mathematicians. In 1994, Andrew Wiles finally provided a rigorous proof using advanced techniques from elliptic curves and modular forms—methods far beyond those available in Fermat's era. Wiles was awarded the Abel Prize in 2016, with the citation describing his work as a “stunning advance” in mathematics. The Beal conjecture, formulated in 1993, generalizes Fermat's Last Theorem. It states that if  $A^x + B^y = C^z$  holds for positive integers  $A, B, C, x, y, z$  with  $x, y, z > 2$ , then  $A, B$ , and  $C$  must share a common prime factor. In this paper, we prove the Beal conjecture using elementary methods involving parametrization of quadratic Diophantine equations, divisibility properties, and congruence relations. Our approach potentially offers a solution closer in spirit to the mathematical tools available in Fermat's time.

**Keywords:** generalized Fermat equation; prime divisors; coprimality; congruence

**MSC:** 11D41; 11A41; 11A05; 11A07

## 1. Introduction

Fermat's Last Theorem, first stated by Pierre de Fermat in the 17<sup>th</sup> century, claims that there are no positive integer solutions to the equation  $a^n + b^n = c^n$  whenever  $n \in \mathbb{N}$  (where  $\mathbb{N}$  denotes the positive integers) is greater than 2. In a margin note left on his copy of Diophantus's *Arithmetica*, Fermat claimed to have discovered a proof which the margin was too small to contain [1]. Later mathematicians such as Leonhard Euler and Sophie Germain made significant contributions to its study [2,3], and 20<sup>th</sup>-century work by Ernst Kummer proved the theorem for a specific class of exponents [4]. However, a complete proof remained elusive.

In 1994, British mathematician Andrew Wiles announced a proof of Fermat's Last Theorem. His work was complex and multifaceted, drawing on advanced topics such as elliptic curves and modular forms—areas of mathematics that did not exist during Fermat's lifetime. After initial errors were corrected, Wiles's work was accepted as the long-awaited proof [5]. The proof was described as a “stunning advance” in the citation for Wiles's Abel Prize in 2016. His work also established much of the Taniyama–Shimura conjecture (now the modularity theorem) and introduced powerful modularity lifting techniques applicable to numerous other problems [6]. The sophisticated tools employed by Wiles differ vastly from any methods Fermat could have used in the 17<sup>th</sup> century.

In 1993, Andrew Beal, an American banker and amateur mathematician, formulated a conjecture while exploring generalizations of Fermat's Last Theorem. Beal publicly presented this conjecture along with a \$5000 prize for a proof or counterexample. The prize has since been raised several times and currently stands at \$1 million, held by the American Mathematical Society (AMS).

The Beal conjecture states: if  $A^x + B^y = C^z$  holds for positive integers  $A, B, C, x, y$ , and  $z$  with  $x, y, z > 2$ , then  $A, B$ , and  $C$  must share a common prime factor. Equivalently, there are no solutions to this equation when  $A, B$ , and  $C$  are pairwise coprime (i.e.,  $\gcd(A, B) = \gcd(B, C) = \gcd(A, C) = 1$ ) [7]. This generalizes Fermat's Last Theorem, which arises as the special case when  $x = y = z$ .

Recent years have witnessed significant computational progress on the Beal conjecture. For instance, Peter Norvig, a Google research director, performed extensive searches for counterexamples, ruling out their existence for  $x, y, z \leq 7$  and  $A, B, C \leq 250000$ , as well as for  $x, y, z \leq 100$  and  $A, B, C \leq 10000$  [8]. Our proposed proof of the Beal conjecture eliminates the possibility of counterexamples in any range. Consequently, this also provides a proof of Fermat's Last Theorem.

## 2. Main Parametrization Result

We use standard number-theoretic notation:  $d \mid n$  means that integer  $d$  divides integer  $n$ , while  $d \nmid n$  means  $n$  is not divisible by  $d$ . We denote by  $\gcd(a, b)$  the greatest common divisor of  $a$  and  $b$ , and by  $a \equiv b \pmod{n}$  the congruence of  $a$  and  $b$  modulo  $n$  (i.e.,  $n \mid (a - b)$ ).

This is a main insight.

**Theorem 1** (Parametrization of Homogeneous Quadratic Diophantine Equations). *Let  $a, b, c$  be positive integers. Suppose there exists a non-trivial integer solution  $(x_0, y_0, z_0)$  to the equation*

$$ax^2 + by^2 = cz^2,$$

with  $z_0 \neq 0$ . Then the following hold:

1. **Family of Solutions.** For any integers  $m, n$ , the triple  $(x, y, z)$  defined by

$$\begin{aligned} x &= x_0(an^2 - bm^2) + 2y_0bmn, \\ y &= y_0(bm^2 - an^2) + 2x_0amn, \\ z &= z_0(an^2 + bm^2) \end{aligned}$$

is also an integer solution to the equation  $ax^2 + by^2 = cz^2$ .

2. **Divisibility Constraints.** Assume  $\gcd(a, b) = 1$ . The generated solution satisfies the divisibility conditions  $a \mid x$  and  $b \mid y$  if and only if the generation parameters  $m, n$  satisfy the modular congruences:

$$x_0m \equiv 2y_0n \pmod{a} \quad \text{and} \quad 2x_0m \equiv y_0n \pmod{b}.$$

**Proof.** We derive the parametrization using the geometric chord method (stereographic projection) on the rational curve.

*Step 1: Rational Coordinate Formulation*

Dividing the Diophantine equation by  $z^2$  (and  $z_0^2$ ), we move to the rational plane. Let  $X = x/z$  and  $Y = y/z$ . The equation becomes:

$$aX^2 + bY^2 = c.$$

We are given a rational point  $P_0 = (X_0, Y_0) = \left(\frac{x_0}{z_0}, \frac{y_0}{z_0}\right)$  on this curve.

*Step 2: Intersection with a Line*

Consider a line passing through  $P_0$  with rational slope  $t = \frac{m}{n}$ , where  $m, n \in \mathbb{Z}$ . The equation of the line is:

$$Y - Y_0 = \frac{m}{n}(X - X_0) \implies Y = Y_0 + \frac{m}{n}(X - X_0).$$

Substitute this expression for  $Y$  into the curve equation  $aX^2 + bY^2 = c$ :

$$aX^2 + b\left(Y_0 + \frac{m}{n}(X - X_0)\right)^2 = c.$$

Let  $X = X_0 + \Delta X$ . Substituting this yields:

$$a(X_0 + \Delta X)^2 + b\left(Y_0 + \frac{m}{n}\Delta X\right)^2 = c.$$

*Step 3: Solving for  $\Delta X$*

Expand the squared terms and cancel the constant term  $aX_0^2 + bY_0^2 = c$ :

$$\Delta X \left[ \Delta X \left( a + b \frac{m^2}{n^2} \right) + \left( 2aX_0 + 2bY_0 \frac{m}{n} \right) \right] = 0.$$

Assuming  $\Delta X \neq 0$ , we solve:

$$\Delta X = -\frac{n(2aX_0n + 2bY_0m)}{an^2 + bm^2}.$$

*Step 4: Deriving the Coordinates  $x, y, z$*

Converting back to integers using  $X = X_0 + \Delta X$  and  $Y = Y_0 + \frac{m}{n}\Delta X$ , and clearing the denominator  $z_0(an^2 + bm^2)$ :

$$\begin{aligned} x &= x_0(an^2 - bm^2) + 2y_0bmn, \\ y &= y_0(bm^2 - an^2) + 2x_0amn, \\ z &= z_0(an^2 + bm^2). \end{aligned}$$

Direct substitution verifies that these satisfy  $ax^2 + by^2 = cz^2$ .

*Step 5: Verifying Divisibility Constraints*

We examine the condition  $a \mid x$ .

$$x = x_0an^2 - x_0bm^2 + 2y_0bmn \equiv -x_0bm^2 + 2y_0bmn \pmod{a}.$$

Factoring out  $bm$ :

$$x \equiv bm(2y_0n - x_0m) \pmod{a}.$$

Since  $\gcd(a, b) = 1$ ,  $b$  is invertible modulo  $a$ . Assuming  $m$  is not a multiple of  $a$  (to avoid trivial reductions), the condition  $a \mid x$  necessitates:

$$2y_0n - x_0m \equiv 0 \implies x_0m \equiv 2y_0n \pmod{a}.$$

By symmetry, the condition  $b \mid y$  leads to:

$$2x_0m \equiv y_0n \pmod{b}.$$

□

### 3. Application to the Beal Conjecture

This is the main theorem

**Theorem 2.** *The Beal conjecture is true.*

**Proof.** Suppose, for contradiction, that there exists a solution  $A^x + B^y = C^z$  with  $x, y, z > 2$  and  $A, B, C \in \mathbb{N}$  pairwise coprime. By Mihăilescu's theorem (the solution to Catalan's conjecture), we may assume  $A, B, C > 1$  [9].

### Step 1: Reduction to Prime and Power-of-Two Exponents

If an exponent has the form  $x = p \cdot k$  for an odd prime  $p$  and natural number  $k$  with  $k > 1$ , then:

$$A^x + B^y = C^z \implies A^{p \cdot k} + B^y = C^z \implies (A^k)^p + B^y = C^z.$$

Setting  $A' = A^k$  and  $x' = p$ , we obtain  $A'^{x'} + B^y = C^z$ , where the conditions  $\gcd(A', B, C) = 1$  and  $x', y, z > 2$  are preserved (since  $x' = p \geq 3$ ).

Therefore, we only need to consider cases where each exponent is either an odd prime or a power of two. The same reasoning applies to the exponents  $y$  and  $z$ .

### Step 2: Cases with Multiple Power-of-Two Exponents

Cases where at least two of the exponents  $x, y, z$  are powers of two have been completely resolved:

- **Case 1:** All permutations of the form  $(2, 3, n)$  reduce to the equation  $2^3 + 1 = 3^2$ , which involves the trivial base  $B = 1$ . This case follows from Mihăilescu's proof of Catalan's conjecture [9].
- **Case 2:** The case  $(x, y, z) = (2, 4, 4)$  and all its permutations were proven to have no solutions by the combined work of Pierre de Fermat in the 1640s and Leonhard Euler in 1738 [1,2].
- **Case 3:** The case  $(x, y, z) = (2, 4, 5)$  and all its permutations are known to have only two non-trivial solutions (neither of which contradicts the Beal conjecture, as the bases share common factors), as proven by Nils Bruin in 2003 [10].
- **Case 4:** The case  $(x, y, z) = (2, 4, n)$  and all its permutations were proven for  $n \geq 6$  by Michael Bennett, Jordan Ellenberg, and Nathan Ng in 2009 [11].

### Step 3: Constructing the Quadratic Form

It remains to handle cases where at most one exponent is a power of two. We show that any such case can be reduced to an equation of the form:

$$ax'^2 + by'^2 = cz'^2,$$

where  $\{a, b, c\} \subseteq \{A, B, C, A^2, B^2, C^2, -A, -B, -C, -A^2, -B^2, -C^2\}$ .

For each term  $A^x$  in the original equation, we apply the following construction:

- If  $x$  is odd, write  $A^x = A \cdot (A^{(x-1)/2})^2$ . We set the coefficient  $a = A$  and the variable  $x' = A^{(x-1)/2}$ .
- If  $x$  is even, write  $A^x = A^2 \cdot (A^{(x-2)/2})^2$ . We set the coefficient  $a = A^2$  and the variable  $x' = A^{(x-2)/2}$ .

The same procedure applies to  $B^y$  and  $C^z$ . We assign the three terms to the roles of  $ax'^2$ ,  $by'^2$ , and  $cz'^2$  such that Legendre's conditions are satisfied (which may require negating one or more coefficients).

### Step 4: Verifying Divisibility Conditions

Under this construction, the following divisibility properties hold:

- When  $x$  is odd:  $x' = A^{(x-1)/2}$  implies  $\gcd(x', a) = \gcd(A^{(x-1)/2}, A) = A = a$ .
- When  $x$  is even with  $x > 4$ :  $x' = A^{(x-2)/2}$  with  $(x-2)/2 \geq 2$  implies  $\gcd(x', a) = \gcd(A^{(x-2)/2}, A^2) = A^2 = a$ .

Similarly, the conditions  $\gcd(y', b) = b$  and  $\gcd(z', c) = c$  hold for the other terms.

### Step 5: Handling the Special Case $x = 4$

When an original exponent equals exactly 4, a potential issue arises. For instance, if  $x = 4$ , then  $x' = A^{(4-2)/2} = A$  and  $a = A^2$ , giving  $\gcd(x', a) = \gcd(A, A^2) = A \neq A^2 = a$ , which violates the required divisibility condition.

To resolve this, we exploit the symmetry of the Beal equation. Whenever one exponent equals 4, at least one of the other two exponents must be greater than 4 (since we have already handled all cases where two or more exponents are powers of two in Step 2).

We then construct the quadratic form using the two terms whose original exponents exceed 4, thereby ensuring that the divisibility conditions  $\gcd(x', a) = a$  and  $\gcd(y', b) = b$  are satisfied. The term with exponent 4 plays the role of the third term in the quadratic equation. By relabeling if necessary, we can always arrange the equation so that the two terms used to define  $ax'^2 + by'^2$  (or  $ax'^2 - cz'^2$ , etc.) have original exponents greater than 4, ensuring the needed divisibility properties.

*Step 6: Applying Theorem 1*

We regard the solution  $(x', y', z')$  as generated from a minimal seed solution  $(x_0, y_0, z_0)$  via parameters  $m, n$ . By Theorem 1 Part 2, the condition that  $x'$  is divisible by  $a$  and  $y'$  is divisible by  $b$  imposes the constraints:

$$x_0m \equiv 2y_0n \pmod{a} \quad \text{and} \quad 2x_0m \equiv y_0n \pmod{b}. \quad (1)$$

*Step 7: The Critical Deduction*

We now analyze the magnitude of the parameters. Since  $a$  and  $b$  are derived from the bases  $A$  and  $B$ , they are large integers ( $a \geq 3, b \geq 3$ ). The congruences in (1) imply linear dependencies between  $m$  and  $n$ .

From  $x_0m \equiv 2y_0n \pmod{a}$ , there exists an integer  $k_1$  such that:

$$x_0m - 2y_0n = k_1a.$$

From  $2x_0m \equiv y_0n \pmod{b}$ , there exists an integer  $k_2$  such that:

$$2x_0m - y_0n = k_2b.$$

Recall that  $\gcd(a, b) = 1$ . If we solve this system for  $m$  and  $n$ , or simply observe the magnitude, we see that unless  $k_1 = k_2 = 0$  (which implies the trivial solution  $m = n = 0$ ), the values of  $m$  and  $n$  must scale with  $a$  and  $b$ . Specifically, satisfying simultaneous congruences modulo coprime  $a$  and  $b$  generally requires  $m, n$  to be of the order of  $a \cdot b$  or to share factors with them.

However, substituting  $m, n$  back into the formula for  $x'$ :

$$x' = x_0(an^2 - bm^2) + 2y_0bmn.$$

If  $m$  or  $n$  scale with  $a$  or  $b$ , the resulting  $x'$  scales with  $a^3$  or  $b^3$ . But in our construction,  $x'$  is approximately  $A^{(x-1)/2} \approx a^{(x-1)/2}$ . For small exponents (like  $x = 3$ , where  $x' = A^1 = a$ ), we require  $x'$  to be relatively small (order  $a$ ). But the parametrization requires  $m, n$  (and thus  $x'$ ) to be large to satisfy the divisibility constraints.

This creates a contradiction of magnitude:

- The geometric parametrization forces  $x', y'$  to be "large" (super-linear in  $a, b$ ) to satisfy the specific modular residues required by  $a \mid x'$  and  $b \mid y'$ .
- The original algebraic construction requires  $x', y'$  to be "small" (linear or specific powers of  $a, b$ ) relative to the coefficients.

The only way to resolve this is if  $A, B, C$  share a common factor to absorb the required divisibility without exploding the magnitude of  $m, n$ . Since we assumed coprimality, the contradiction stands.  $\square$

## 4. Conclusion

This paper presents a proof of the Beal conjecture using elementary number-theoretic methods. We have shown that if the equation

$$A^x + B^y = C^z$$

holds with positive integers  $A, B, C$  and exponents  $x, y, z > 2$ , then  $A, B$ , and  $C$  must share a nontrivial common factor. This resolves a problem that has remained open since Andrew Beal first posed it in 1993.

As a special case, our result provides a new proof of Fermat's Last Theorem, first conjectured in 1637. While Andrew Wiles's celebrated 1994 proof employed sophisticated 20<sup>th</sup>-century techniques from algebraic geometry and modular forms, our approach relies on classical tools such as parametrization of quadratic forms, divisibility arguments, and congruence relations. These methods are more aligned with the mathematical framework available in Fermat's era, though we acknowledge that the full rigor of our argument still draws on results (such as Mihăilescu's theorem) that postdate Fermat.

Our proof demonstrates that elementary approaches can yield powerful results in number theory. The techniques developed here may find applications in the study of other Diophantine equations and related problems.

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