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Vector Potential, Magnetic Field, Mutual Inductance, Magnetic Force, Torque and Stiffness Calculation between Current-Carrying Arc Segments with Inclined Axes in Air

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Abstract: In this paper we give the improved and new analytical and semi-analytical expression for calculating the magnetic vector potential, magnetic field, magnetic force, mutual inductance, torque, and stiffness between two inclined current-carrying arc segments in air. The expressions are obtained either in the analytical form over the incomplete elliptic integrals of the first and the second time or by the single numerical integration of some elliptical integrals of the first and the second kind. The validity of the presented formulas is proved from the special cases when the inclined circular loops are treated. We mention that all formulas are obtain by the integral approach except the stiffness which is found by the derivative of the magnetic force.

Keywords: Vector Potential; Magnetic Field; Mutual Inductance; Magnetic Force; Torque; Stiffness.

1. Introduction

Analytical and semi-analytical methods in the calculation of parameters of electric circuits and force interaction between their elements play an important role in power transfer, wireless communication, sensing and actuation, are applied in different fields of science, including electrical and electronic engineering, medicine, physics, nuclear magnetic resonance, mechatronics, and robotics. Several efficient numerical methods implemented in the commercially available software currently provides an accurate and fast solution for the calculation of parameters of electrical circuits. However, analytical methods allow to obtain the result in the form of a final formula with a finite number of input parameters, which when applicable may significantly reduce computation effort and facilitate mathematical analysis. The calculation of the magnetic vector potential, magnetic field, mutual inductance, magnetic force, torque and stiffness are used in many applications including magnetic force interaction, electromagnetic levitation, superconducting levitation, wireless power transfer, electromagnetic actuation, micromachined contactless inductive suspensions, hybrid suspensions, biomedical applications, topology optimization, nuclear magnetic resonance, indoor positioning systems, navigation sensors, and magneto-inductive wireless communications [1-]. In this paper we calculate all previously mentioned electromagnetic quantities in the analytical and semi-analytical form. We treat the current-carrying arc segment with inclined axes in air. All expressions are obtained in the form of the incomplete elliptic integrals of the first and second kind and in the form of the single integrals whose kernel functions are also the incomplete elliptic integrals of the first and second kind. We mention that some formulas are improved such as the calculation of the magnetic vector potential and magnetic field. The new formulas are given for the mutual inductance, the magnetic force, the torque, and the stiffness. In these formulas the angles of current-carrying arcs are arbitrary. The validity of all formulas is verified with the corresponding calculations for the inclined circular loops. For the convenience of a reader, all derived formulas including were programmed by using the Mathematica. The Mathematica files with the implemented formulas are available from the author as supplementary materials to this paper.

2. Basic expressions

Let us take into consideration two current-carrying arc segments as showed in Fig.1, where the center of the larger segment (primary coil) of the radius R_P is placed at the plane *XOY* whose center is *O* (0,0,0). The smaller circular segment (secondary coil) of the radius R_S is placed in an inclined plane whose general equation is,

$$\lambda \equiv ax + by + cz + d = 0 \tag{1}$$

where *a*, *b* and *c* are the components of the normal $\vec{N}$ on the inclined plane in the center of the secondary circular segment *C* (x_C, y_C, z_C). The segments are with the currents R_P R_S, respectively. For circular segments (see Figure 1) we define, [22-24]:

1) The primary circular segment of radius R_P is placed in the plane *XOY* (*Z* = 0) with the center at O (0, 0, 0). An arbitrary point *P* (x_P, y_P, z_P) of this segment has parametric coordinates,

$$x_P = R_P \cos(t), y_P = R_P \sin(t), \quad z_P = 0, \quad t \in (\varphi_1, \varphi_2) \tag{2}$$

2) The differential of the primary circular segment is given by,

$$d\vec{l}_P = R_P \{-\sin(t), \cos(t), 0\} dt, \quad t \in (\varphi_1, \varphi_2) \tag{3}$$

3) The primary circular segment of radius R_S is placed in the inclined plane (1) with the center at *C* (x_C, y_C, z_C). The unit vector **N** (the unit vector of the axis z') at the point C which is the center of the secondary circular segment) laying in the plane λ is defined by,

$$\vec{N} = \left\{\frac{a}{L}, \frac{b}{L}, \frac{c}{L}\right\}, \quad L = \sqrt{a^2 + b^2 + c^2} \tag{4}$$

4) The unit vector between two points *C* and *S* they are placed in the plane (1) is,

$$\vec{u} = \{u_x, u_y, u_z\} = \left\{-\frac{ab}{Ll}, \frac{l}{L}, -\frac{bc}{Ll}\right\}, l = \sqrt{a^2 + c^2} \tag{5}$$

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5) We define the unit vector $\boldsymbol{v}$ as the cross product of the unit vectors $\boldsymbol{N}$ and $\boldsymbol{u}$ as follows,

$$\vec{v} = \vec{N} \times \vec{u} = \{v_x, v_y, z_z\} = \left\{-\frac{c}{l}, 0, \frac{a}{l}\right\} \quad (6)$$

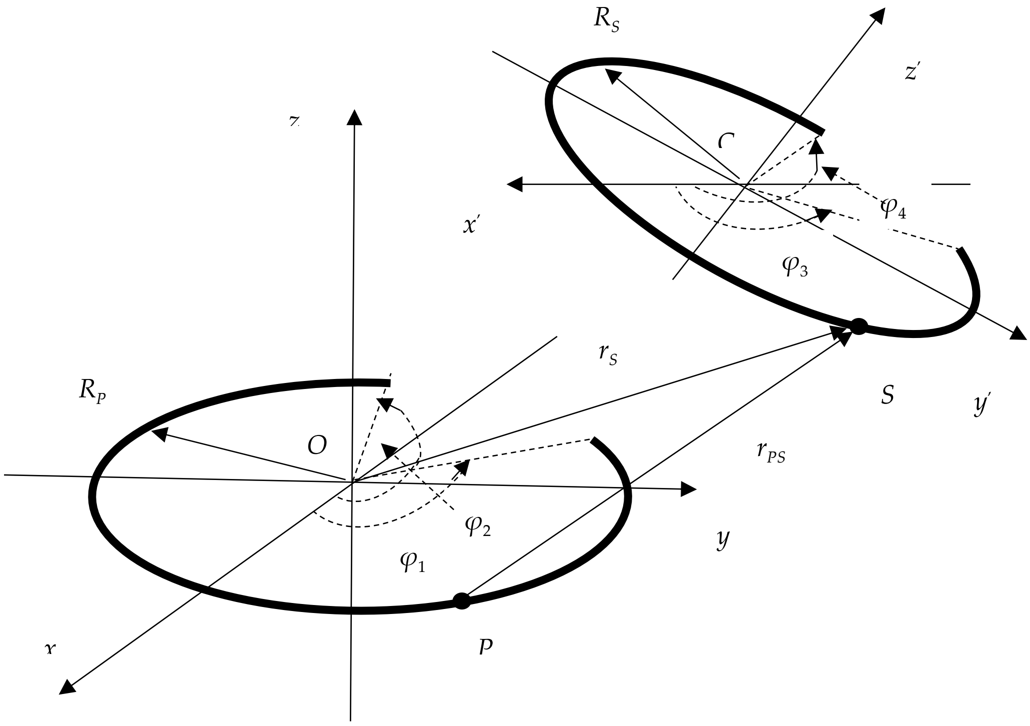

Figure1. Two inclined current-carrying segments

6) An arbitrary point S (x_S y_S, z_S) of the secondary circular segment has parametric coordinates,

$$x_s = x_C + R_s u_x \cos(\theta) + R_s v_x \sin(\theta)$$

$$y_s = y_C + R_s u_y \cos(\theta) + R_s v_y \sin(\theta) \quad (7)$$

$$z_s = z_C + R_s u_z \cos(\theta) + R_s v_z \sin(\theta)$$

$$\theta \in (\varphi_3, \varphi_4)$$

This is well-known parametric equation of circle in $3D$ space. The filamentary circular segments are the part of this cercle.

7) The differential element of the secondary circular segment is given by,

$$d\vec{l_S} = R_S \{l_{xS}, l_{yS}, l_{zS}\} d\theta, \quad \theta \in (\varphi_3, \varphi_4) \quad (8)$$

where

$$l_{xS} = -u_x\sin(\theta) + v_x\cos(\theta)$$

$$l_{yS} = -u_y\sin(\theta) + v_y\cos(\theta)$$

$$l_{zS} = -u_z\sin(\theta) + v_z\cos(\theta)$$

3. Magnetic vector potential calculation at the point $S(x_S, y_S, z_S)$

The magnetic vector potential $\vec{A}(S)$ produced by the primary circular arc segment of the radius R_P carrying the current I_P, can be calculated in an arbitrary point S (x_S, y_S, z_S) by,

$$\vec{A}(S) = \frac{\mu_0}{4\pi}\int\frac{I_P d\vec{l}_P}{r_{PS}} \tag{9}$$

where,

$$\vec{r}_{PS} = (x_S - x_P)\vec{\imath} + (y_S - y_P)\vec{\jmath} + (z_S - z_P)\vec{k}$$

$$r_{PS}^2 = (x_S - x_P)^2 + (y_S - y_P)^2 + (z_S - z_P)^2 = x_S^2 + y_S^2 + z_S^2 + R_P^2 -$$

$$-2R_P\sqrt{x_S^2 + y_S^2}\cos(t-\gamma)$$

$$\cos(\gamma) = \frac{x_S}{p}.\quad \sin(\gamma) = \frac{y_S}{p},\qquad p = \sqrt{x_S^2 + y_S^2}$$

$\vec{\imath}$, $\vec{\jmath}$ and $\vec{k}$ the unit vector of axes x, y, and z respectively.

From (3) and (9) we have,

$$A_x(S) = -\frac{\mu_0 I_P R_P}{4\pi}\int_{\varphi_1}^{\varphi_2}\frac{\sin(t)}{r_{PS}}dt \tag{10}$$

$$A_y(S) = \frac{\mu_0 I_P R_P}{4\pi}\int_{\varphi_1}^{\varphi_2}\frac{\cos(t)}{r_{PS}}dt \tag{11}$$

$$A_z(S) = 0 \tag{12}$$

Let us introduce the following substitution $t - \gamma = \pi - 2\beta$.

(10), (11) and (12) become,

$$A_x(S) = -\frac{\mu_0 I_P\sqrt{R_P}}{4\pi p\sqrt{p}}k\int_{\beta_1}^{\beta_2}\frac{\sin(\gamma - 2\beta)}{\Delta}d\beta \tag{13}$$

$$A_y(S) = \frac{\mu_0 I_P\sqrt{R_P}}{4\pi p\sqrt{p}}k\int_{\beta_1}^{\beta_2}\frac{\cos(\gamma - 2\beta)}{\Delta}d\beta \tag{14}$$

$$A_z(S) = 0 \tag{15}$$

$$\Delta = \sqrt{1-k^2 sin^2(\beta)}\,,\qquad k^2 = \frac{4R_P p}{[R_P+p]^2+z_S^2},\qquad p=\sqrt{x_S^2+y_S^2}$$

The final solutions for (8), (9) and (10) can be obtained analytically in the form of the incomplete elliptic integrals of the first and the second kind and the simple elementary functions (Appendix A).

Finally,

$$A_x(S) = -\frac{\mu_0 I_P\sqrt{R_P}}{4\pi k p\sqrt{p}} I_x \tag{16}$$

$$A_y(S) = \frac{\mu_0 I_P\sqrt{R_P}}{4\pi k p\sqrt{p}} I_y \tag{17}$$

$$A_z(S) = 0 \tag{18}$$

where,

$$I_x = \{y_S[(k^2-2)F(\beta,k)+2E(\beta,k)]+2x_S\Delta\}\Big|_{\beta_1}^{\beta_2}$$

$$I_y = \{x_S[(k^2-2)F(\beta,k)+2E(\beta,k)]-2y_S\Delta\}\Big|_{\beta_1}^{\beta_2}$$

$$\beta_1 = \frac{\pi}{2}+\frac{\gamma-\varphi_1}{2},\quad \beta_2 = \frac{\pi}{2}+\frac{\gamma-\varphi_2}{2}$$

$F(\beta,k)$ and $E(\beta,k)$, [40-41] are the incomplete elliptic integrals of the first and the second kind.

These expressions are valid for $z=0$ and $x_S \neq R_P\cos(t)\,, y_S \neq R_P\sin(t)$.

3.1 *Special cases*

3.1 1. $\varphi_1 = 0,\ \ \varphi_2 = 2\pi$

$$A_x(S) = -A_0 \sin(\gamma) \tag{19}$$

$$A_y(S) = A_0 \cos(\gamma) \tag{20}$$

$$A_z(S) = 0 \tag{21}$$

$$A_0 = \frac{\mu_0 I_P\sqrt{R_P}}{2\pi k\sqrt{p}}[(2-k^2)K(k)-2E(k)],\qquad k^2 = \frac{4R_P p}{[R_P+p]^2+z_S^2}$$

where $K(k)$ and $E(k)$,[40-41] are the complete integrals of the first and the second kind. Expressions (19), (20) and (21) are valid for $z_S = 0$.

3.1 2. Z -axis ($x_S = y_S = 0, z_S \neq 0$)

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$$A_x(z_S) = \frac{\mu_0 I_P R_P}{4\pi\sqrt{z_S^2 + R_P^2}}[\cos(\varphi_2) - \cos(\varphi_1)] \quad (22)$$

$$A_y(z_S) = \frac{\mu_0 I_P R_P}{4\pi\sqrt{z_S^2 + R_P^2}}[\sin(\varphi_2) - \sin(\varphi_1)] \quad (23)$$

$$A_z(S) = 0 \quad (24)$$

3.1 3. $x_S = R_P\cos(t), y_S = R_P\sin(t),\ z_S = 0, \varphi \in (\varphi_1, \varphi_2)$, (See Figure 2)

This is the singular case. The point S is between φ_1 and φ_2 on the cercle.

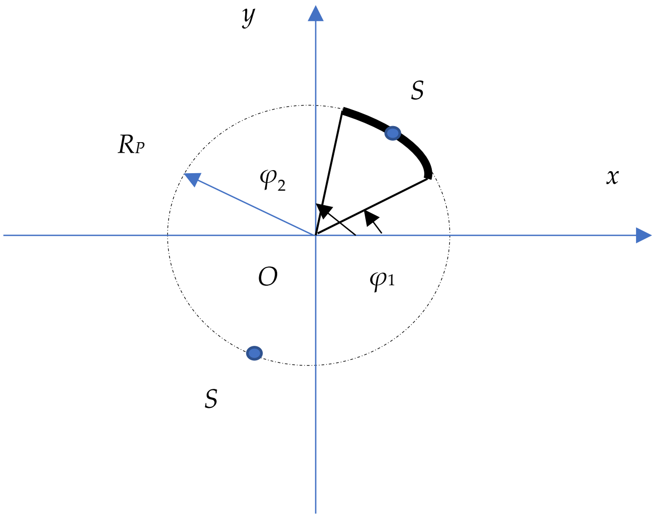

Figure 2. The point S lies on the arc segment with the arbitrary angels where $x_S^2 + y_S^2 = R_P^2$

3.1 4. $x_S = R_P\cos(t), y_S = R_P\sin(t),\ z_S = 0, \varphi \in (\varphi_2, \varphi_1 + 2\pi)$, (See Figure 2)

$$A_x(S) = -\frac{\mu_0 I_P}{4\pi}\left\{\sin(\gamma)log \mid tan\frac{\varphi_1 - \gamma}{\varphi_2 - \gamma} \mid - 2sin\frac{\varphi_1 + \gamma}{2} + 2sin\frac{\varphi_2 + \gamma}{2}\right\} \quad (25)$$

$$A_y(S) = \frac{\mu_0 I_P}{4\pi}\left\{\sin(\gamma)log \mid tan\frac{\varphi_1 - \gamma}{\varphi_2 - \gamma} \mid + 2cos\frac{\varphi_1 + \gamma}{2} - 2cos\frac{\varphi_2 + \gamma}{2}\right\} \quad (26)$$

$$A_z(S) = 0 \quad (27)$$

The point S is between φ_2 and $\varphi_1 + 2\pi$ on the cercle.
Thus, all results are obtained in the closed form over the incomplete elliptic integrals of the first and the second kind as well as over some elementary functions.

4. Magnetic field calculation at the point S (x_S ,y_S, z_S)

The magnetic field $\vec{B}(S)$ produced by the primary circular segment of the radius R_P carrying the current I_P, can be calculated in an arbitrary point S (x_S y_S, z_S) by,

$$\vec{B}(S) = \frac{\mu_0 I_P}{4\pi}\int \frac{d\vec{l}_P \times \vec{r}_{PS}}{r_{PS}^3} \quad (28)$$

From (2), (3) and (28) the components of the magnetic field are,

$$B_x(S) = \frac{\mu_0 I_P R_P z_S}{4\pi} \int_{\varphi_1}^{\varphi_2} \frac{\cos(t)}{r_{PS}^3} dt \tag{29}$$

$$B_y(S) = \frac{\mu_0 I_P R_P z_S}{4\pi} \int_{\varphi_1}^{\varphi_2} \frac{\sin(t)}{r_{PS}^3} dt \tag{30}$$

$$B_z(S) = \frac{\mu_0 I_P R_P}{4\pi} \int_{\varphi_1}^{\varphi_2} \frac{R_P - \sqrt{x_S^2 + y_S^2}\cos(t-\gamma)}{r_{PS}^3} dt \tag{31}$$

where r_{PS}, γ and p are previously given.

Let us introduce the following substitution $t - \gamma = \pi - 2\beta$.
(29), (30) and (31) become,

$$B_x(S) = \frac{\mu_0 I_P R_P z_S}{4\pi} \int_{\varphi_1}^{\varphi_2} \frac{\cos(t)}{r_{PS}^3} dt \tag{32}$$

$$B_y(S) = \frac{\mu_0 I_P R_P z_S}{4\pi} \int_{\varphi_1}^{\varphi_2} \frac{\sin(t)}{r_{PS}^3} dt \tag{33}$$

$$B_z(S) = -\frac{\mu_0 I_P R_P k^3}{16\pi p R_P \sqrt{p R_P}} \int_{\beta_1}^{\beta_2} \frac{R_P + \sqrt{x_S^2 + y_S^2}\cos(2\beta)}{\Delta^3} d\beta \tag{34}$$

where β_1 and β_2 are previously given.

The final solutions for (32), (33) and (34) can be obtained analytically in the form the incomplete elliptic integrals of the first and second kind and simple elementary functions (See Appendix B).

$$B_x(S) = \frac{\mu_0 I_P z_S k}{16\pi p^2 \sqrt{R_P p}(1-k^2)} \, I_{xx} \tag{35}$$

$$B_y(S) = \frac{\mu_0 I_P z_S k}{16\pi p^2 \sqrt{R_P p}(1-k^2)} \, I_{yy} \tag{36}$$

$$B_z(S) = -\frac{\mu_0 I_P k}{16\pi p^2 \sqrt{R_P p}(1-k^2)} \, I_{yy} \tag{37}$$

where

$$I_{xx} = x_S \left\{ (k^2-2)E(\beta,k) + (2-2k^2)F(\beta,k) + k^2(2-k^2)\frac{\sin(\beta)\cos(\beta)}{\Delta} \right\} \Bigg|_{\beta_1}^{\beta_2} +$$

$$+ \frac{2y_S}{\Delta}(1-k^2) \Bigg|_{\beta_1}^{\beta_2}$$

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$$I_{yy} = y_S \left\{ (k^2 - 2)E(\beta, k) + (2 - 2k^2)F(\beta, k) + k^2(2 - k^2)\frac{\sin(\beta)\cos(\beta)}{\Delta} \right\} \Big|_{\beta_1}^{\beta_2} -$$

$$-\frac{2x_S}{\Delta}(1 - k^2) \Big|_{\beta_1}^{\beta_2}$$

$$I_{zz} = \{[k^2(R_p + p) - 2p]E(\beta, k) + (2p - 2pk^2)F(\beta, k) +$$

$$+k^2(2p - (R_p + p)k^2)\frac{\sin(\beta)\cos(\beta)}{\Delta} \Big\} \Big|_{\beta_1}^{\beta_2}$$

where Δ and k^2 are previously given.

Thus, for the given point $S(x_S, y_S, z_S)$ the magnetic field produced by the circular segment with the current I_P can be calculated analytically over the incomplete elliptic integrals of the first and the second kind (35) (26) and (37).

4.1 *Special cases*

4.1 1. $z_S = 0$

$$B_x(S) = 0 \tag{38}$$

$$B_y(S) = 0 \tag{39}$$

$$B_z(S) = -\frac{\mu_0 I_P k_0}{16\pi p\sqrt{R_P p}(1 - k_0^2)} I_z(k_0) \tag{40}$$

$$k_0^2 = \frac{4R_P p}{[R_P + p]^2}, \quad x_S^2 + y_S^2 = R_p^2$$

$I_z(k_0)$ is given by I_{zz} from (34).

4.1 2. $z_S = 0\,,\ x_S^2 + y_S^2 = R_p^2$ and $\varphi \in [\varphi_1, \varphi_2\,]$

This is the singular case, (see Figure 2) where the point S is between φ_1 and φ_2.

4.1 3. $z_S = 0\,,\ x_S^2 + y_S^2 = R_p^2$ and $\varphi \in (\varphi_2, \varphi_1 + 2\pi)$

$$B_x(S) = 0 \tag{41}$$

$$B_y(S) = 0 \tag{42}$$

$$B_z(S) = \frac{\mu_0 I_P}{4\pi R_p}\, log \mid tan \frac{\varphi_1 - \gamma}{\varphi_2 - \gamma} \mid \tag{43}$$

The point S is between φ_2 and $\varphi_1 + 2\pi$ on the cercle (See Figure 2).

4.1 4. $Z - axis\ S(0,0,z_S)$

$$B_x(S) = \frac{\mu_0 I_P R_p z_T}{4\pi \sqrt[2]{(R_P^2 + z_S^2)^3}} [\sin(\beta_2) - \sin(\beta_1)] \quad (44)$$

$$B_y(S) = \frac{\mu_0 I_P R_p z_T}{4\pi \sqrt[2]{(R_P^2 + z_S^2)^3}} [\cos(\beta_1) - \cos(\beta_2)] \quad (45)$$

$$B_z(S) = \frac{\mu_0 I_P R_P^2}{4\pi \sqrt[2]{(R_P^2 + z_S^2)^3}} [\beta_2 - \beta_1] \quad (46)$$

4.1 5. $\varphi_1 = 0$ and $\varphi_2 = 2\pi$

$$B_x(S) = B_0 \frac{z_S}{p} \left\{ \left[\frac{R_P^2 - p^2 - z_S^2}{(R_P - p)^2 + z_S^2} \right] E(k) - K(k) \right\} \cos(\gamma) \quad (47)$$

$$B_y(S) = B_0 \frac{z_S}{p} \left\{ \left[\frac{R_P^2 - p^2 - z_S^2}{(R_P - p)^2 + z_S^2} \right] E(k) - K(k) \right\} \sin(\gamma) \quad (48)$$

$$B_y(S) = B_0 \left\{ \left[\frac{R_P^2 + p^2 + z_S^2}{(R_P - p)^2 + z_S^2} \right] E(k) + K(k) \right\} \quad (49)$$

$$B_0 = \frac{\mu_0 I_P k}{4\pi \sqrt{R_P p}}, \quad k^2 = \frac{4 R_P p}{[R_P + p]^2 + z_S^2}, \quad p = \sqrt{x_S^2 + y_S^2}$$

This is well-known expressions [11] obtained in the form of the complete elliptic integrals of the first and second kind $K(k)$ and $E(k)$.

5. Magnetic force calculation between two inclined current-carrying arc segments

The magnetic force between two inclined arc segments carrying currents I_P and I_S can be calculated by,

$$\vec{F} = \frac{\mu_0 I_P I_S}{4\pi} \int_{\varphi_1}^{\varphi_2} \int_{\varphi_3}^{\varphi_4} \frac{d\vec{l}_s \times (d\vec{l}_P \times \vec{r}_{PS})}{r_{PS}^3} \quad (50)$$

where $\vec{r}_{PS}$ the vector between the point P of the primary arc segment and the point S of the second arc segment (oriented to S) and $d\vec{l}_P$ and $d\vec{l}_s$ are the elementary current-carrying element of the primary and the secondary arc segment given by (3) and (7), (See Figure 1).

Equations (50) can be written as follows,

$$\vec{F} = I_S \int_{\varphi_3}^{\varphi_4} d\vec{l}_S \times \vec{B}(S) \quad (51)$$

where $\vec{B}(S)$ is the magnetic field produced by the primary current I_P in the first arc segment, acting at the point S of the second arc segment.

Previously, we calculated the magnetic field whose components are given by equations (35) (36) and (37). Using (7), (35), (36), (35) and (51) the components of the magnetic forces are as follows,

$$F_x = I_S R_S \int_{\varphi_3}^{\varphi_4} \left[l_{yS} B_z(S) - l_{zS} B_y(S)\right] d\theta \tag{52}$$

$$F_y = -I_S R_S \int_{\varphi_3}^{\varphi_4} \left[l_{xS} B_z(S) - l_{zS} B_x(S)\right] d\theta \tag{53}$$

$$F_z = I_S R_S \int_{\varphi_3}^{\varphi_4} \left[l_{xS} B_y(S) - l_{yS} B_x(S)\right] d\theta \tag{54}$$

Thus, the calculation of the magnetic force is obtained by the simple integration where the kernel functions are given in the analytical form over the incomplete elliptic integrals of the first and the second kind.

5.1 ***Special cases***

5.1 1. $a = c = 0$

This case is the singular case. The first arc segment lies in the plane $z = 0$ and the second in the plane y = constant. There are two possibilities for this case.

5.1 2. $\vec{u} = \{-1,0,0\}$, $\vec{v} = \{0,0,-1\}$

5.1 3. $\vec{u} = \{0,0,-1\}$, $\vec{v} = \{-1,0,0\}$

These vectors must be used in equations (52), (53) and (54).

6. Magnetic torque calculation between two inclined current-carrying arc segments

Torque is defined as the cross product of a displacement and a force. The displacement is from the center for taking torque, which is arbitrarily defined, to the point S of application of the force to the body experiencing the torque.

$$d\,\vec{\tau} = \vec{r}_{CS} \times d\,\vec{F}(S) \tag{55}$$

In (55) $\vec{r}_{CS} = (x_S - x_C)\vec{\imath} + (y_S - y_C)\vec{\jmath} + (z_S - z_C)\vec{k}$ is the vector of displacement between the center C of the second arc segment and the point S of the application of the second arc segment.

Previously, we calculated the magnetic force between two arc current0carrying segments where we used the analytical expressions of the magnetic field at the at the point S of the second arc segment. The magnet field is produced by the current in the primary arc segment. We use the same reasoning for the torque and from (55) we have,

$$d\,\vec{\tau} = I_S R_S\,\vec{r}_{CS} \times \left(d\vec{l}_S \times \vec{B}(S)\right) \tag{56}$$

or,

$$\vec{\tau} = I_S R_S \int_{\varphi_3}^{\varphi_4} \vec{r}_{CS} \times \left(d\vec{l}_S \times \vec{B}(S)\right) \tag{57}$$

Using (7), (35), (36), (37) and developing the double cross product in (57) we obtain the final components of the torque.

$$\tau_x = I_S R_S \int_{\varphi_3}^{\varphi_4} J_x d\theta \tag{58}$$

$$\tau_y = I_S R_S \int_{\varphi_3}^{\varphi_4} J_y d\theta \tag{59}$$

$$\tau_z = I_S R_S \int_{\varphi_3}^{\varphi_4} J_x d\theta \tag{60}$$

where,

$$J_x = -[(y_S - y_C)l_{yS} + (z_S - z_C)l_{zS}]B_x(S) + (y_S - y_C)l_{xS}B_y(S) + (z_S - z_C)l_{xS}B_z(S)$$

$$J_y = \ (x_S - x_C)l_{yS}B_x(S) - [(z_S - z_C)l_{zS} + (x_S - x_C)l_{xS}]B_y(S) + (z_S - z_C)l_{yS}B_z(S)$$

$$J_z = \ (x_S - x_C)l_{zS}B_x(S) + (y_S - y_C)l_{zS}B_y(S) - [(x_S - x_C)l_{zS} + (y_S - y_C)l_{yS}]B_z(S)$$

Thus, the calculation of the magnetic torque is obtained by the simple integration where the kernel functions are given in the analytical form over the incomplete elliptic integrals of the first and the second kind.

6.1 *Special cases*

6.1 1. $a = c = 0$

This case is the singular case. The first arc segment lies in the plane $z = 0$ and the second in the plane y = constant. There are two possibilities for this case.

6.1 2. $\vec{u} = \{-1{,}0{,}0\}$, $\vec{v} = \{0{,}0,-1\}$

6.1 3. $\vec{u} = \{0{,}0,-1\}$, $\vec{v} = \{-1{,}0{,}0\}$

These vectors must be used in equations (58), (59) and (60).

7. Mutual inductance calculation between two current-carrying arc segments with inclined axes

The mutual inductance between current-carrying arc segments with inclined axes in air can be calculated by,

$$M = \frac{\mu_0}{4\pi} \int_{\varphi_1}^{\varphi_2} \int_{\varphi_3}^{\varphi_4} \frac{d\vec{l_P} \cdot d\vec{l_S}}{r_{PS}} \tag{61}$$

where $d\vec{l_P}, d\vec{l_S}$ and r_{PS} are previously given.

From, (3), (7) and (61) the mutual inductance can by calculated by,

$$M = \frac{\mu_0 R_P R_S}{4\pi} \int_{\varphi_1}^{\varphi_2} \int_{\varphi_3}^{\varphi_4} \frac{-l_{xs}\sin(t) + l_{ys}\cos(t)}{\sqrt{x_S^2 + y_S^2 + z_S^2 + R_P^2 - 2R_P\sqrt{x_S^2 + y_S^2}\cos(t-\gamma)}} dt d\theta \tag{62}$$

We take the substitution $t - \gamma = \pi - 2\beta$ that leads to final solution for the mutual inductance (See Appendix C).

$$M = \frac{\mu_0 R_S \sqrt{R_P}}{4\pi} \int_{\varphi_3}^{\varphi_4} \frac{V}{kp\sqrt{p}} d\theta \tag{63}$$

where,

$$V = [l_{ys}x_S - l_{xs}\, y_S]\{[(k^2 - 2)F(\beta,k) + 2E(\beta,k)]\} \Big|_{\beta_1}^{\beta_2} - 2\Delta[l_{ys}y_S + l_{xs}\, x_S] \Big|_{\beta_1}^{\beta_2}$$

Thus, the calculation of the mutual inductance is obtained by the simple integration where the kernel functions are given in the analytical form over the incomplete elliptic integrals of the first and the second kind.

7.1 *Special cases*

7.1 1. $a = c = 0$

This case is the singular case. The first arc segment lies in the plane $z = 0$ and the second in the plane y = constant. There are two possibilities for this case.

7.1 2. $\vec{u} = \{-1,0,0\}$, $\vec{v} = \{0,0,-1\}$

7.1 3. $\vec{u} = \{0,0,-1\}$, $\vec{v} = \{-1,0,0\}$

These vectors must be used in equation (63).

All previous electromagnetic quantities are obtained by using the integral approach.

8. Stiffness calculation between two inclined current-carrying arc segments

The stiffness is the extent to which an object resists deformation in response to an applied force. Knowing the magnetic force between two inclined current-carrying arc segments the corresponding stiffness between them can be calculated by the derivate of the corresponding components as follows [],

$$k_{xx} = -\frac{\partial F_x}{\partial x}, \quad k_{xy} = -\frac{\partial F_x}{\partial y}, \quad k_{xz} = -\frac{\partial F_x}{\partial z} \tag{64}$$

$$k_{yy} = -\frac{\partial F_y}{\partial y}, \quad k_{yx} = -\frac{\partial F_y}{\partial x}, \quad k_{yz} = -\frac{\partial F_y}{\partial z} \tag{65}$$

$$k_{zz} = -\frac{\partial F_z}{\partial z}, \quad k_{zx} = -\frac{\partial F_z}{\partial x}, \quad k_{zy} = -\frac{\partial F_z}{\partial y} \tag{66}$$

Thus, the first derivative of the corresponding force components over the corresponding variable leads to the corresponding stiffness. Obviously, it is not easy work because of the complicate kernel functions which are the analytical functions given in the form of incomplete elliptic integrals of the first and the second kind and some elementary functions. Even though it is the tedious work we give only the stiffness k_{zz} from (66) which is the axial stiffness. This developed formula can serve the potential readers to make other stiffness, for example, by Mathematica or MATLAB programming, The calculation of other stiffness will be subject of our future work. In this paper we give the benchmark example for calculating the axial stiffness between two coaxial current circular loops.

The magnetic force between two coaxial circular loops is,

$$F_x = 0 \tag{67}$$

$$F_y = 0 \tag{68}$$

$$F_z = \frac{\mu_0 I_P I_S}{4\sqrt{R_P R_S}} \frac{zk}{1-k^2} \Phi(k) \tag{69}$$

where,

$$k^2 = \frac{4R_P R_S}{[R_P + R_S]^2 + z^2}$$

$$\Phi(k) = 2(1-k^2)K(k) - (2-k^2)E(k)$$

I_P and I_S are the currents in the primary and secondary loop.
R_P and R_S are the corresponding radii of loops.

Obviously, we can find analytically only the stiffness k_{zz} because others are zero.

This stiffness k_{zz} is given by,

$$k_{zz(coaxial\ loops)} = -\frac{\partial F_z}{\partial z} \tag{70}$$

or,

$$k_{zz(coaxial\ loops)} = -\frac{\mu_0 I_P I_S}{4\sqrt{R_P R_S}} T_0 \tag{71}$$

where,

$$\frac{dk}{dz} = \frac{zk^3}{4R_P R_S} \frac{1+k^2}{(1-k^2)^2} \tag{72}$$

$$T_0 = \frac{d}{dz}\left[\frac{zk}{1-k^2}\Phi\{k\}\right] = \frac{k}{1-k^2}\Phi\{k\} + z\frac{d}{dk}\left[\frac{k}{1-k^2}\right]\frac{dk}{dz}\Phi(k) + \frac{zk}{1-k^2}\frac{d\Phi\{k\}}{dk}\frac{dk}{dz} =$$

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$$= \frac{k}{1-k^2}\Phi\{k\} - \frac{z^2k^3}{4R_PR_S}\frac{1+k^2}{(1-k^2)^2}\Phi\{k\} - \frac{z^2k^4}{4R_PR_S}\frac{1}{1-k^2}\frac{d\Phi\{k\}}{dk} \tag{73}$$

From (71), (72) and (73) the axial stiffness k_{zz} is,

$$k_{zz(coaxial\ loops)} = -\frac{\mu_0 I_P I_S}{4\sqrt{R_PR_S}}\frac{k}{1-k^2}\left\{\left[1-\frac{z^2k^2}{4R_PR_S}\frac{1+k^2}{(1-k^2)}\right]\Phi\{k\} - \frac{3z^2k^4}{4R_PR_S}\Psi(k)\right\} \tag{74}$$

where,

$$\Psi(k) = E(k) - K(k)$$

As mentioned before, this formula (74) will serve as the benchmark example to verify the validity of the general expression for the stiffness k_{zz}. In Appendix D we give the complete expressions of this axial stiffness.

Here, we give only final expressions of k_{zz}.

$$k_{zz} = -\frac{\partial F_z}{\partial z_S} = -\frac{\mu_0 I_P I_S R_S}{16\pi\sqrt{R_P}}\int_{\varphi_3}^{\varphi_4}\frac{k}{(1-k^2)\sqrt[2]{p^5}}\left[l_{xS}T_{zz1} - l_{yS}T_{zz2}\right]d\theta \tag{75}$$

where,

$$T_{zz1} = I_{yy} - \frac{z_S^2k^3}{pR_P}\frac{1+k^2}{1-k^2}I_{yy} - \frac{z_S^2k^4}{pR_P}V_y$$

$$T_{zz2} = I_{xx} - \frac{z_S^2k^2}{pR_P}\frac{1+k^2}{1-k^2}I_{xx} - \frac{z_S^2k^4}{pR_P}V_x$$

$$I_{xx} = x_S\,S + 2y_S\,(1-k^2)\left[\frac{1}{\Delta}\right]$$

$$I_{yy} = y_S\,S - 2x_S\,(1-k^2)\left[\frac{1}{\Delta}\right]$$

$$S = (k^2-2)E(\beta,k) + (2-2k^2)F(\beta,k) + k^2(2-k^2)\frac{\sin(2\beta)}{2\Delta}$$

$$V_y = y_S\,b_1 + 2x_S\,b_2$$

$$V_x = x_S\,b_1 - 2y_S\,b_2$$

$$b_1 = 3[E(\beta,k) - F(\beta,k)]\Big|_{\beta_1}^{\beta_2} + \left\{\left[\frac{\sin(2\beta)}{\Delta}\right](1-2k^2) + \left[\frac{\sin(2\beta)sin^2(\beta)}{\Delta^3}\right]\frac{k^2(2-k^2)}{2}\right\}\Big|_{\beta_1}^{\beta_2}$$

$$b_2 = \left[\frac{2}{\Delta}\right]\Big|_{\beta_1}^{\beta_2} - (1-k^2)\left[\frac{sin^2(\beta)}{\Delta^3}\right]\Big|_{\beta_1}^{\beta_2}$$

8.1 ***Special cases***

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8.1 1. $a = c = 0$

This case is the singular case. The first arc segment lies in the plane $z = 0$ and the second in the plane y = constant. There are two possibilities for this case.

8.1 2. $\vec{u} = \{-1,0,0\}$, $\vec{v} = \{0,0,-1\}$

8.1 3. $\vec{u} = \{0,0,-1\}$, $\vec{v} = \{-1,0,0\}$

These vectors must be used in equation ().

9. Numerical validation

To verify the validity of the new formulas we applied the following set of examples. The special cases are discussed. We compared the results of the presented formulas with those known in the literature.

Example 1. Calculate the magnetic vector potential produced by the current-carrying arc segment of the radius $R_P = 3$ at the point S (x_S, y_S, z_S) = S (3, 4, 5). All dimensions are in meters. The current $I_P = 1$ A.

Let us begin with the circular loop for which is $\varphi_1 = 0$ and $\varphi_2 = 2\pi$.

From (16), (17) and (18) we have the components, and the total magnetic vector potential as follows.

$$A_x(S) = 28.61844373019504\, nTm$$
$$A_y(S) = 21.46383279764628\, nTm$$
$$A_z(S) = 0$$
$$A(S) = 35.7730546627438\, nTm$$

From [11] we obtained the same results for the total magnetic vector potential. Obviously, it is the well-known formulas for the current loop.

Let us take $\varphi_1 = \pi/3$ and $\varphi_2 = 5\pi/4$. From (16), (17) and (18) we have,

$$A_x(S) = 60.73902566793771\, nTm$$
$$A_y(S) = -54.76725580732807\, nTm$$
$$A_z(S) = 0$$
$$A(S) = 81.78436004368871\, nTm$$

Thus, this examples for the arbitrary angles may serve as the benchmark example.

Example 2. Calculate the magnetic field produced by the current-carrying arc segment of the radius $R_P = 3$ at the point S (x_S, y_S, z_S) = S (3, 4, 5). All dimensions are in meters. The current $I_P = 1$ A.

Let us begin with the circular loop for which is $\varphi_1 = 0$ and $\varphi_2 = 2\pi$.
From (35), (36) and (37) we have the components, and the total magnetic as follows,

$$B_x(S) = 6.590422756026894\ nT$$
$$B_y(S) = 8.787230341369193\ nT$$
$$B_z(S) = 5.554432293082448\ nT$$
$$B(S) = 12.30856641830695\ nT$$

Even though the magnetic field produced by the circular loop can be considered as axisymmetric so that we need to calculate only the radial and azimuthal component. Applying equations from [] these components as well as the total magnetic field are as follows,

$$B_r(S) = 10.98403792671149\ nT$$
$$B_z(S) = 5.554432293082448\ nT$$
$$B(S) = 12.30856641830695\ nT$$

From the previous calculations the radial component of the magnetic field is,

$$B_r(S) = \sqrt{B_x^2(S) + B_y^2(S)} = 10.98403792671149\ nT$$

Thus, we show the validity of our equations (35), (36) and (37).
From [11] we obtained the same results for the magnetic field. Obviously, it is the well-known formula for the current loop.

Now, let us apply these equations for the same problem but with the different positions of angles, for example, $\varphi_1 = \pi/6$ and $\varphi_2 = 3\pi/4$. We obtain,

$$B_x(S) = 3.204077158320579\ nT$$
$$B_y(S) = 11.48651408884254\ \text{nT}$$
$$B_z(S) = -3.013457271456703\ nT$$
$$B(S) = 12.29987971797063\ nT$$

Thus, this examples for the arbitrary angles may serve as the benchmark example. As we can see the calculations of the magnetic vector potential and the magnetic field of the current-carrying segments with arbitrary angles are obtained in the closed form and expressed by the incomplete elliptic integrals of the first and the second kind. In the case of the circular loops these calculations are the well-known and obtained over the complete elliptical integral of the first and the second kind.

Example 3. Calculate the magnetic force between two arc currying-courant segments whose radii are $R_P = 0.2$ m and $R_S = 0.1$ m, respectively. The first arc segment is placed in the plane XOY and the second in the plane XOY and the second in the plane $x + y + z = 0.3$ with the center C (0.1; 0.1;0.1) which lies in this plane. The currents are unit.

We begin with two inclined circular loops.

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From [39] the components of the magnetic force are as follows,

$$F_{xRen} = -0.10807277\ nN$$
$$F_{yRen} = -0.10807276\ nN$$
$$F_{zRen} = -1.4073547\ nN$$

From [31] the components of the magnetic force are as follows,

$$F_{xPoletkin} = -0.108072965612845\ nN$$
$$F_{yPolrtkin} = -0.108072965612845\ nN$$
$$F_{zPoletkin} = -1.40737206031365\ nN$$

From [23, 24] the components of the magnetic force are as follows,

$$F_x = -0.1080729656128444\ nN$$
$$F_y = -0.1080729656128444\ nN$$
$$F_z = -1.407372060313649\ nN$$

From the work presented in this paper, using (52), (53) and (54) we have,

$$F_x = -0.1080729656128444\ nN$$
$$F_y = -0.1080729656128444\ nN$$
$$F_z = -1.407372060313649\ nN$$

Thus, we confirmed the validity of the approach presented here.

Now, let us apply these equations for the same problem but with the different positions of the segment angles, for example, $\varphi_1 = \varphi_3 = \pi/6$ and $\varphi_2 = \varphi_4 = 3\pi/4$. We obtain,

$$F_x = -137.7416772905457\ nN$$
$$F_y = -6.783844980209707\ nN$$
$$F_z = 32.30984917651751\ nN$$

Example 4. The center of the primary coil of the radius R_P = 0.4 m is O (0; 0; 0) and the center of the secondary coil of the radius R_S = 0.05 m is C (0.1 m; 0.15 m; 0.0 m). The secondary coil is in the plane 3x + 2y + z = 0.6. Calculate the magnetic force between coils. All currents are unit. The angles of segments are respectively $\varphi_1 = 0$, $\varphi_2 = 2\pi$ and $\varphi_3 = 0$, $\varphi_4 = 19\pi/10$, $195\pi/100$, $19999\ \pi/10000$, 2π. Investigate four cases for angle φ_4.

The first coil is the current loop. Using the presented method here we have:
For $\varphi_4 = 19\pi/10$,

$$F_x = -1.030225970922242\ nN$$
$$F_y = -5.151227163000918\ nN$$

$$F_z = 27.14297688555945\ nN$$

For φ4 = 195π/100,

$$F_x = 2.692181753461003\ nN$$
$$F_y = 1.173665675174731 nN$$
$$F_z = 27.52894004960609\ \text{nN}$$

For φ4 = 19999π/10000,

$$F_x = 4.171134702846683\ nN$$
$$F_y = 6.514234771668451\ nN$$
$$F_z = 27.71528704863114\ nN$$

For φ4 = 2π,

$$F_x = 4.171776672650815\ nN$$
$$F_y = 6.523855691357912\ \ nN$$
$$F_z = 27.71549975211961\ nN$$

The last results for φ4 = 2π, are obtained in [23] and [24].

Thus, we show that the presented formula for the magnetic force between two inclined current-carrying segments with arbitrary angels are correct that is proved by the limit case for two inclined circular loops.

Example 5. The center of the primary coil of the radius R_P = 0.3 m is O (0; 0; 0) and the center of the secondary coil of the radius R_S = 0.3 m is C (0.1 m; −0.3 m; 0.2 m). The secondary coil is in the plane x − 2y + z = 0.9. All currents are unit but of the opposite sign. The angles of segments are respectively φ_1 = 0, φ_2 = π, 3π/2, 7π/24, 90π/46, 1999π/1000, 2 π and φ3 = 0, φ4 = π, 3π/2, 7π/24, 90π/46, 1999π/1000, 2 π. Calculate the magnetic force between these current segments.

Using the presented method here we have,

$\varphi_1 = 0$, $\varphi_2 = \pi$, $\varphi_3 = 0$, $\varphi_4 = \pi$

$$F_x = 0.1434856008022091\ \ \mu N$$
$$F_y = -0.1326852649109414\ \ \mu N$$
$$F_z = 0.02679590119992052\ \ \mu N$$

$\varphi_1 = 0$, $\varphi_2 = 3\pi/2$, $\varphi_3 = 0$, $\varphi_4 = 3\pi/2$

$$F_x = 0.125846805955475\ \mu N$$
$$F_y = -0.1412059414594633\ \ \mu N$$
$$F_z = -0.008568308516912724\ \ \mu N$$

$\varphi_1 = 0$, $\varphi_2 = 7\pi/4$, $\varphi_3 = 0$, $\varphi_4 = 7\pi/4$

$$F_x = 0.1971372403346838\ \mu N$$
$$F_y = -0.3359342255993592\ \mu N$$
$$F_z = -0.1029523519216523\ \mu N$$

$\varphi_1 = 0$, $\varphi_2 = 90\pi/46$, $\varphi_3 = 0$, $\varphi_4 = 90\pi/46$

$$F_x = 0.2298232863299166 \mu N$$
$$F_y = -0.5316472767567059\ \mu N$$
$$F_z = -0.094553128442032\ \mu N$$

$\varphi_1 = 0$, $\varphi_2 = 1999\pi/1000$, $\varphi_3 = 0$, $\varphi_4 = 1999\pi/1000$

$$F_x = 0.2292493650352244 \mu N$$
$$F_y = -0.5614719226361647\ \mu N$$
$$F_z = -0.09253078729453428\ \mu N$$

Let us take the limit case of two inclined current loops. This approach gives,

$$F_x = 0.2292455704933025\ \mu N$$
$$F_y = -0.5621415690326643\ \mu N$$
$$F_z = -0.09249247340323912\ \mu N$$

The last results are obtained in [23] and [24].

Thus, when segments lead to the circular loops, we can see the results that converge to those of the circular loops.

Example 6. The center of the primary coil of the radius $R_P = 1$ m is O (0; 0; 0) and the center of the secondary coil of the radius $R_S = 0.5$m is C (2 m; 2 m; 2 m). The secondary coil is in the plane y = 2. Coils are with perpendicular axes radius $R_S = 0.5$m is C (2 m; 2 m; 2 m). The secondary coil is in the plane y = 2. Coils are with perpendicular axes. Calculate the magnetic force between coils. All currents are unit.

This case is the singular case because $a = c = 0$. Let us begin with two perpendicular current loops, [23] and [24] in which we found,

$$F_x = -4.901398177052345\ nN$$
$$F_y = -1.984872313200137\ nN$$
$$F_z = -2.582265710169336\ nN$$

By [39],

$$F_{xRen} = -4.9013835\ nN$$
$$F_{yRen} = -1.9848816\ nN$$

$$F_{zRen} = -2.5821969\ nN$$

By this work and using 5.1.2 [$\vec{u} = \{-1,0,0\}$, $\vec{v} = \{0,0,-1\}$] and (52), (53) and (54) we have,

$$F_x = 4.901398177052345\ nN$$
$$F_y = 1.984872313200137\ nN$$
$$F_z = 2.582265710169336\ nN$$

By this work and using 5.1.3 [$\vec{u} = \{0,0,-1\}$, $\vec{v} = \{-1,0,0\}$] and (52), (53) and (54) we have,

$$F_x = -4.901398177052345\ nN$$
$$F_y = -1.984872313200137\ nN$$
$$F_z = -2.582265710169336\ nN$$

Thus, we obtained for the case 5.1.3 the same results as in [23], [24] and [31]. For the case 5.1.2 we obtain the same results as in [23] and [24] but with opposite signs for each component, because in them we did not take into consideration these unit vectors.

Let us take the case 5.1.3. and $\varphi_1 = \pi/6$, $\varphi_2 = 5\pi/6$, $\varphi_3 = \pi/4$, $\varphi_4 = 5\pi/4$. The approach presented here, gives,

$$F_x = -12.06294047887778\ nN$$
$$F_y = 5.242872781049669\ nN$$
$$F_z = 7.708406091689127\ nN$$

Let us take the case 5.1.3. and $\varphi_1 = \pi/1000$, $\varphi_2 = 1999\pi/1000$, $\varphi_3 = \pi/1000$, $\varphi_4 = 1999\pi/1000$. The approach presented here, gives,

$$F_x = -4.901398087973561\ nN$$
$$F_y = -1.977166719062928\ nN$$
$$F_z = -2.553525470247053\ nN$$

Changing the angles of the segments we approach to limit case (See the first calculation in this example).

Thus, this singular case where the angles are arbitrary, can be used as the benchmark example which in the limit where the current coils are the current loops as proved previously.

Example 7. Calculate the torque between two inclined current-carrying arc segments for which is $R_P = 0.2$ m and $R_S = 0.1$ m. The first arc segment is placed in the plane XOY and the second in the plane $x + y + z = 0.3$ with the center C (0.1; 0.1;0.1) m which lies in this plane. The currents are unit.

Let us begin with two circular loops for which is $\varphi_1 = 0$, $\varphi_2 = 2\pi$, $\varphi_3 = 0$ and $\varphi_4 = 2\pi$.
From [39] the components of the magnetic force are as follows,

$$T_{xRen} = -27.861249\ nN\ m$$
$$T_{yRen} = 27.861249\ nNm$$
$$T_{zRen} = 0$$

From [31] the components of the magnetic force are as follows,

$$T_{xPoletkin} = -27.8620699713\ nNm$$
$$T_{yPolrtkin} = 27.8620699713\ \ nNm$$
$$T_{zPoletkin} = -5.65233285126159 \times 10^{-14}\ nNm \approx 0$$

Using the approach presented in this paper, equations (58), (59) and (60) we have,

$$T_{xThis-Work} = -27.86206997129496\ nNm$$
$$T_{yThis\_work} = 27.86206997129496\ \ nNm$$
$$T_{zThis-Work} = 5.007385868157401 \times 10^{-64}\ nNm \approx 0$$

All results are in an excellent agreement. Thus, we confirmed the validity of the approach presented here.

Let us take $\varphi_1 = \pi/12$, $\varphi_2 = \pi$, $\varphi_3 = 0$ and $\varphi_4 = 2\pi$.

The approach presented here gives,

$$T_{xThis-Work} = -0.4295228631728361\ nNm$$
$$T_{yThis\_work} = 0.3155545746006545\ nNm$$
$$T_{zThis-Work} = 0.1139682885721816\ nNm$$

Example 8. The center of the primary coil of the radius R_P = 0.3 m is O (0; 0; 0) and the center of the secondary coil of the radius R_S = 0.3 m is C (0.1 m; −0.3 m; 0.2 m). The secondary coil is in the plane x − 2y + z = 0.9. Calculate the magnetic force between coils. All currents are unit but of the opposite sign. The angles of segments are respectively $\varphi_1 = \pi/2$, $\varphi_2 = \pi$ and $\varphi_3 = 3\pi/2$, $\varphi_4 = 2\pi$. Calculate the magnetic torque between these current segments.

The approach presented here gives,

$$T_{xThis-Work} = -8.953546268649248\ nNm$$
$$T_{yThis\_work} = -6.014106918957883\ nNm$$
$$T_{zThis-Work} = -3.074667569266484\ nNm$$

Example 9. Let us consider two arc segments of the radii R_P = 40 cm and R_S = 10 cm. The primary arc segment lies in the plane z = 0 and it is centered at O (0;0;0). The secondary arc segment lies in the plane y = 20 cm, with its center located at C (0;20;10) cm. Calculate the torque between two arc segments with $\varphi_1 = 0$, $\varphi_2 = \pi$, $\varphi_3 = 0$, $\varphi_4 = \pi$.

This case is the singular case because $a = c = 0$. Let us begin with two inclined circular loops.

Using 6.1.2 [$\vec{u} = \{-1,0,0\}$, $\vec{v} = \{0,0,-1\}$] and (58), (59) and (60) we have,

$$T_{xThis-Work} = -0.498395165432447\ nNm$$
$$T_{yThis\_work} = 0$$
$$T_{zThis-Work} = 3.696785155039511 \times 10^{-128}\ nNm \approx 0$$

Using 6.1.3 [$\vec{u} = \{0,0,-1\}$, $\vec{v} = \{-1,0,0\}$] and (52), (53) and (54) we have,

$$T_{xThis-Work} = 0.498395165432447\ nNm$$
$$T_{yThis\_work} = 0$$
$$T_{zThis-Work} = -3.696785155039511 \times 10^{-128}\ nNm \approx 0$$

Thus, we obtained with the case 6.1.3 and the case 6.1,2 the same results but with opposite signs for each component. It was explained in the previous examples where the singularities appear.

Let take the case 6.1.2. and $\varphi_1 = 0$, $\varphi_2 = \pi$, $\varphi_3 = 0$, $\varphi_4 = \pi$. The approached here gives,

$$T_{xThis-Work} = -24.91975827162235\ nNm$$
$$T_{yThis\_work} = 0$$
$$T_{zThis-Work} = -0.9803004730404883\ nNm$$

Let take the case 6.1.3. and $\varphi_1 = 0$, $\varphi_2 = \pi$, $\varphi_3 = 0$, $\varphi_4 = \pi$. The approached here gives,

$$T_{xThis-Work} = 24.91975827162235\ nNm$$
$$T_{yThis\_work} = 0$$
$$T_{zThis-Work} = 0.9803004730404883\ nNm$$

These results were expected.

Example 10. The center of the primary coil of the radius $R_P = 1$ m is O (0;0;0) and the center of the secondary coil of the radius $R_S = 0.5$ m is C (2;2;2) m. The secondary coil is in the plane $y = 2$ m that means that the coils are with perpendicular axes. Calculate the magnetic torque between coils for which is $\varphi_1 = 0$, $\varphi_2 = \pi$, $\varphi_3 = \pi$ and $\varphi_4 = 2\pi$. All currents are unit.

This case is the singular case because $a = c = 0$. Let us begin with two perpendicular current loops.

Using 6.1.2 [$\vec{u} = \{-1,0,0\}$, $\vec{v} = \{0,0,-1\}$] and (58), (59) and (60) we have.

$$T_{xThis-Work} = -0.3526562725465321\ nNm$$
$$T_{yThis\_work} = 0$$
$$T_{zThis-Work} = 5.833051727704416\ nNm$$

Using 6.1.3 [$\vec{u} = \{0,0,-1\}$, $\vec{v} = \{-1,0,0\}$] and (52), (53) and (54) we have,

$$T_{xThis-Work} = 0.3526562725465321\ nNm$$
$$T_{yThis\_work} = 0$$
$$T_{zThis-Work} = -5.833051727704416\ nNm$$

Thus, we obtained with the case 6.1.3 and the case 6.1.2 the same results but with opposite signs for each component.

By [39],

$$T_{xRen} = 0.35628169\ nNm$$
$$T_{yRen} = -0.40169339 \times 10^{-6}\ nNm \approx 0$$
$$T_{zRen} = -5.8330053\ nNm$$

All results are in an excellent agreement.

For $\varphi_1 = 0$, $\varphi_2 = \pi$, $\varphi_3 = \pi$ and $\varphi_4 = 2\pi$.

Using 6.1.2 [$\vec{u} = \{-1,0,0\}$, $\vec{v} = \{0,0,-1\}$] and (58), (59) and (60) we have.

$$T_{xThis-Work} = 7.245205284708146\ nNm$$
$$T_{yThis\_work} = 0$$
$$T_{zThis-Work} = 3.767244177134524\ nNm$$

Using 6.1.3 [$\vec{u} = \{0,0,-1\}$, $\vec{v} = \{-1,0,0\}$] and (52), (53) and (54) we have,

$$T_{xThis-Work} = -7.245205284708146\ nNm$$
$$T_{yThis\_work} = 0$$
$$T_{zThis-Work} = -3.767244177134524\ nNm$$

Thus, we obtained with the case 6.1.3 and the case 6.1.2 the same results but with opposite signs for each component that was proved in previous singular cases.

Example 11. Calculate the mutual inductance between two inclined current-carrying arc segments for which is R_P =0.2 m and R_S = 0.1 m. The first arc segment is placed in the plane XOY and the second in the plane $x + y + z = 0.3$ with the center C (0.1; 0.1;0.1) which lies in this plane.
Let us begin with $\varphi_1 = 0$, $\varphi_2 = 2\pi$, $\varphi_3 = 0$ and $\varphi_4 = 2\pi$.
Applying (63) the mutual inductance is,

$$M = 81.31862021231823\ nH$$

We find the same result in [22].

Now, let us change the positions of the arc segments, for example, $\varphi_1 = 0$, $\varphi_2 = \pi/2$, $\varphi_3 = \pi$ and $\varphi_4 = 3\pi/2$. Applying (63) the mutual inductance is,

$$M = 17.38258810896817\ nH$$

Example 12. Let us consider two arc segments of the radii $R_P = 40$ cm and $R_S = 10$ cm. The primary arc segment lies in the plane z = 0 and it is centered at *O* (0;0;0). The secondary arc segment lies in the plane $y = 20$ cm, with its center is located at *C* (0;20;10) cm. Calculate the mutual inductance between two arc segments.

This is the singular case. Let us begin with two circular loops for which is $\varphi_1 = 0$, $\varphi_2 = 2\pi$, $\varphi_3 = 0$ and $\varphi_4 = 2\pi$.

$$M = -10.72715167866112\ nH$$

We find the same result in [22].

For $y = -20$ cm the mutual inductance is,

$$M = 10.72715167866112\ nH$$

For $y = 0$ cm the mutual inductance is

$$M = 0$$

Example 13. Let us consider two arc segments of the radii $R_P = 40$ cm and $R_S = 10$ cm, which are mutually perpendicular each other. The primary arc segment lies in the plane z = 0 and it is centered at *O* (0;0;0) the center of the secondary coil is located at origin, thus *C* = *O* (0;0;0). Calculate the mutual inductance between two arc segments, [4444].

Let us begin with two circular loops for which is $\varphi_1 = 0$, $\varphi_2 = 2\pi$, $\varphi_3 = 0$ and $\varphi_4 = 2\pi$.
Here, we taste tree cases 1) $a = 1$, $b = c = 1$; 2) $a = 0$, $b = 1$, $c = 0$; 3) $a = b = 0$, $c = 1$.
For all cases the mutual inductance [22] gives,

$$M = 0$$

By this work we obtained the same value. It means that for any position of the secondary loop the mutual inductance is zero when the center of the second loop is positioned in the origin *O*. The same results are obtained in [37].

Example 14. Let us consider the previous example, but the center of the secondary coil occupies a position on the XOY surface with the following coordinates $x_C = y_C = 10$ cm and $z_C = 0$, [37]. Calculate the mutual inductance between two arc segments/

Let us begin with two circular loops for which is $\varphi_1 = 0$, $\varphi_2 = 2\pi$, $\varphi_3 = 0$ and $\varphi_4 = 2\pi$.

From this approach the mutual inductance gives

$$M = 1.78729016039874 \times 10^{-134}\ nH \approx 0$$

In [37] the mutual is zero.

Example 15. Calculate the stiffness between two coaxial circular loops for which is R_P = 2 m and R_S = 1m. The axial distance between loops is 1 m.

Obviously, there is only the stiffness k_{zz} and other stiffness are zero because the coaxial loops.

From (74) the stiffness is,

$$k_{zz} = -\frac{\partial F_z}{\partial z} = -0.2064021172440473 \times 10^{-6}\ N/m$$

From developed general formula (75),

$$k_{zz} = -\frac{\partial F_z}{\partial z} = -0.2064021172440473 \times 10^{-6}\ N/m$$

Thus, with the benchmark formula we confirmed the validity of the general formula for k_{zz}.

Let us take $\varphi_1 = 0$, $\varphi_2 = \pi/2$, $\varphi_3 = 3\pi/2$ and $\varphi_4 = 2\pi$.

From general formula (75),

$$k_{zz} = -\frac{\partial F_z}{\partial z} = -0.7609053909589719 \times nN/m$$

Example 16. Calculate the stiffness between two inclined current-carrying arc segments for which is R_P = 0.2 m and R_S = 0.1 m. The first arc segment is placed in the plane XOY and the second in the plane $x + y + z = 0.3$ with the center C (0.1; 0.1;0.1) m which lies in this plane.

Let us begin with two circular loops for which is $\varphi_1 = 0$, $\varphi_2 = 2\pi$, $\varphi_3 = 0$ and $\varphi_4 = 2\pi$.

By this approach (75) we have,

$$k_{zz} = -\frac{\partial F_z}{\partial z} = -57.36862305837861 \times 10^{-6}\ N/m$$

This example could be used as the benchmark example to teste other methods in which the axial stiffness is calculated. Now, let us take $\varphi_1 = \pi/4$, $\varphi_2 = \pi/2$, $\varphi_3 = 3\pi/4$ and $\varphi_4 = 3\pi/2$.

By this approach (75) we have,

$$k_{zz} = -\frac{\partial F_z}{\partial z} = -28.70523534358855 \times 10^{-6}\ N/m$$

Example 17. Let us consider two arc segments of the radii R_P = 40 cm and R_S = 10 cm. The primary arc segment lies in the plane z = 0 and it is centered at O (0.;0;0) m. The secondary arc segment lies in the plane y = 20 cm, with its center is located at C (10;20;10) cm. Calculate the stiffness between two arc segments.

This case is the singular case because $a = c = 0$. Let us begin with two perpendicular current circular loops.

Using 8.1.2 [$\vec{u} = \{-1,0,0\}$, $\vec{v} = \{0,0,-1\}$] and (75) we have.

$$k_{zzThis-Work} = -1.322488731905245\ \mu N/m$$

Using 8.1.3 [$\vec{u} = \{0,0,-1\}$, $\vec{v} = \{-1,0,0\}$] and (75) we have,

$$k_{zzThis-Work} = 1.322488731905245\ \mu N/m$$

Thus, we obtained with the case 8.1.2 and the case 8.1.3 the same results but with opposite signs for each component.
Now, let us take $\varphi_1 = \pi$, $\varphi_2 = 2\pi$, $\varphi_3 = \pi$ and $\varphi_4 = 2\pi$.

Using 8.1.2 [$\vec{u} = \{-1,0,0\}$, $\vec{v} = \{0,0,-1\}$] and (75) we have.

$$k_{zzThis-Work} = 34.21629087092779\ nN/m$$

Using 8.1.3 [$\vec{u} = \{0,0,-1\}$, $\vec{v} = \{-1,0,0\}$] and (75) we have,

$$k_{zzThis-Work} = -34.21629087092779\ nN/m$$

Conclusions

In this paper we give some ameliorated and new formulas for calculating important electromagnetic quantities such as the magnetic vector potential, the magnetic field, the magnetic force, the mutual inductance, and the stiffness between two inclined current-carrying arc segments in air. The angles of arc segments are arbitrary. All formulas are developed in the close form over the incomplete elliptic integrals of the first and the second kind (the magnetic vector potential and the magnetic field) and in the simple integral form whose kernel functions are also given in the close form over the incomplete elliptic integrals of the first and the second kind (the magnetic force, the magnetic torque, the mutual inductance, and the stiffness). As we know these electromagnetic quantities appear for the first time in this form in literature. They are given in very simple form so that the potential readers can easily program them with MATLAB or Mathematica. All formulas are verified by the well-known formulas for inclined current circular loops. The presented formulas for calculating the magnetic force, the magnetic force, the mutual inductance, and the stiffness between the current-carrying arc segments with the arbitrary angles can be used as the benchmark examples for testing other methods.

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Appendix A

$$I_1=\int_{\beta_1}^{\beta_2}\frac{\sin(\gamma-2\beta)}{\Delta}d\beta=\sin(\gamma)\int_{\beta_1}^{\beta_2}\frac{\cos(2\beta)}{\Delta}d\beta-\cos(\gamma)\int_{\beta_1}^{\beta_2}\frac{\sin(2\beta)}{\Delta}d\beta=$$

$$=\frac{y_S}{p}\int_{\beta_1}^{\beta_2}\frac{1-2sin^2(\beta)}{\Delta}d\beta-2\frac{x_S}{p}\int_{\beta_1}^{\beta_2}\frac{\sin(\beta)\cos(\beta)}{\Delta}d\beta$$

From [] and [] we obtain the final expression in the analytical form,

$$I_1=\frac{1}{pk^2}\{y_S[(k^2-2)F(\beta,k)+2E(\beta,k)]+2x_S\Delta\}\Big|_{\beta_1}^{\beta_2}$$

$$I_2=\int_{\beta_1}^{\beta_2}\frac{\cos(\gamma-2\beta)}{\Delta}d\beta=\cos(\gamma)\int_{\beta_1}^{\beta_2}\frac{\cos(2\beta)}{\Delta}d\beta+\sin(\gamma)\int_{\beta_1}^{\beta_2}\frac{\sin(2\beta)}{\Delta}d\beta=$$

$$=\frac{x_S}{p}\int_{\beta_1}^{\beta_2}\frac{1-2sin^2(\beta)}{\Delta}d\beta+2\frac{y_S}{p}\int_{\beta_1}^{\beta_2}\frac{\sin(\beta)\cos(\beta)}{\Delta}d\beta$$

$$I_2=\frac{1}{pk^2}\{x_S[(k^2-2)F(\beta,k)+2E(\beta,k)]-2y_S\Delta\}\Big|_{\beta_1}^{\beta_2}$$

Appendix B

$$I_3=\int_{\beta_1}^{\beta_2}\frac{\cos(\gamma-2\beta)}{r_{PS}^3}d\beta=\left\{\cos(\gamma)\int_{\beta_1}^{\beta_2}\frac{\cos(2\beta)}{r_{TP}^3}d\beta+\sin(\gamma)\int_{\beta_1}^{\beta_2}\frac{\sin(2\beta)}{r_{TP}^3}d\beta\right\}=$$

$$=\frac{x_S}{p}\int_{\beta_1}^{\beta_2}\frac{1-2sin^2(\beta)}{\Delta^3}d\beta+\frac{y_S}{p}\int_{\beta_1}^{\beta_2}\frac{\sin(2\beta)}{\Delta^3}d\beta=\frac{x_s}{pk^2(1-k^2)}\{(k^2-2)E(\beta,k)+$$

$$+(2-2k^2)F(\beta,k)+k^2(2-k^2)\,k^2(2-k^2)\frac{\sin(\beta)\cos(\beta)}{\Delta}\}\Big|_{\beta_1}^{\beta_2}$$

$$I_3=\frac{x_s}{pk^2(1-k^2)}\{(k^2-2)E(\beta,k)+(2-2k^2)F(\beta,k)+$$

$$+k^2(2-k^2)\frac{\sin(\beta)\cos(\beta)}{\Delta}\}\Big|_{\beta_1}^{\beta_2}+\frac{2y_S}{pk^2\Delta}\Big|_{\beta_1}^{\beta_2}$$

$$I_4=\int_{\beta_1}^{\beta_2}\frac{\sin(\gamma-2\beta)}{r_{PS}^3}d\beta\left\{\sin(\gamma)\int_{\beta_1}^{\beta_2}\frac{\cos(2\beta)}{r_{TP}^3}d\beta-\cos(\gamma)\int_{\beta_1}^{\beta_2}\frac{\sin(2\beta)}{r_{TP}^3}d\beta\right\}=$$

$$=\frac{y_s}{pk^2(1-k^2)}\left\{(k^2-2)E(\beta,k)+(2-2k^2)F(\beta,k)+k^2(2-k^2)\frac{\sin(\beta)\cos(\beta)}{\Delta}\right\}\Big|_{\beta_1}^{\beta_2}-$$

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$$-\frac{2x_S}{pk^2\Delta}\Big|_{\beta_1}^{\beta_2}$$

$$I_5=\int_{\beta_1}^{\beta_2}\frac{R_P+\sqrt{x_S^2+y_S^2}\cos(2\beta)}{\Delta^3}d\beta=\int_{\beta_1}^{\beta_2}\frac{R_P+p}{\Delta^3}d\beta-2p\int_{\beta_1}^{\beta_2}\frac{\cos(2\beta)}{\Delta^3}d\beta=$$

$$=\frac{1}{k^2(1-k^2)}\{[k^2(R_p+p)-2p]E(\beta,k)+(2p-2pk^2)F(\beta,k)+$$

$$+k^2(2p-(R_p+p)k^2)\frac{\sin(\beta)\cos(\beta)}{\Delta}\}\Big|_{\beta_1}^{\beta_2}$$

Appendix C

$$I_6=\int_{\varphi_1}^{\varphi_2}\frac{-l_{Sx}\sin(t)+l_{Sy}\cos(t)}{\sqrt{x_S^2+y_S^2+z_S^2+R_P^2-2R_P\sqrt{x_S^2+y_S^2}\cos(t-\gamma)}}dt$$

The substitution $t-\gamma=\pi-2\beta$ gives,

$$I_6=-\frac{2l_{xS}}{\sqrt{\left(R_P+\sqrt{x_S^2+y_S^2}\right)^2+z_S^2}}\left\{\sin(\gamma)\int_{\beta_1}^{\beta_2}\frac{\cos(2\beta)}{\Delta}d\beta-\cos(\gamma)\int_{\beta_1}^{\beta_2}\frac{\sin(2\beta)}{\Delta}d\beta\right\}+$$

$$+\frac{2l_{yS}}{\sqrt{\left(R_P+\sqrt{x_S^2+y_S^2}\right)^2+z_S^2}}\left\{\cos(\gamma)\int_{\beta_1}^{\beta_2}\frac{\cos(2\beta)}{\Delta}d\beta+\sin(\gamma)\int_{\beta_1}^{\beta_2}\frac{\sin(2\beta)}{\Delta}d\beta\right\}=$$

$$=-\frac{l_{xS}k}{p\sqrt{R_Pp}}\left\{y_S\int_{\beta_1}^{\beta_2}\frac{1-2sin^2(\beta)}{\Delta}d\beta-2x_S\int_{\beta_1}^{\beta_2}\frac{\sin(\beta)\cos(\beta)}{\Delta}d\beta\right\}+$$

$$+\frac{l_{yS}k}{p\sqrt{R_Pp}}\left\{x_S\int_{\beta_1}^{\beta_2}\frac{1-2sin^2(\beta)}{\Delta}d\beta+2y_S\int_{\beta_1}^{\beta_2}\frac{\sin(\beta)\cos(\beta)}{\Delta}d\beta\right\}=$$

$$=\frac{l_{Sy}}{kp\sqrt{R_Pp}}[l_{ys}x_S-l_{xs}\,y_S]\{[(k^2-2)F(\beta,k)+2E(\beta,k)]\}\Big|_{\beta_1}^{\beta_2}-2\Delta[l_{ys}y_S+l_{xs}\,x_S]\Big|_{\beta_1}^{\beta_2}$$

Appendix D

$$k_{zz}=-\frac{\partial F_z}{\partial z_S}=-I_SR_S\frac{\partial}{\partial z_S}\int_{\varphi_3}^{\varphi_4}[l_{xS}B_y(S)-l_{yS}B_x(S)]\,d\theta=$$

$$=-I_SR_S\int_{\varphi_3}^{\varphi_4}\left[\frac{\partial l_{xS}}{\partial z_S}B_y(S)+l_{xS}\frac{\partial B_y(S)}{\partial z_S}-\frac{\partial l_{yS}}{\partial z_S}B_x(S)-l_{yS}\frac{\partial B_x(S)}{\partial z_S}\right]d\theta=$$

$$=-I_SR_S\int_{\varphi_3}^{\varphi_4}\left[l_{xS}\frac{\partial B_y(S)}{\partial z_S}-l_{yS}\frac{\partial B_x(S)}{\partial z_S}\right]d\theta=\left[\text{because }\frac{\partial l_{xS}}{\partial z_S}=\frac{\partial l_{yS}}{\partial z_S}=0\right]=$$

$$=-\frac{\mu_0I_PI_SR_S}{16\pi\sqrt{R_P}}\int_{\varphi_3}^{\varphi_4}\left[l_{xS}\frac{\partial\left(\frac{z_Sk}{(1-k^2)\sqrt[2]{p^5}}I_{yy}\right)}{\partial z_S}-l_{yS}\frac{\partial\left(\frac{z_Sk}{(1-k^2)\sqrt[2]{p^5}}I_{xx}\right)}{\partial z_S}\right]d\theta=$$

$$=-\frac{\mu_0I_PI_SR_S}{16\pi\sqrt{R_P}}\int_{\varphi_3}^{\varphi_4}\frac{k}{(1-k^2)\sqrt[2]{p^5}}[l_{xS}T_{zz1}-l_{yS}T_{zz2}]\,d\theta$$

$$T_{z1}=\frac{\partial}{\partial z_S}\left[\frac{z_Sk}{(1-k^2)}I_{yy}\right]=\frac{k}{(1-k^2)}I_{yy}+\frac{\partial}{\partial k}\left[\frac{k}{(1-k^2)}\right]\frac{\partial k}{\partial z_S}I_{yy}+\frac{z_Sk}{(1-k^2)}\frac{\partial I_{yy}}{\partial z_S}$$

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$$\frac{\partial k}{\partial z_S} = -\frac{z_S k^3}{4pR_P}$$

$$T_{z1} = \frac{k}{(1-k^2)} T_{zz1}$$

$$T_{zz1} = c_1 I_{yy} + z_S \frac{\partial (I_{yy})}{\partial z_S}$$

$$T_{zz2} = c_1 I_{xx} + z_S \frac{\partial (I_{xx})}{\partial z_S}$$

I_{xx} and I_{yy} are given in () and ().

$$c_1 = 1 - \frac{z_S^2 k^2}{4pR_P} \frac{1+k^2}{1-k^2}$$

$$\frac{\partial (I_{yy})}{\partial z_S} = \frac{\partial}{\partial z_S}\{y_S(A_2 - A_1) - 2x_S(1-k^2)(\Delta_2^{-1} - \Delta_1^{-1})\} = y_S \frac{\partial}{\partial z_S}(A_2 - A_1) -$$

$$-\frac{x_S z_S}{pR_P} k^4 (\Delta_2^{-1} - \Delta_1^{-1}) + \frac{x_S z_S}{2pR_P} k^4 (1-k^2)[sin^2(\beta_2)\Delta_2^{-3} - sin^2(\beta_1)\Delta_1^{-3}]$$

$$\frac{\partial A_2}{\partial z_S} = 2kE(\beta_2, k)\frac{\partial k}{\partial z_S} + (k^2 - 2)\frac{dE(\beta_2,k)}{dk}\frac{\partial k}{\partial z_S} + (2 - 2k^2)F(\beta_2, k)\frac{\partial k}{\partial z_S} -$$

$$-4k\frac{dF(\beta_2,k)}{dk}\frac{\partial k}{\partial z_S} + \frac{\sin(2\beta_2)}{2}\frac{\partial k}{\partial z_S}\left[4k(1-k^2)\frac{1}{\Delta_2} + k^3(2-k^2)\frac{sin^2(\beta_2)}{\Delta_2^3}\right] =$$

$$= \frac{\partial k}{\partial z_S}\left\{2k[E(\beta_2, k) - 2F(\beta_2, k)] + (k^2 - 2)\frac{dE(\beta_2,k)}{dk} + (2 - 2k^2)\frac{dF(\beta_2,k)}{dk} + \right.$$

$$\left. + \frac{k\sin(2\beta_2)}{2\Delta_2}\left[4(1-k^2) + k^2(2-k^2)\frac{sin^2(\beta_2)}{\Delta_2^2}\right]\right\}$$

$$\frac{\partial (I_{yy})}{\partial z_S} = y_S\, b_1 + 2x_S\, b_2$$

$$b_1 = 3[E(\beta, k) - F(\beta, k)]\Big|_{\beta_1}^{\beta_2} + \left\{\left[\frac{\sin(2\beta)}{\Delta}\right](1-2k^2) + \left[\frac{\sin(2\beta)sin^2(\beta)}{\Delta^3}\right]\frac{k^2(2-k^2)}{2}\right\}\Big|_{\beta_1}^{\beta_2}$$

$$T_{zz1} = I_{yy=} + z_S \frac{\partial (I_{xx})}{\partial z_S} T_{zz1} = I_{yy} - \frac{z_S^2 k^3}{pR_P}\frac{1+k^2}{1-k^2} I_{yy} - \frac{z_S^2 k^4}{pR_P}\frac{\partial (I_{yy})}{\partial z_S}$$

Similarly,

$$T_{zz2} = I_{xx} - \frac{z_S^2 k^3}{pR_P}\frac{1+k^2}{1-k^2} I_{xx} - \frac{z_S^2 k^4}{pR_P}\frac{\partial (I_{xx})}{\partial z_S}$$

$$\frac{\partial (I_{xx})}{\partial z_S} = x_S\, b_1 - 2y_S b_2$$

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