The Complete Proof of the Riemann Hypothesis

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Abstract Robin criterion states that the Riemann Hypothesis is true if and only if the inequality $\sigma(n) < e^{\gamma} \times n \times \log\log n$ holds for all $n > 5040$, where $\sigma(n)$ is the sum-of-divisors function and $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. We prove that the Robin inequality is true for all $n > 5040$ which are not divisible by any prime number between 2 and 953. Using this result, we show there is a contradiction just assuming the possible smallest counterexample $n > 5040$ of the Robin inequality. In this way, we prove that the Robin inequality is true for all $n > 5040$ and thus, the Riemann Hypothesis is true.

Keywords Riemann hypothesis · Robin inequality · sum-of-divisors function · prime numbers

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1 Introduction

In mathematics, the Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$ [7]. As usual $\sigma(n)$ is the sum-of-divisors function of n [3]:

$$\sum_{d \mid n} d$$

where $d \mid n$ means the integer d divides to n and $d \nmid n$ means the integer d does not divide to n. Define $f(n)$ to be $\frac{\sigma(n)}{n}$. Say $\mathsf{Robins}(n)$ holds provided

$$f(n) < e^{\gamma} \times \log\log n.$$

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The constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant, and log is the natural logarithm. The importance of this property is:

Theorem 1.1 Robins(n) *holds for all* $n > 5040$ *if and only if the Riemann Hypothesis is true [7].*

It is known that Robins(n) holds for many classes of numbers n.

Theorem 1.2 Robins(n) *holds for all* $n > 5040$ *that are not divisible by* 2 *[3].*

On the one hand, we prove that Robins(n) holds for all $n > 5040$ that are not divisible by any prime number between 3 and 953. Let $q_1 = 2, q_2 = 3, \ldots, q_m$ denote the first m consecutive primes, then an integer of the form $\prod_{i=1}^{m} q_i^{a_i}$ with $a_1 \geq a_2 \geq \cdots \geq a_m \geq 0$ is called an Hardy-Ramanujan integer [3]. A natural number n is called superabundant precisely when, for all $m < n$

$$f(m) < f(n).$$

Theorem 1.3 *If* n *is superabundant, then* n *is an Hardy-Ramanujan integer [2].*

Theorem 1.4 *The smallest counterexample of the Robin inequality greater than* 5040 *must be a superabundant number [1].*

On the other hand, we prove the nonexistence of such counterexample and therefore, the Riemann Hypothesis is true.

2 A Central Lemma

These are known results:

Lemma 2.1 *[3]. For* $n > 1$*:*

$$f(n) < \prod_{q \mid n} \frac{q}{q-1}. \tag{2.1}$$

Lemma 2.2 *[4].*

$$\prod_{k=1}^{\infty} \frac{1}{1 - \frac{1}{q_k^2}} = \zeta(2) = \frac{\pi^2}{6}. \tag{2.2}$$

The following is a key lemma. It gives an upper bound on $f(n)$ that holds for all n. The bound is too weak to prove Robins(n) directly, but is critical because it holds for all n. Further the bound only uses the primes that divide n and not how many times they divide n.

Lemma 2.3 *Let* $n > 1$ *and let all its prime divisors be* $q_1 < \cdots < q_m$*. Then,*

$$f(n) < \frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i}.$$

Proof We use that lemma 2.1:

$$f(n) < \prod_{i=1}^{m} \frac{q_i}{q_i - 1}.$$

Now for $q > 1$,

$$\frac{1}{1 - \frac{1}{q^2}} = \frac{q^2}{q^2 - 1}.$$

So

$$\begin{aligned} \frac{1}{1 - \frac{1}{q^2}} \times \frac{q+1}{q} &= \frac{q^2}{q^2 - 1} \times \frac{q+1}{q} \\ &= \frac{q}{q-1}. \end{aligned}$$

Then by lemma 2.2,

$$\prod_{i=1}^{m} \frac{1}{1 - \frac{1}{q_i^2}} < \zeta(2) = \frac{\pi^2}{6}.$$

Putting this together yields the proof:

$$\begin{aligned} f(n) &< \prod_{i=1}^{m} \frac{q_i}{q_i - 1} \\ &\leq \prod_{i=1}^{m} \frac{1}{1 - \frac{1}{q_i^2}} \times \frac{q_i + 1}{q_i} \\ &< \frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i}. \end{aligned}$$

3 About the p-adic order

In basic number theory, for a given prime number p, the p-adic order of a natural number n is the highest exponent $\nu_p \geq 1$ such that p^{ν_p} divides n. This is a known result:

Lemma 3.1 *In general, we know that* $\mathsf{Robins}(n)$ *holds for a natural number* $n > 5040$ *that satisfies either* $\nu_2(n) \leq 19$, $\nu_3(n) \leq 12$ *or* $\nu_7(n) \leq 6$, *where* $\nu_p(n)$ *is the p-adic order of n [5].*

We know the following lemmas:

Lemma 3.2 *[5]. Let* $\prod_{i=1}^{m} q_i^{a_i}$ *be the representation of n as a product of primes* $q_1 < \cdots < q_m$ *with natural numbers as exponents* $a_1, \ldots, a_m$. *Then,*

$$f(n) = \left(\prod_{i=1}^{m} \frac{q_i}{q_i - 1} \right) \times \prod_{i=1}^{m} \left(1 - \frac{1}{q_i^{a_i + 1}} \right).$$

Lemma 3.3 *[5]. Let $n > e^{e^{23.762143}}$ and let all its prime divisors be $q_1 < \cdots < q_m$, then*

$$\left(\prod_{i=1}^{m} \frac{q_i}{q_i - 1}\right) < \frac{1771561}{1771560} \times e^{\gamma} \times \log\log n.$$

Lemma 3.4 Robins(n) *holds for all $10^{10^{10}} \geq n > 5040$ [5].*

Putting together all these results, then we obtain that

Lemma 3.5 Robins(n) *holds for $n > 5040$ when $\nu_{31}(n) \leq 3$.*

Proof From lemma 3.2, we note that

$$f(n) = \left(\prod_{i=1}^{m} \frac{q_i}{q_i - 1}\right) \times \prod_{i=1}^{m} \left(1 - \frac{1}{q_i^{a_i+1}}\right) \leq \left(\prod_{i=1}^{m} \frac{q_i}{q_i - 1}\right) \times \left(1 - \frac{1}{31^{\nu_{31}(n)+1}}\right)$$

when $\nu_{31}(n) \leq 3$. We only need to look at the case where $\nu_{31}(n) = 3$ since the weaker cases follow because

$$\left(1 - \frac{1}{31^{1+1}}\right) < \left(1 - \frac{1}{31^{2+1}}\right) < \left(1 - \frac{1}{31^{3+1}}\right).$$

In this way, we obtain that

$$f(n) \leq \left(\prod_{i=1}^{m} \frac{q_i}{q_i - 1}\right) \times \left(1 - \frac{1}{31^{3+1}}\right) = \frac{923520}{923521} \times \left(\prod_{i=1}^{m} \frac{q_i}{q_i - 1}\right)$$

when $\nu_{31}(n) \leq 3$. With lemma 3.3, we have for $n > e^{e^{23.762143}}$

$$\frac{923520}{923521} \times \left(\prod_{i=1}^{m} \frac{q_i}{q_i - 1}\right) < \frac{923520}{923521} \times \frac{1771561}{1771560} \times e^{\gamma} \times \log\log n < e^{\gamma} \times \log\log n$$

since $\frac{923520}{923521} \times \frac{1771561}{1771560} < 1$. In light of lemma 3.4 and the fact that $e^{e^{23.762143}} < 10^{10^{10}}$, we then conclude that Robins(n) holds for $n > 5040$ when $\nu_{31}(n) \leq 3$.

4 A Particular Case

We can easily prove that Robins(n) is true for certain kind of numbers:

Lemma 4.1 Robins(n) *holds for $n > 5040$ when $q \leq 7$, where q is the largest prime divisor of n.*

Proof Let $n > 5040$ and let all its prime divisors be $q_1 < \cdots < q_m \leq 5$, then we need to prove

$$f(n) < e^{\gamma} \times \log\log n$$

that is true when

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} \leq e^{\gamma} \times \log\log n$$

according to the lemma 2.1. For $q_1 < \cdots < q_m \leq 5$,

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} \leq \frac{2 \times 3 \times 5}{1 \times 2 \times 4} = 3.75 < e^{\gamma} \times \log\log(5040) \approx 3.81.$$

However, we know for $n > 5040$

$$e^{\gamma} \times \log\log(5040) < e^{\gamma} \times \log\log n$$

and therefore, the proof is complete when $q_1 < \cdots < q_m \leq 5$. The remaining case is for $n > 5040$ when all its prime divisors are $q_1 < \cdots < q_m \leq 7$. Robins$(n)$ holds for $n > 5040$ when $\nu_7(n) \leq 6$ according to the lemma 3.1 [5]. Hence, it is enough to prove this for those natural numbers $n > 5040$ when $7^7 \mid n$. For $q_1 < \cdots < q_m \leq 7$,

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} \leq \frac{2 \times 3 \times 5 \times 7}{1 \times 2 \times 4 \times 6} = 4.375 < e^{\gamma} \times \log\log(7^7) \approx 4.65.$$

However, for $n > 5040$ and $7^7 \mid n$, we know that

$$e^{\gamma} \times \log\log(7^7) \leq e^{\gamma} \times \log\log n$$

and as a consequence, the proof is complete when $q_1 < \cdots < q_m \leq 7$.

5 A Better Bound

This is a known result:

Lemma 5.1 *[8]. For $x > 1$:*

$$\sum_{q \leq x} \frac{1}{q} < \log\log x + B + \frac{1}{\log^2 x} \tag{5.1}$$

where

$$B = 0.2614972128\cdots$$

denotes the (Meissel-)Mertens constant [8].

We show a better result:

Lemma 5.2 *For $x \geq 11$, we have*

$$\sum_{q \leq x} \frac{1}{q} < \log\log x + \gamma - 0.12.$$

Proof Let's define $H = \gamma - B$. The lemma 5.1 is the same as

$$\sum_{q \leq x} \frac{1}{q} < \log\log x + \gamma - (H - \frac{1}{\log^2 x}).$$

For $x \geq 11$,

$$(H - \frac{1}{\log^2 x}) > (0.31 - \frac{1}{\log^2 11}) > 0.12$$

and thus,

$$\sum_{q \leq x} \frac{1}{q} < \log\log x + \gamma - (H - \frac{1}{\log^2 x}) < \log\log x + \gamma - 0.12.$$

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6 On a Square Free Number

We know the following results:

Lemma 6.1 *[3]. For* $0 < a < b$*:*

$$\frac{\log b - \log a}{b - a} = \frac{1}{(b-a)} \int_a^b \frac{dt}{t} > \frac{1}{b}. \tag{6.1}$$

Lemma 6.2 *[3]. For* $q > 0$*:*

$$\log(q+1) - \log q = \int_q^{q+1} \frac{dt}{t} < \frac{1}{q}. \tag{6.2}$$

We recall that an integer n is said to be square free if for every prime divisor q of n we have $q^2 \nmid n$ [3]. Robins(n) holds for all $n > 5040$ that are square free [3].

Lemma 6.3 *For a square free number*

$$n = q_1 \times \cdots \times q_m$$

such that $q_1 < q_2 < \cdots < q_m$ *are odd prime numbers,* $q_m \geq 11$ *and* $3 \nmid n$*, then:*

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \sigma(n) \leq e^\gamma \times n \times \log\log(2^{19} \times n).$$

Proof By induction with respect to $\omega(n)$, that is the number of distinct prime factors of n [3]. Put $\omega(n) = m$ [3]. We need to prove the assertion for those integers with $m = 1$. From a square free number n, we obtain

$$\sigma(n) = (q_1 + 1) \times (q_2 + 1) \times \cdots \times (q_m + 1) \tag{6.3}$$

when $n = q_1 \times q_2 \times \cdots \times q_m$ [3]. In this way, for every prime number $q_i \geq 11$, then we need to prove

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (1 + \frac{1}{q_i}) \leq e^\gamma \times \log\log(2^{19} \times q_i). \tag{6.4}$$

For $q_i = 11$, we have

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (1 + \frac{1}{11}) \leq e^\gamma \times \log\log(2^{19} \times 11)$$

is actually true. For another prime number $q_i > 11$, we have

$$(1 + \frac{1}{q_i}) < (1 + \frac{1}{11})$$

and

$$\log\log(2^{19} \times 11) < \log\log(2^{19} \times q_i)$$

which clearly implies that the inequality (6.4) is true for every prime number $q_i \geq 11$. Now, suppose it is true for $m - 1$, with $m \geq 2$ and let us consider the assertion for those square free n with $\omega(n) = m$ [3]. So let $n = q_1 \times \cdots \times q_m$ be a square free number and assume that $q_1 < \cdots < q_m$ for $q_m \geq 11$.

Case 1: $q_m \geq \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) = \log(2^{19} \times n)$.

By the induction hypothesis we have

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (q_1+1) \times \cdots \times (q_{m-1}+1) \leq e^{\gamma} \times q_1 \times \cdots \times q_{m-1} \times \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1})$$

and hence

$$\frac{\pi^2}{6} \times \frac{3}{2} \times (q_1+1) \times \cdots \times (q_{m-1}+1) \times (q_m+1) \leq$$

$$e^{\gamma} \times q_1 \times \cdots \times q_{m-1} \times (q_m+1) \times \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1})$$

when we multiply the both sides of the inequality by (q_m+1). We want to show

$$e^{\gamma} \times q_1 \times \cdots \times q_{m-1} \times (q_m+1) \times \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1}) \leq$$

$$e^{\gamma} \times q_1 \times \cdots \times q_{m-1} \times q_m \times \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) = e^{\gamma} \times n \times \log\log(2^{19} \times n).$$

Indeed the previous inequality is equivalent with

$$q_m \times \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) \geq (q_m+1) \times \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1})$$

or alternatively

$$\frac{q_m \times (\log\log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) - \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1}))}{\log q_m} \geq$$

$$\frac{\log\log(2^{19} \times q_1 \times \cdots \times q_{m-1})}{\log q_m}.$$

We can apply the inequality in lemma 6.1 just using $b = \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m)$ and $a = \log(2^{19} \times q_1 \times \cdots \times q_{m-1})$. Certainly, we have

$$\log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) - \log(2^{19} \times q_1 \times \cdots \times q_{m-1}) =$$

$$\log \frac{2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m}{2^{19} \times q_1 \times \cdots \times q_{m-1}} = \log q_m.$$

In this way, we obtain

$$\frac{q_m \times (\log\log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) - \log\log(2^{19} \times q_1 \times \cdots \times q_{m-1}))}{\log q_m} >$$

$$\frac{q_m}{\log(2^{19} \times q_1 \times \cdots \times q_m)}.$$

Using this result we infer that the original inequality is certainly satisfied if the next inequality is satisfied

$$\frac{q_m}{\log(2^{19} \times q_1 \times \cdots \times q_m)} \geq \frac{\log\log(2^{19} \times q_1 \times \cdots \times q_{m-1})}{\log q_m}$$

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which is trivially true for $q_m \geq \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m)$ [3].

Case 2: $q_m < \log(2^{19} \times q_1 \times \cdots \times q_{m-1} \times q_m) = \log(2^{19} \times n)$.

We need to prove

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \frac{\sigma(n)}{n} \leq e^{\gamma} \times \log\log(2^{19} \times n).$$

We know $\frac{3}{2} < 1.503 < \frac{4}{2.66}$. Nevertheless, we could have

$$\frac{3}{2} \times \frac{\sigma(n)}{n} \times \frac{\pi^2}{6} < \frac{4 \times \sigma(n)}{3 \times n} \times \frac{\pi^2}{2 \times 2.66}$$

and therefore, we only need to prove

$$\frac{\sigma(3 \times n)}{3 \times n} \times \frac{\pi^2}{5.32} \leq e^{\gamma} \times \log\log(2^{19} \times n)$$

where this is possible because of $3 \nmid n$. If we apply the logarithm to the both sides of the inequality, then we obtain

$$\log(\frac{\pi^2}{5.32}) + (\log(3+1) - \log 3) + \sum_{i=1}^{m} (\log(q_i + 1) - \log q_i) \leq \gamma + \log\log\log(2^{19} \times n).$$

In addition, note that $\log(\frac{\pi^2}{5.32}) < \frac{1}{2} + 0.12$. However, we know

$$\gamma + \log\log q_m < \gamma + \log\log\log(2^{19} \times n)$$

since $q_m < \log(2^{19} \times n)$. We use that lemma 6.2 for each term $\log(q+1) - \log q$ and thus,

$$0.12 + \frac{1}{2} + \frac{1}{3} + \frac{1}{q_1} + \cdots + \frac{1}{q_m} \leq 0.12 + \sum_{q \leq q_m} \frac{1}{q} \leq \gamma + \log\log q_m$$

where $q_m \geq 11$. Hence, it is enough to prove

$$\sum_{q \leq q_m} \frac{1}{q} \leq \gamma + \log\log q_m - 0.12$$

but this is true according to the lemma 5.2 for $q_m \geq 11$. In this way, we finally show the lemma is indeed satisfied.

7 Robin on Divisibility

$\mathsf{Robins}(n)$ holds for every $n > 5040$ that is not divisible by 2 [3]. We extend this property to other prime numbers:

Lemma 7.1 $\mathsf{Robins}(n)$ *holds for all* $n > 5040$ *when* $3 \nmid n$. *More precisely: every possible counterexample* $n > 5040$ *of the Robin inequality must comply with* $(2^{20} \times 3^{13}) \mid n$.

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Proof We will check the Robin inequality is true for every natural number $n = q_1^{a_1} \times q_2^{a_2} \times \cdots \times q_m^{a_m} > 5040$ such that $q_1, q_2, \cdots, q_m$ are distinct prime numbers, $a_1, a_2, \cdots, a_m$ are natural numbers and $3 \nmid n$. We know this is true when the greatest prime divisor of $n > 5040$ is lesser than or equal to 7 according to the lemma 4.1. Therefore, the remaining case is when the greatest prime divisor of $n > 5040$ is greater than or equal to 11. We need to prove

$$f(n) < e^{\gamma} \times \log\log n$$

that is true when

$$\frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i} \leq e^{\gamma} \times \log\log n$$

according to the lemma 2.3. Using the formula (6.3) for the square free numbers, then we obtain that is equivalent to

$$\frac{\pi^2}{6} \times \frac{\sigma(n')}{n'} \leq e^{\gamma} \times \log\log n$$

where $n' = q_1 \times \cdots \times q_m$ is the square free kernel of the natural number n [3]. The Robin inequality has been proved for all integers n not divisible by 2 (which are bigger than 10) [3]. Hence, we only need to prove the Robin inequality is true when $2 \mid n'$. In addition, we know that $\mathsf{Robins}(n)$ holds for every $n > 5040$ when $\nu_2(n) \leq 19$ according to the lemma 3.1 [5]. Consequently, we only need to prove that $\mathsf{Robins}(n)$ holds for $n > 5040$ when $2^{20} \mid n$ and thus,

$$e^{\gamma} \times n' \times \log\log(2^{19} \times \frac{n'}{2}) \leq e^{\gamma} \times n' \times \log\log n$$

because of $2^{19} \times \frac{n'}{2} \leq n$ where $2^{20} \mid n$ and $2 \mid n'$. So,

$$\frac{\pi^2}{6} \times \sigma(n') \leq e^{\gamma} \times n' \times \log\log(2^{19} \times \frac{n'}{2}).$$

According to the formula (6.3) for the square free numbers and $2 \mid n'$, then,

$$\frac{\pi^2}{6} \times 3 \times \sigma(\frac{n'}{2}) \leq e^{\gamma} \times 2 \times \frac{n'}{2} \times \log\log(2^{19} \times \frac{n'}{2})$$

which is the same as

$$\frac{\pi^2}{6} \times \frac{3}{2} \times \sigma(\frac{n'}{2}) \leq e^{\gamma} \times \frac{n'}{2} \times \log\log(2^{19} \times \frac{n'}{2})$$

where this is true according to the lemma 6.3 when $3 \nmid \frac{n'}{2}$. In addition, we know that $\mathsf{Robins}(n)$ holds for every $n > 5040$ when $\nu_3(n) \leq 12$ according to the lemma 3.1 [5]. Hence, we only need to prove that $\mathsf{Robins}(n)$ holds for every $n > 5040$ when $2^{20} \mid n$ and $3^{13} \mid n$. To sum up, the proof is complete.

Let's state the following known properties:

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Lemma 7.2 *$\sigma(n)$ and $f(n)$ are multiplicatives [3]. Besides, for a prime number q and a positive integer $a \geq 0$, we have that $\sigma(q^a) = \frac{q^{a+1}-1}{q-1}$ [3]. We know that $f(q^a) < \frac{q}{q-1}$ and $f(q^{a+1}) > f(q^a)$ for all primes q and all $a \geq 0$.*

Lemma 7.3 *$\textsf{Robins}(n)$ holds for all $n > 5040$ when $5 \nmid n$ or $7 \nmid n$.*

Proof We need to prove

$$f(n) < e^{\gamma} \times \log\log n$$

when $(2^{20} \times 3^{13}) \mid n$. Suppose that $n = 2^a \times 3^b \times m$, where $a \geq 20$, $b \geq 13$, $2 \nmid m$, $3 \nmid m$ and $5 \nmid m$ or $7 \nmid m$. Therefore, we need to prove

$$f(2^a \times 3^b \times m) < e^{\gamma} \times \log\log(2^a \times 3^b \times m).$$

We know

$$f(2^a \times 3^b \times m) = f(3^b) \times f(2^a \times m)$$

since f is multiplicative [3]. In addition, we know $f(3^b) < \frac{3}{2}$ for every natural number b [3]. In this way, we have

$$f(3^b) \times f(2^a \times m) < \frac{3}{2} \times f(2^a \times m).$$

However, that would be equivalent to

$$\frac{3}{2} \times f(2^a \times m) = \frac{9}{8} \times f(3) \times f(2^a \times m) = \frac{9}{8} \times f(2^a \times 3 \times m)$$

where $f(3) = \frac{4}{3}$ since f is multiplicative [3]. Nevertheless, we have

$$\frac{9}{8} \times f(2^a \times 3 \times m) < f(5) \times f(2^a \times 3 \times m) = f(2^a \times 3 \times 5 \times m)$$

and

$$\frac{9}{8} \times f(2^a \times 3 \times m) < f(7) \times f(2^a \times 3 \times m) = f(2^a \times 3 \times 7 \times m)$$

where $5 \nmid m$ or $7 \nmid m$, $f(5) = \frac{6}{5}$ and $f(7) = \frac{8}{7}$. We know the Robin inequality is true for $2^a \times 3 \times 5 \times m$ and $2^a \times 3 \times 7 \times m$ when $a \geq 20$, since this is true for every natural number $n > 5040$ when $\nu_3(n) \leq 12$ according to the lemma 3.1 [5]. Hence, we would have

$$f(2^a \times 3 \times 5 \times m) < e^{\gamma} \times \log\log(2^a \times 3 \times 5 \times m) < e^{\gamma} \times \log\log(2^a \times 3^b \times m)$$

and

$$f(2^a \times 3 \times 7 \times m) < e^{\gamma} \times \log\log(2^a \times 3 \times 7 \times m) < e^{\gamma} \times \log\log(2^a \times 3^b \times m)$$

when $b \geq 13$.

Lemma 7.4 *$\textsf{Robins}(n)$ holds for all $n > 5040$ when a prime number $11 \leq q \leq 47$ complies with $q \nmid n$.*

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Proof We know that $\mathsf{Robins}(n)$ holds for every $n > 5040$ when $\nu_7(n) \leq 6$ according to the lemma 3.1 [5]. We need to prove

$$f(n) < e^{\gamma} \times \log\log n$$

when $(2^{20} \times 3^{13} \times 7^7) \mid n$. Suppose that $n = 2^a \times 3^b \times 7^c \times m$, where $a \geq 20$, $b \geq 13$, $c \geq 7$, $2 \nmid m$, $3 \nmid m$, $7 \nmid m$, $q \nmid m$ and $11 \leq q \leq 47$. Therefore, we need to prove

$$f(2^a \times 3^b \times 7^c \times m) < e^{\gamma} \times \log\log(2^a \times 3^b \times 7^c \times m).$$

We know

$$f(2^a \times 3^b \times 7^c \times m) = f(7^c) \times f(2^a \times 3^b \times m)$$

since f is multiplicative [3]. In addition, we know $f(7^c) < \frac{7}{6}$ for every natural number c [3]. In this way, we have

$$f(7^c) \times f(2^a \times 3^b \times m) < \frac{7}{6} \times f(2^a \times 3^b \times m).$$

However, that would be equivalent to

$$\frac{7}{6} \times f(2^a \times 3^b \times m) = \frac{49}{48} \times f(7) \times f(2^a \times 3^b \times m) = \frac{49}{48} \times f(2^a \times 3^b \times 7 \times m)$$

where $f(7) = \frac{8}{7}$ since f is multiplicative [3]. In addition, we know

$$\frac{49}{48} \times f(2^a \times 3^b \times 7 \times m) < f(q) \times f(2^a \times 3^b \times 7 \times m) = f(2^a \times 3^b \times 7 \times q \times m)$$

where $q \nmid m$, $f(q) = \frac{q+1}{q}$ and $11 \leq q \leq 47$. Nevertheless, we know the Robin inequality is true for $2^a \times 3^b \times 7 \times q \times m$ when $a \geq 20$ and $b \geq 13$, since this is true for every natural number $n > 5040$ when $\nu_7(n) \leq 6$ according to the lemma 3.1 [5]. Hence, we would have

$$\begin{aligned} f(2^a \times 3^b \times 7 \times q \times m) &< e^{\gamma} \times \log\log(2^a \times 3^b \times 7 \times q \times m) \\ &< e^{\gamma} \times \log\log(2^a \times 3^b \times 7^c \times m) \end{aligned}$$

when $c \geq 7$ and $11 \leq q \leq 47$.

Lemma 7.5 $\mathsf{Robins}(n)$ *holds for all* $n > 5040$ *when a prime number* $53 \leq q \leq 953$ *complies with* $q \nmid n$.

Proof We know that $\mathsf{Robins}(n)$ holds for every $n > 5040$ when $\nu_{31}(n) \leq 3$ according to the lemma 3.5. We need to prove that

$$f(n) < e^{\gamma} \times \log\log n$$

when $(2^{20} \times 3^{13} \times 31^4) \mid n$. Suppose that $n = 2^a \times 3^b \times 31^c \times m$, where $a \geq 20$, $b \geq 13$, $c \geq 4$, $2 \nmid m$, $3 \nmid m$, $31 \nmid m$, $q \nmid m$ and $53 \leq q \leq 953$. Therefore, we need to prove that

$$f(2^a \times 3^b \times 31^c \times m) < e^{\gamma} \times \log\log(2^a \times 3^b \times 31^c \times m).$$

We know that

$$f(2^a \times 3^b \times 31^c \times m) = f(31^c) \times f(2^a \times 3^b \times m)$$

since f is multiplicative [3]. In addition, we know that $f(31^c) < \frac{31}{30}$ for every natural number c [3]. In this way, we have that

$$f(31^c) \times f(2^a \times 3^b \times m) < \frac{31}{30} \times f(2^a \times 3^b \times m).$$

However, that would be equivalent to

$$\frac{31}{30} \times f(2^a \times 3^b \times m) = \frac{961}{960} \times f(31) \times f(2^a \times 3^b \times m) = \frac{961}{960} \times f(2^a \times 3^b \times 31 \times m)$$

where $f(31) = \frac{32}{31}$ since f is multiplicative [3]. In addition, we know that

$$\frac{961}{960} \times f(2^a \times 3^b \times 31 \times m) < f(q) \times f(2^a \times 3^b \times 31 \times m) = f(2^a \times 3^b \times 31 \times q \times m)$$

where $q \nmid m$, $f(q) = \frac{q+1}{q}$ and $53 \leq q \leq 953$. Nevertheless, we know the Robin inequality is true for $2^a \times 3^b \times 31 \times q \times m$ when $a \geq 20$ and $b \geq 13$, since this is true for every natural number $n > 5040$ when $\nu_{31}(n) \leq 3$ according to the lemma 3.5. Hence, we would have that

$$f(2^a \times 3^b \times 31 \times q \times m) < e^{\gamma} \times \log\log(2^a \times 3^b \times 31 \times q \times m)$$
$$< e^{\gamma} \times \log\log(2^a \times 3^b \times 31^c \times m)$$

when $c \geq 4$ and $53 \leq q \leq 953$.

8 Helpful Lemmas

In mathematics, the Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{q \leq x} \log q$$

where $q \leq x$ means all the prime numbers q that are less than or equal to x.

Lemma 8.1 *[8]. For $x \geq 41$:*

$$\theta(x) > (1 - \frac{1}{\log(x)}) \times x.$$

Besides, we know that

Lemma 8.2 *[8]. For $x \geq 286$:*

$$\prod_{q \leq x} \frac{q}{q-1} < e^{\gamma} \times (\log x + \frac{1}{2 \times \log(x)}).$$

For the counting prime function $\pi(x)$, we know that

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Lemma 8.3 *[8]. For $x \geq 17$:*

$$\frac{x}{\log x} < \pi(x) < 1.25506 \times \frac{x}{\log x}.$$

The following lemma is crucial in our proof

Lemma 8.4 *[6]. For $x > -1$:*

$$\frac{x}{x+1} \leq \log(1+x) \leq x.$$

The smallest counterexample of the Robin inequality greater than 5040 complies with

Lemma 8.5 *If $n > 5040$ is the smallest counterexample of the Robin inequality, then $q < \log n$ where q denotes the largest prime factor of n [3].*

We show some tools that could help us in the final proof.

Lemma 8.6 *Let $q \geq 2$ be a prime and let $b \geq 0$ be a positive integer. If $q^a \| n$, then*

$$f(q^b \times n) = f(n) \times \frac{q^{a+b+1} - 1}{q^{a+b+1} - q^b}$$

where $q^a \| n$ signifies that q^a divides n, but q^{a+1} does not divide n.

Proof We assume that $q^a \| n$. Since $\sigma(n)$ and $f(n)$ are multiplicatives according to the lemma 7.2, then we would only need to study $f(q^{a+b})$ where we know from lemma 7.2 that $\sigma(q^a) = \frac{q^{a+1}-1}{q-1}$. Then,

$$\begin{aligned} f(q^{a+b}) &= \frac{q^{a+b+1} - 1}{q^{a+b} \times (q-1)} \times \frac{q^{a+1} - 1}{q^a \times (q-1)} \times \frac{q^a \times (q-1)}{q^{a+1} - 1} \\ &= f(q^a) \times \frac{q^{a+b+1} - 1}{q^{a+b} \times (q-1)} \times \frac{q^a \times (q-1)}{q^{a+1} - 1} \\ &= f(q^a) \times \frac{q^{a+b+1} - 1}{q^b} \times \frac{1}{q^{a+1} - 1} \\ &= f(q^a) \times \frac{q^{a+b+1} - 1}{q^{a+b+1} - q^b}. \end{aligned}$$

Let's see another inequalities:

Lemma 8.7 *If $n > 5040$ is the smallest counterexample of the Robin inequality, then*

$$\frac{\log \log n}{\log q} < \left(1 + \frac{1}{2 \times \log^2 q}\right)$$

and

$$\frac{\log \log \log n}{\log q} < \frac{\log \log q}{\log q} + \frac{1}{2 \times \log^3 q}$$

when we assume that $q \geq 953$ is the largest prime factor of n.

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Proof Let $\prod_{i=1}^{m} q_i^{a_i}$ be the representation of n as a product of the first m consecutive primes $q_1 < \cdots < q_m$ with natural numbers as exponents $a_1, \ldots, a_m$. According to the theorems 1.3 and 1.4, the primes $q_1 < \cdots < q_m$ must be the first m consecutive primes since $n > 5040$ should be an Hardy-Ramanujan integer. We assume that $q_m \geq 953$. For $q_m \geq 953$, we have that

$$\prod_{q \leq q_m} \frac{q}{q-1} < e^{\gamma} \times (\log q_m + \frac{1}{2 \times \log(q_m)})$$

because of the lemma 8.2. We use that lemma 2.1 to show that

$$e^{\gamma} \times \log\log n \leq f(n) < \prod_{q \leq q_m} \frac{q}{q-1} < e^{\gamma} \times (\log q_m + \frac{1}{2 \times \log(q_m)})$$

since we assume that n is a counterexample of the Robin inequality. In this way, we obtain that

$$\log\log n < (\log q_m + \frac{1}{2 \times \log(q_m)})$$

which is the same as

$$\frac{\log\log n}{\log q_m} < (1 + \frac{1}{2 \times \log^2(q_m)}).$$

Besides, if we apply the logarithm to the both sides of the inequality, then

$$\log\log\log n < \log \left(\log q_m \times (1 + \frac{1}{2 \times \log^2(q_m)}) \right)$$

that is equivalent to

$$\log\log\log n < \log\log q_m + \log(1 + \frac{1}{2 \times \log^2(q_m)}).$$

We use that lemma 8.4 to show that

$$\log(1 + \frac{1}{2 \times \log^2(q_m)}) \leq \frac{1}{2 \times \log^2(q_m)}.$$

Therefore, we finally have that

$$\frac{\log\log\log n}{\log q_m} < \frac{\log\log q_m}{\log q_m} + \frac{1}{2 \times \log^3 q_m}.$$

Let's show another inequality

Lemma 8.8 *For $q_m \geq 953$, we have that*

$$\sum_{q \leq q_m} \frac{\log\log q}{q_m} > \frac{1}{\log q_m}$$

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Proof This is the same as

$$\sum_{q \leq q_m} \log\log q > \frac{q_m}{\log q_m}.$$

According to the lemma 8.3, it is enough to show that

$$\sum_{q \leq q_m} \log\log q \geq \pi(q_m) > \frac{q_m}{\log q_m}$$

when $q_m \geq 953$. We know that for all primes $q_i > q_m \geq 953$, then

$$\log\log q_i > 1.$$

Hence, it is enough to prove that

$$\sum_{q \leq q_m} \log\log q \geq \sum_{q \leq 953} \log\log q \geq \pi(953).$$

We compute that

$$\sum_{q \leq 953} \log\log q > 274.$$

However, we know that $q_{274} = 1759 > 953$ and thus,

$$274 \geq \pi(953).$$

Therefore, the proof is done.

9 Proof of Main Theorems

Theorem 9.1 $\mathsf{Robins}(n)$ *holds for all* $n > 5040$ *when a prime number* $q \leq 953$ *complies with* $q \nmid n$.

Proof This is a compendium of the results from the theorem 1.2 and the lemmas 7.1, 7.3, 7.4 and 7.5.

Theorem 9.2 *Let* $\prod_{i=1}^{m} q_i^{a_i}$ *be the representation of n as a product of the first m consecutive primes* $q_1 < \cdots < q_m$ *with natural numbers as exponents* $a_1, \ldots, a_m$. *We obtain a contradiction just assuming that* $n > 5040$ *is the smallest integer such that* $\mathsf{Robins}(n)$ *does not hold.*

Proof According to the theorems 1.3 and 1.4, the primes $q_1 < \cdots < q_m$ must be the first m consecutive primes since $n > 5040$ should be an Hardy-Ramanujan integer. From the theorem 9.1, we know that necessarily $q_m \geq 953$. Under our assumption, we know that

$$f(n) \geq e^{\gamma} \times \log\log n.$$

For $b = 1$ and the lemma 8.6, we know that

$$f(n) = f(q_i \times m) = f(m) \times \frac{q_i^{a_i+2} - 1}{q_i^{a_i+2} - q_i}$$

for every prime q_i that divides n where $m = \frac{n}{q_i}$. If we subtract $f(m)$ to both sides of the inequality, then we obtain that

$$f(n) - f(m) \geq e^{\gamma} \times \log\log n - f(m).$$

Then,

$$\begin{aligned}
f(n) - f(m) &= f(m) \times \frac{q_i^{a_i+2} - 1}{q_i^{a_i+2} - q_i} - f(m) \\
&= f(m) \times \left(\frac{q_i^{a_i+2} - 1}{q_i^{a_i+2} - q_i} - 1\right) \\
&= f(m) \times (\frac{q_i - 1}{q_i^{a_i+2} - q_i}) \\
&= f(m) \times (\frac{q_i - 1}{q_i \times (q_i^{a_i+1} - 1)}) \\
&= f(m) \times (\frac{1}{q_i \times \sigma(q_i^{a_i})}) \\
&= f(m') \times f(q_i^{a_i-1}) \times (\frac{1}{q_i \times \sigma(q_i^{a_i})}) \\
&= f(m') \times \frac{\sigma(q_i^{a_i-1})}{q_i^{a_i-1}} \times (\frac{1}{q_i \times \sigma(q_i^{a_i})}) \\
&< f(m') \times \frac{\sigma(q_i^{a_i})}{q_i^{a_i}} \times (\frac{1}{q_i \times \sigma(q_i^{a_i})}) \\
&= f(m') \times \frac{1}{q_i^{a_i+1}}
\end{aligned}$$

where $m' = \frac{n}{q_i^{a_i}}$ and we know that $q_i^{a_i} \| n$ and $\frac{\sigma(q_i^{a_i})}{q_i^{a_i}} > \frac{\sigma(q_i^{a_i-1})}{q_i^{a_i-1}}$ because of the lemma 7.2. In this way, we have that

$$f(m') \times \frac{1}{q_i^{a_i+1}} \geq e^{\gamma} \times \log\log n - f(m).$$

We know that $\mathsf{Robins}(m')$ and $\mathsf{Robins}(m)$ hold, since $n > 5040$ is the smallest integer such that $\mathsf{Robins}(n)$ does not hold. Consequently, we only need to prove that

$$\begin{aligned}
e^{\gamma} \times \log\log m' \times \frac{1}{q_i^{a_i+1}} &> f(m') \times \frac{1}{q_i^{a_i+1}} \\
&\geq e^{\gamma} \times \log\log n - f(m) \\
&> e^{\gamma} \times \log\log n - e^{\gamma} \times \log\log m.
\end{aligned}$$

As result, we have that

$$\log\log m' \times \frac{1}{q_i^{a_i+1}} > \log\log(q_i \times m) - \log\log m$$

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since $m = \frac{n}{q_i}$. We know that

$$\begin{aligned}\log\log(q_i \times m) - \log\log m &= \log(\log q_i + \log m) - \log\log m \\ &= \log\left(\log m \times (1 + \frac{\log q_i}{\log m})\right) - \log\log m \\ &= \log\log m + \log(1 + \frac{\log q_i}{\log m}) - \log\log m \\ &= \log(1 + \frac{\log q_i}{\log m}).\end{aligned}$$

In addition, we know that

$$\log(1 + \frac{\log q_i}{\log m}) \geq \frac{\log q_i}{\log n}$$

using the lemma 8.4. Certainly, we will have that

$$\log(1 + \frac{\log q_i}{\log m}) \geq \frac{\frac{\log q_i}{\log m}}{\frac{\log q_i}{\log m} + 1} = \frac{\log q_i}{\log q_i + \log m} = \frac{\log q_i}{\log n}.$$

As a consequence, we would have

$$\log\log m' \times \frac{1}{q_i^{a_i+1}} > \frac{\log q_i}{\log n}$$

which is equivalent to

$$\log n \times \log\log m' > q_i^{a_i+1} \times \log q_i.$$

However, we know that

$$\log n \times \log\log n > \log n \times \log\log m'$$

and thus

$$\log n \times \log\log n > q_i^{a_i+1} \times \log q_i.$$

For $n > 10^{10^{10}}$, we have that $\log n \times \log\log n > 1$ according to the lemma 3.4. Moreover, for $q_i \geq 3$, then $q_i^{a_i+1} \times \log q_i > 1$. In addition, for $q_1 = 2$, we have that $q_1^{a_1+1} \times \log q_1 > 1$ since $a_1 \geq 20$ due to the lemma 3.1. Since the both sides of the inequality is greater that 1 for all primes q_i which divides n, then we can multiply the inequalities to obtain

$$(\log n \times \log\log n)^{\pi(q_m)} > n \times N_m \times \prod_{i=1}^{m} \log q_i$$

where $N_m = \prod_{i=1}^{m} q_i$ is the primorial number of order m. If we apply the logarithm to the both sides of the inequality, then we would have

$$\pi(q_m) \times (\log\log n + \log\log\log n) > \log n + \log N_m + \sum_{i=1}^{m} \log\log q_i$$

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which is equivalent to

$$\pi(q_m) \times (\log\log n + \log\log\log n) > \log n + \theta(q_m) + \sum_{i=1}^{m} \log\log q_i.$$

If we apply the lemma 8.3, then we would have

$$1.25506 \times \frac{q_m}{\log q_m} \times (\log\log n + \log\log\log n) > \log n + \theta(q_m) + \sum_{i=1}^{m} \log\log q_i.$$

Let's introduce the lemma 8.1 in this inequality and thus

$$1.25506 \times \frac{q_m}{\log q_m} \times (\log\log n + \log\log\log n) > \log n + (1 - \frac{1}{\log q_m}) \times q_m + \sum_{i=1}^{m} \log\log q_i.$$

In addition, we can transform this into

$$1.25506 \times \frac{q_m}{\log q_m} \times (\log\log n + \log\log\log n) > q_m + (1 - \frac{1}{\log q_m}) \times q_m + \sum_{i=1}^{m} \log\log q_i$$

because of the lemma 8.5. If we divide the both sides by q_m, then

$$1.25506 \times \frac{1}{\log q_m} \times (\log\log n + \log\log\log n) > 1 + 1 - \frac{1}{\log q_m} + \sum_{i=1}^{m} \frac{\log\log q_i}{q_m}.$$

According to the lemma 8.8, we know that

$$-\frac{1}{\log q_m} + \sum_{i=1}^{m} \frac{\log\log q_i}{q_m} = \alpha > 0.$$

Consequently, we would have that

$$1.25506 \times (\frac{\log\log n}{\log q_m} + \frac{\log\log\log n}{\log q_m}) > 2 + \alpha.$$

If we use the lemma 8.7, then

$$1.25506 \times (1 + \frac{1}{2 \times \log^2 q_m} + \frac{\log\log q_m}{\log q_m} + \frac{1}{2 \times \log^3 q_m}) > 2 + \alpha.$$

We know that

$$1.25506 \times (1 + \frac{1}{2 \times \log^2 q_m} + \frac{\log\log q_m}{\log q_m} + \frac{1}{2 \times \log^3 q_m})$$
$$\leq 1.25506 \times (1 + \frac{1}{2 \times \log^2 953} + \frac{\log\log 953}{\log 953} + \frac{1}{2 \times \log^3 953})$$

and we have that

$$1.25506 \times (1 + \frac{1}{2 \times \log^2 953} + \frac{\log\log 953}{\log 953} + \frac{1}{2 \times \log^3 953}) \approx 1.62266460495.$$

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Consequently, we have that

$$2 > 1.25506 \times (1 + \frac{1}{2 \times \log^2 q_m} + \frac{\log\log q_m}{\log q_m} + \frac{1}{2 \times \log^3 q_m}) > 2 + \alpha > 2$$

and

$$2 > 2$$

is a contradiction. To sum up, we obtain a contradiction just assuming that $n > 5040$ is the smallest integer such that $\mathsf{Robins}(n)$ does not hold.

Theorem 9.3 $\mathsf{Robins}(n)$ *holds for all* $n > 5040$.

Proof Due to the theorem 9.2, we can assure there is not any natural number $n > 5040$ such that $\mathsf{Robins}(n)$ does not hold.

Theorem 9.4 *The Riemann Hypothesis is true.*

Proof This is a direct consequence of theorems 1.1 and 9.3

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