Renormalizable and unitary Lorentz invariant model of quantum gravity

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Abstract

We analyze the $R + R^2$ model of quantum gravity where terms quadratic in the curvature tensor are added to the General Relativity action. This model was recently proved to be a self-consistent quantum theory of gravitation, being both renormalizable and unitary. The model can be made practically indistinguishable from General Relativity at astrophysical and cosmological scales by the proper choice of paarameters.

1   Introduction

Creation of quantum gravity still remains a prominent task of modern physics.

The problem is due to well known non-renormalizability of General Relativity. In the work [1] it was shown by direct calculations that at the one loop level General Relativity is renormalizable without matter fields but becomes unrenormalizable after inclusion of matter fields. Then also by explicit calculations it was demonstrated [2] that General Relativity is non-renormalizable at the two-loop level even without matter fields.

In [3] renormalizability of the $R+R^2$-theory was proved. The proof used a specific covariant gauge for simplicity. For general gauges an assumption was made that ultraviolet divergences have the so called cohomological structure. This hypothesis was proved for a class of background gauges in the work [4]. Hence we consider renormalizability of $R + R^2$ gravity with four derivatives of the metric as well established.

We will also call this model briefly as quadratic gravity.

But in the works [3, 5] it was also stated that quadratic gravity is not physical because it violates unitarity or causality. So this model was commonly considered as unphysical.

Quite recenyly quadratic quantum gravity was proved to be in fact unitary [6]. Thus the $R + R^2$ model is a candidate for the quantum theory of gravitation.

In the present paper we discuss in detail the exact form of the Lagrangian of quadratic gravity, the questions of unitarity, stability of the vacuum state and the behavior of the model at astrophysical and cosmological scales.

2   Main part

We consider the relativistic $R + R^2$ action including all terms quadratic in the Riemann tensor $R_{\mu\nu}$

$$S_{sym} = \int d^D x \mu^{-2\epsilon} \sqrt{-g} \left(-M_{Pl}^2 R + \alpha R_{\mu\nu} R^{\mu\nu} + \beta R^2 + \delta R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + M_{Pl}^2 \Lambda \right),$$

(1)

here the R-term is the Einstein-Hilbert Lagrangian. The Λ-term is not essential in perturbation theory which we consider.

$M_{Pl}^2 = 1/(16\pi G)$ is the Planck mass squared, $R_{\mu\nu\rho\sigma}$ is the Riemann tensor, $R_{\mu\nu}$ is the Ricci tensor and R is the Ricci scalar. α, β and δ are coupling

constants, $D = 4 - 2\epsilon$ is the dimension of the space-time within dimensional regularization [7]. ϵ is the regularizarion parameter and μ is the parameter with the dimension of a mass in dimensional regularization.

The Riemann tensor reads

$$R^{\rho}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda} \Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda} \Gamma^{\lambda}_{\mu\sigma}, \tag{2}$$

here are the Christoffell symbols

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left(\partial_\nu g_{\mu\beta} + \partial_\mu g_{\nu\beta} - \partial_\beta g_{\mu\nu} \right). \tag{3}$$

Let us underline that dimensional regularization [7] is presently the only known regularization of ultraviolet divergences preserving gauge invariance of gravity.

The term contaning the coupling δ in the Lagrangian (1) is omitted in the literature, see e.g. [3, 4, 8]. This is because of the Gauss-Bonnet identity

$$\int d^4x \sqrt{-g} \left(R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \right) = 0. \tag{4}$$

The identity is valid only in four-dimensional space. But the dimension of the space-time in dimensional regularization is $4 - 2\epsilon$. Thus it seems that the term with the coupling δ must be preserved in the action to have renormalizability.

From the other side it is in principle possible that one will invent four-dimensional regularizatiom which preserves gauge invariance of gravitational Lagrangian. Then the term with the coupling δ should be omitted. The number of coupling constants in the Lagrangian most probably should not depend on the choice of regularization. In this case the term with the coupling δ should be omitted in dimensional regularization also. The point can be checked with direct calculations of counterterms of the Lagrangian. Corresponding calculations are rather involved even at the one-loop level. This is a subject for a seperate publication.

We will work within perturbation theory. Thus a linearized theory is considered around the flat space metric

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{5}$$

here the convention in four dimensions is $\eta_{\mu\nu} = diag(+1, -1, -1, -1)$. Within dimensional regularization $\eta_{\mu\nu}\eta^{\mu\nu} = D$. Indexes are raised and lowered by means of the tensor $\eta_{\mu\nu}$.

Gauge transformations of gravity are generated by diffeomorphisms $x^\mu \to x^\mu + \zeta^\mu(x)$ and have the form

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \zeta_\nu + \partial_\nu \zeta_\mu + \left(h_{\lambda\mu}\partial_\nu + h_{\lambda\nu}\partial_\mu + (\partial_\lambda h_{\mu\nu})\right)\zeta^\lambda, \qquad (6)$$

whith arbitrary functions $\zeta_\mu(x)$.

Following standard Faddeev-Popov quantization [9], see also [10], one adds to the Lagrangian a gauge fixing term which can br chosen e.g. in the form

$$S_{gf} = -\frac{1}{2\xi}\int d^D x F_\mu \partial_\nu \partial^\nu F^\mu, \qquad (7)$$

here $F^\mu = \partial_\nu h^{\nu\mu}$, ξ is the gauge parameter. Physical results, of course, do not depend on the allowed choice of the form of the gauge fixing term.

One should also add the ghost term

$$S_{ghost} = \int d^D x d^D y \overline{C}_\mu(x)\frac{\delta F^\mu(x)}{\delta \zeta_\nu(y)} C_\nu(y) = \qquad (8)$$

$$\int d^D x \partial^\nu \overline{C^\mu}\left[\partial_\nu C_\mu + \partial_\mu C_\nu + h_{\lambda\mu}\partial_\nu C^\lambda + h_{\lambda\nu}\partial_\mu C^\lambda + (\partial_\lambda h_{\mu\nu})C^\lambda\right],$$

where $\overline{C}$ and C are ghost fields. Then one gets the generating functional of graviton Green functions

$$Z(J) = N^{-1}\int dh_{\mu\nu}dC_\lambda d\overline{C_\rho}\exp\left[i\left(S_{sym} + S_{gf} + S_{ghost} + d^D x \mu^{-2\epsilon}J_{\mu\nu}h^{\mu\nu}\right)\right], \qquad (9)$$

here N is the normalization factor of the functional integral in the usual notaton, $J_{\mu\nu}$ is as usual the source of gravitons.

We work within perturbation theory, hence one makes the shift of the fields

$$h_{\mu\nu} \to M_{Pl}\mu^{-\epsilon}h_{\mu\nu}. \qquad (10)$$

Perturbative expansion is in inverse powers of the Plank mass or in other words in powers of the Newton coupling constant $G \propto 1/M_{Pl}^2$.

Let us obtain the graviton propagator. One takes the part of the Lagrangian quadratic in $h_{\mu\nu}$ and makes the Fourier transform

$$Q_{\mu\nu\rho\sigma} = \frac{1}{4}\int d^D k \; h^{\mu\nu}(-k)\left[\left(k^2 + M_{Pl}^{-2}k^4(\alpha + 4\delta)\right)P_{\mu\nu\rho\sigma}^{(2)}\right.$$

$$\left. + k^2\left(-2 + 4M_{Pl}^{-2}k^2(\alpha + 3\beta + \delta)\right)P_{\mu\nu\rho\sigma}^{(0-s)} \qquad (11)\right.$$

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$$+\frac{1}{\xi}M_{Pl}^{-2}k^4\left(P_{\mu\nu\rho\sigma}^{(1)}+2P_{\mu\nu\rho\sigma}^{(0-w)}\right)\Bigg]h^{\rho\sigma}(k),$$

$P_{\mu\nu\rho\sigma}^{(i)}$ beung projectors to the spin-2, spin-1 and spin-0 components of the field $h_{\mu\nu}$:

$$P_{\mu\nu\rho\sigma}^{(2)}=\frac{1}{2}\left(\Theta_{\mu\rho}\Theta_{\nu\sigma}+\Theta_{\mu\sigma}\Theta_{\nu\rho}\right)-\frac{1}{3}\Theta_{\mu\nu}\Theta_{\rho\sigma},\tag{12}$$

$$P_{\mu\nu\rho\sigma}^{(1)}=\frac{1}{2}\left(\Theta_{\mu\rho}\omega_{\nu\sigma}+\Theta_{\mu\sigma}\omega_{\nu\rho}+\Theta_{\nu\rho}\omega_{\mu\sigma}+\Theta_{\nu\sigma}\omega_{\mu\rho}\right),\tag{13}$$

$$P_{\mu\nu\rho\sigma}^{(0-s)}=\frac{1}{3}\Theta_{\mu\nu}\Theta_{\rho\sigma},\tag{14}$$

$$P_{\mu\nu\rho\sigma}^{(0-w)}=\omega_{\mu\nu}\omega_{\rho\sigma}.\tag{15}$$

Here $\Theta_{\mu\nu}=\eta_{\mu\nu}-k_\mu k_\nu/k^2$ and $\omega_{\mu\nu}=k_\mu k_\nu/k^2$ are transverse and longitudinal projectors correspondingly.

We note that the expression (11) differs from the similar one in [8] by the absence of ϵ-dependent terms.

To get the graviton propagator $D_{\mu\nu\rho\sigma}$ one inverts the matrix in square brackets of the expression (11):

$$[Q]_{\mu\nu\kappa\lambda}D^{\kappa\lambda\rho\sigma}=\frac{1}{2}(\delta_\mu^\rho\delta_\nu^\sigma+\delta_\mu^\sigma\delta_\nu^\rho).\tag{16}$$

Then the propagator has the form

$$D_{\mu\nu\rho\sigma}=\frac{1}{i(2\pi)^D}\Bigg[\frac{4}{k^2}\left(\frac{1}{1+M_{Pl}^{-2}k^2(\alpha+4\delta)}\right)P_{\mu\nu\rho\sigma}^{(2)}\tag{17}$$

$$-\frac{2}{k^2}\left(\frac{1+2\epsilon\frac{1-M_{Pl}^{-2}k^2(\alpha+4\beta)}{1+M_{Pl}^{-2}k^2(\alpha+4\delta)}}{1-\epsilon-M_{Pl}^{-2}k^2\left((2\alpha+6\beta+2\delta)-\epsilon(\alpha+4\beta)\right)}\right)P_{\mu\nu\rho\sigma}^{(0-s)}$$

$$+4\xi\frac{1}{M_{Pl}^{-2}k^4}\left(P_{\mu\nu\rho\sigma}^{(1)}+\frac{1}{2}P_{\mu\nu\rho\sigma}^{(0-w)}\right)\Bigg].$$

Then one performs partial fractioning. The propagator takes the forn

$$D_{\mu\nu\rho\sigma}=\frac{1}{i(2\pi)^D}\Bigg[4P_{\mu\nu\rho\sigma}^{(2)}\left(\frac{1}{k^2}-\frac{1}{k^2-M_{Pl}^2/(-\alpha-4\delta)}\right)\tag{18}$$

$$-2\frac{P_{\mu\nu\rho\sigma}^{(0-s)}}{1-\epsilon}\left(1+2\epsilon\frac{1-M_{Pl}^{-2}k^2(\alpha+4\beta)}{1+M_{Pl}^{-2}k^2(\alpha+4\delta)}\right)$$

$$\left(\frac{1}{k^2} - \frac{1}{k^2 - M_{Pl}^2(1-\epsilon)/(2\alpha + 6\beta + 2\delta - \epsilon(\alpha + 4\beta))}\right)$$

$$+\frac{4\xi}{M_{Pl}^{-2}k^4}\left(P_{\mu\nu\rho\sigma}^{(1)} + \frac{1}{2}P_{\mu\nu\rho\sigma}^{(0-w)}\right)\Bigg].$$

In four dimensions one obtains the following graviton propagator

$$D_{\mu\nu\rho\sigma} = \frac{4}{i(2\pi)^D}\left[\frac{P_{\mu\nu\rho\sigma}^{(2)} - \frac{1}{2}P_{\mu\nu\rho\sigma}^{(0-s)}}{k^2} - \frac{P_{\mu\nu\rho\sigma}^{(2)}}{k^2 - M_{Pl}^2/(-\alpha - 4\delta)}\right. \qquad (19)$$

$$\left. +\left(\frac{1}{2}\right)\frac{P_{\mu\nu\rho\sigma}^{(0-s)}}{k^2 - M_{Pl}^2/(2\alpha + 6\beta + 2\delta)} + \frac{\xi}{M_{Pl}^{-2}k^4}\left(P_{\mu\nu\rho\sigma}^{(1)} + \frac{1}{2}P_{\mu\nu\rho\sigma}^{(0-w)}\right)\right],$$

We will now consider classical quadratic gravity. In this case for a point particle having the energy-momentum tensor $T_{\mu\nu} = \delta_\mu^0\delta_\nu^0 M\delta^3(x)$ one gets the gravitational field [5]

$$V(r) = \frac{M}{2\pi M_{Pl}^2}\left(-\frac{1}{4r} + \frac{e^{-m_2 r}}{3r} - \frac{e^{-m_0 r}}{12r}\right). \qquad (20)$$

$m_2^2 = M_{Pl}^2/(-\alpha - 4\delta)$ and $m_0^2 = M_{Pl}^2/(2\alpha + 6\beta + 2\delta)$ are squared masses correspondingly of massive spin-2 and spin-0 gravitons. Cupling constants α, β and δ can be chosen to obtain positive masses. In [3, 5] it was noted that masses can be chosen large enough to have an agreement with experiments.

Our propagator (19) reproduces the expression (20). One can see it by means of the calculation of the tree level Feynman diagram corresponding to an exchange of two point-like particles by a graviton.

The graviton propagator in the work [3] dos not produce the expression (20). It contains some technical errors. To see this one puts in the $R + R^2$ Lagrangian coupling constants equal to zero except the Newton coupling. The Lagrangian is reduced then to General Relativity. Hence the graviton propagator should also be reduced to one of General Relativity:

$$D_{\mu\nu\rho\sigma}(k) = \frac{1}{i(2\pi)^4}\frac{1/2\eta_{\mu\rho}\eta_{\nu\sigma} + 1/2\eta_{\mu\sigma}\eta_{\nu\rho} - 1/2\eta_{\mu\nu}\eta_{\rho\sigma} + terms \propto k}{k^2}, \qquad (21)$$

where the gauge condition with $\xi = 0$ is taken for simplicity.

Our propagator (19) reproduces the propagator 21 in this limit. The propagator of the work [3] has the factor 1 instead of 1/2 in the third term of the numerator of (19) in the corresponding limit.

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The second term in the expression (19) for the graviton propagator has the non-standard minus sign. Hrnce one considers it as the massive spin-2 ghost. For renormalizability of quadratic gravity one must shift poles of all propagators in Feynman diagrams in the same way $k^2 \to k^2 + i0$. Hence the spin-2 ghost must be considered as a state with negative metric [3]. That is why violation of either unitarity or causality within the $R + R^2$ model was claimed in [3, 5].

But this massive spin-2 ghost is unstable. It unavoidably decays in two or more physical massless gravitons. The width of the decay is small. However independently of the value of this decay width spin-2 ghost particles do not appear as asymptotic states of the S-matrix elements. Only physical gravitons appear as external particles of the S-matrix amplitudes. Thus one concludes that unitarity is preserved in the $R + R^2$ model.

There is a statement about instability of theories with ghosts , i.e. their Hamiltonians are unbounded from below and they do not have stable vacuum states. This question was raised in [11] within Quantum Mechanics, see also [12] for the brief review. But this statement is proved only for Quantum Mechanical systems. Quantum Field Theory is quite a different story and renomalizability plus unitarity is enough to have a consistent theory.

To see this let us consider the graviton propagator in the opeator formalism:

$$D_{\mu\nu\rho\sigma}(x-y) = \frac{\delta^2}{\delta J_{\mu\nu}(x)\delta J_{\rho\sigma}(y)} Z(J) = <0|T\left[h_{\mu\nu}(x)h_{\rho\sigma}(y)\right]|0> . \quad (22)$$

One transforms it to the momentum space and inserts the sum over the complete set of momentum eigenstates between two graviton fields. The states with negative norms in the sum have the extra factor -1. It gives the negative residue for the massive spin-2 ghost pole.

There is another way to produce the negative residue for the spin-2 ghost. One can prescribe negative energy to this ghost. The expansion of the graviton fields into the creation and annihilation operators produces normalization factors $1/\sqrt{-2k_0}$. This is the reason for the negative residue for the spin-2 ghost. In this case of negative energy the Hamiltonian would be indeed unbounded and the vacuum state would be unstable.

But as it was mentioned above, one should choose the variant with negarive metric in order to have renormalizability in the theory [3]. Thus one has the consistent theory with the stable ground state.

It should be mentioned that the S-matrix by construction automatically satisfies the unitarity relation

$$S^+S = 1 \tag{23}$$

in theories having Hermitian Lagrangians [13].

To see it one considers the S-matrix in the operator formalism

$$S = T\left(e^{i\int L(x)dx}\right). \tag{24}$$

One introduces a function $g(x)$ having the values in the interval $(0,1)$. This function describes intensity of interactions. Interactions are switched off if $g(x) = 0$. If $g(x) = 1$ then interactions are switched on. Interactions are switched on partly if $0 < g(x) < 1$. One substitutes the product $L(x)g(x)$ for the Lagrangian $L(x)$. The S-matrix becomes the functional

$$S(g) = T\left(exp \; i\int L(x)g(x)dx\right). \tag{25}$$

One splits the interaction region characterised by the function $g(x)$ into an infinitely large number of infinitely thin segments Δ_i using the space-like surfaces $t = const$.

Then one gets

$$S(g) = T\left(exp \; i\int L(x)g(x)dx\right) = T\left(exp \; i\sum_j \int_{\Delta_j} L(x)g(x)dx\right) = \tag{26}$$

$$T\left(\prod_j exp \; i\int_{\Delta_j} L(x)g(x)dx\right).$$

$S(g)$ is defined as the limit

$$S(g) = \lim_{\Delta_j \to 0} T\left(\prod_j \left(1 + i\int_{\Delta_j} L(x)g(x)dx\right)\right). \tag{27}$$

The r.h.s. of (27) is a product taken in the chronological order of the segments Δ_j. Each factor in this product is unitary up to small terms of higher orders for sufficiently small Δ_j. These higher orders can be neglected in the

considered limit. Hence the whole product is unitary. Unitarity of $S(g)$ and of the matrix

$$S = \lim_{g(x) \to 1} S(g) \qquad (28)$$

is proved.

Sometimes one understands the following thing under unitarity. One derives from (23) the famous optical theorem stating that imagenary part of an amplitude of some forward scattering coincides up to a factor with the corresponding total annihilation crosssection

$$Im < i|T|i > = \frac{1}{2} \sum_n < i|T^+|n > < n|T|i >, \qquad (29)$$

where $|i >$ is the scattering state, T is the scattering matrix: $S = 1 + iT$, and one assumes that all physical states $|n >$ form a complete set in the theory

$$\sum_n |n > < n| = 1. \qquad (30)$$

From the other side one can calculate $Im < i|T|i >$ directly from Feynman diagrams using Cutcosky cuts. Then one assumes that the result should coinside with (29). But if it does not happen it does not mean violation of unitarity. It only means that physical states in the theory do not form a complete set (30) and the complete set is formed by physical plus unphysical states.

Unitarity of theories with negative metric states was previously considered in [14, 15], see also references therein. Question of causality were also considered there.

We would like to note that the tree level graviton propagator (19) is modified by the summation of the chain of one-loop insertions. As it was already mentioned above the second term of the propagator (19) has the minus sign. Therefore the summation of the one-loop insertions with the massless graviton in the loop will shift the pole of the spin-2 ghost from the value $k^2 = M_{Pl}^2/(-\alpha - 4\delta)$ to the complex value $k^2 = M_{Pl}^2/(-\alpha - 4\delta) - i\Gamma$. Here Γ is the width of the spin-2 ghost decay into the pare of massless physical gravitons. This complex pole is located on the unphysical Riemann sheet. It is analogous to the known virtual level of the neutron-proton system with opposite spins of nucleons [16].

We would like to underline that we consider not pure R^2 theory but the $R + R^2$ theory where the R^2 terms are added to the Einstein-Hilbert

Lagrangian. Gravitational constants α, β and δ of these terms in the Lagrangian can be chosen suffisiently small to ensure that quadratic gravity will be practically indistinguishable from General Relativity at astrophysical and cosmological scales. This is independently of the discussed above in detail question wether the coupling δ is exactly zero or not. The R^2 terms are introduced only to have renormalizability of quadratic gravity which is valid in particular for arbitrary small couplings α, β and δ.

We have analyzed only purely gravitational $R + R^2$ action. The inclusion of the matter fields in the Lagrangian is straightforward and dose not change conclusions.

3   Conclusions

We have proved unitarity of quantum gravity with the $R + R^2$ action. This model was previously shown to be renormalizable in the work [3]. The parameters of quadratic gravity can be adjusted to ensure that the theory will be practically indistinguishable from General Relativity at astrophysical and cosmological scales.

One can conclude that the $R + R^2$ model is an appropriate candidate for the fundamental quantum theory of gravity.

4   Acknowledgments

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