Galilei Group, Quantum Mechanics, Neutron-Proton Majorana Interaction, and Dark Matter

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Abstract

The Galilei symmetry demands a Bargmann Superselection rule which shows that in nonrelativistic quantum mechanics, it is impossible to have states which are linear superposition of states describing particles of different masses. We show that these kind of masses can not be the masses of particles as obtained in the nonrelativistic limiting case of the Lorentz transformation with $v \ll c$, and in which the nonrelativistc and relativistic quantum mechanical structures are related to each other. In addition, the nonrelativistic quantum mechanical structure of the Bargmann-Superselection-rule kind, has no relativistic counterpart at all. These mathematical conclusions are consolidated by the fact that there exists a Majorana interaction between each neutron-proton pair in nuclei. This Majorana interaction, for the Galilei symmetry to hold, demands that these masses be equal $M_n = M_p$. Hence there do exist two independent and simultaneos nonrelativistc quantum mechanical strucutres in nature. The Majorana interaction besides ignoring spin, neglects all the three forces - the strong, the electromagnetic and the weak. Having only space exchange allowed in it, it is a pure quantum mechanical force. Thus the mass here is pure gravitational, and which is immune to the other three forces. This makes an amazing connection between the gravitational force and quantum mechanics. This pure gravitational mass would manifest itself as the Dark Matter of the universe.

PACS: 03.65.-w ; 03.65.Ta ; 02.20.Qs ; 95.35.+d ; 21.45.Bc ; 21.60.Fw
Keywords: Quantum mechanics, Galilei group, exchange forces, Bargmann Superselection rule, neutron-proton Majorana interaction, Dark Matter

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1. Introduction :

It may not be an exaggeration to say that the Galilei group could be the most perplexing group in physics. After the inital pioneering work by Wigner on the representation theory of the Poincare group [1], it took a while before in 1950's and 1960's, the Galilei group became centre of some activity [2], [3], [4], [5], [6], [7], [8], [9], [10], [11]. In quantum mechanics one deals with the unitary projective representations of the relevant group. Bargmann [3] showed that in most of the cases of interest, like the rotation, the Lorentz, and the Poincare groups, the study of unitary projective representations of the group is reducible to the study of the true unitary representations of the corresponding universal covering group. However the Galilei group behaves very differently. Amazingly, the physical representations of the Galilei group are nontrivial projective representations [3], the true (vector) representations turn out to be devoid of any physical meaning, and this because these do not allow for a sensible notion of localizability [2]. It has also been shown that one can construct a position operator only in the case of nontrivial projective representations [4], [5]. A Bargmann's Superselection rule comes into play, which shows that, the intrinsic role that the mass plays in the relativistic and the nonrelativistic quantum mechanics, is fundamentally different in the two cases [3], [6], [7], [8], [10]. In this paper, we look into these puzzles, and make connection with the actual physical reality.

2. Galilei Group and Quantum Mechanics :

The symmetries of spacetime include rotations, displacements, and transformations between uniformly moving frames of reference, and which at relativistic velocities coresponds to the Lorentz transformations and for velocities small compared to the speed of light, it gives the Galilei transformations. For the last set of all such transformations, we call it the Galilei group. This is focus of our study here. The effect of such transformation is

$$\begin{aligned} \vec{x} &\rightarrow \vec{x'} = R\,\vec{x} + \vec{v}\,t + \vec{a} \\ t &\rightarrow t' = t + b \end{aligned} \tag{1}$$

where $\vec{R}$ is a rotation represented by a 3×3 matrix acting on a three-component vector $\vec{x}$, $\vec{a}$ is a space displacement, $\vec{v}$ is the velocity of a moving coordinate transformation (i.e. the pure Galilei transformation, which is sometimes also called acceleration), and b is a time displacement. Let us specify the general element of the group by [8],

$$G = (R \ , \ \vec{v} \ , \ \vec{a} \ , \ b) \tag{2}$$

Let the time-dependent ϕ function $\Phi(\vec{x}, t)$ of a free particle of mass m, satisfy the Schroedinger equation (with $\hbar = 1$),

$$i \ \frac{\partial}{\partial t}\Phi \ - \ \frac{1}{2m} \ \nabla^2\Phi \ = \ 0 \qquad (\ \nabla = \frac{\partial}{\partial \vec{x}} \ ) \tag{3}$$

In quantum mechanics the Galilei transformation eqn. (2), is represented by a unitary operator $U(G)$. Then the transformed state is given upto a phase factor as $|\vec{x}(t)>'= U(G)|\vec{x}(t)>$. The state in the transformed frame is characterized by a ϕ function $\Phi'(\vec{x}, t)$ which differs from the untransformed ϕ function at most by a phase factor taken at the transformed point [8],

$$\Phi'(\vec{x}, t) \ = \ e^{if(\vec{x}', t')} \ \Phi(\vec{x}', t') \tag{4}$$

Invariance under Galileo transformations means Φ and Φ' must satisfy the same Schroedinger equation. Thus

$$i \ \frac{\partial}{\partial t}\Phi' \ - \ \frac{1}{2m} \ \nabla^2\Phi' \ = \ 0 \tag{5}$$

Constraints on the phase function f arise from the above equation. Due to orthogonality on R and the relations arising in eqn. (2)

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t'} + \vec{v} \cdot \nabla' \ ; \quad \nabla = R\nabla' \ \ thus \ \nabla^2 = \nabla'^2 \tag{6}$$

Substituting eq. (4) in eqn. (5),

$$[-\frac{\partial f}{\partial t'} - \vec{v} \cdot \nabla' f + \frac{1}{2m}(\nabla' f)^2 - \frac{i}{2m}\nabla'^2 f]e^{if}\phi + i[\vec{v} - \frac{1}{m}\nabla' f]e^{if}\nabla'\phi]$$

$$+[i \ \frac{\partial}{\partial t}\Phi \ - \ \frac{1}{2m} \ \nabla^2\Phi]e^{if} = 0 \tag{7}$$

The last term above, vanishes due to eqn. 3. Given that $\phi \ and \ \nabla'\phi$ are independent, one gets two constraints,

$$\nabla' f = m\vec{v} \tag{8}$$

$$\frac{\partial f}{\partial t'} = -\vec{v} \cdot \nabla' f + \frac{1}{2m}(\nabla' f)^2 - \frac{i}{2m}\nabla'^2 f \tag{9}$$

This leads to,

$$\frac{\partial f}{\partial t'} = -\frac{1}{2}mv^2 \tag{10}$$

On integrating eqn. (8) and eqn. (10), we get for the phase factor f,

$$f(\vec{x}', t') = m\vec{v} \cdot \vec{x}' - \frac{1}{2}m\vec{v}^2 t' + C \tag{11}$$

where C is a constant. Most important, we see that here the phase factor f in general cannot be eliminated by some judicious choice of the integration constant C [8].

3. Bargmann Superselection Rule :

The fact that one cannot eliminate the phase factor in eqn. (11) leads to a basic and profound conclusion arrived at by Bargmann [3] that, "It is impossible to have in nonrelativistic quantum mechanics states which are linear superpositions of states describing particles of different masses. This means one cannot grasp in nonrelativistic quantum mechanics states with a mass spectrum, or states describing unstable elementary particles." [8]

To see this, let us consider a linear superposition of two states with different masses m_1 *and* m_2,

$$\phi \;=\; \phi_{m_1} \;+\; \phi_{m_2} \tag{12}$$

and ϕ_1 *and* ϕ_2 transforming according to eqn. (4) (taking C=0) as,

$$\phi'_\alpha(\vec{x}, t) = e^{im_\alpha(\vec{v} \cdot \vec{x}' - \frac{1}{2}\vec{v}^2 t')}\phi_\alpha(\vec{x}', t') \; ; \quad (\alpha = 1, 2) \tag{13}$$

Now we perform the following sequence of transformations: first a translation $\vec{a}$ next a pure Galilei transformation $\vec{v}$, followed by the inverse translation, and lastly the inverse Galilei transformation. These lead to an identity transformation as,

$$\begin{aligned} U \;&=\; U_4\, U_3\, U_2\, U_1 \\ &= (1, -\vec{v}, 0, 0)(1, 0, -\vec{a}, 0)(1, \vec{v}, 0, 0)(1, 0, , \vec{a}, 0) = (1, 0, 0, 0) \end{aligned} \tag{14}$$

The transformed state as per the above operations is [8],

$$\phi_T = \; U\phi \;=\; \quad e^{-m_1 \vec{a} \cdot \vec{v}}\; \phi_{m_1} \;+\; e^{-m_2 \vec{a} \cdot \vec{v}}\; \phi_{m_2} \tag{15}$$

Therefore the above transformation corresponding to the identity, can affect the norm of the superposition in eqn. (15). Thus as per the demand of Galileo invariance, the relative phase of the two states of particles of different masses, is completely arbitrary. To avoid inconsistency one is forced to conclude that a superposition of the type given in eqn. (15), is meaningless, and that there can exist no operators which enforce transitions between states characterized by different masses m_1 *and* m_2. This amounts to existence of the Bargmann superselection rule, which guarantees the strict conservation of mass in this nonrelativistic quantum mechanics. Note that this conclusion is not valid in relativistic quantum mechanics. Levy-Leblond thus points out an important and basic distinction ([6], p.785), "We see here how the mass plays different parts in relativistic and nonrelativistic quantum theories".

Note that in the derivation of the above Bargmann Superselection rule, the spin degree of freedom has been ignored. However one may add spin as an extra interenal degree of freeedom in this nonrelativistic formalism and still the Bargmann Superselection rule continues to hold [6], [7]. However we focus here the no-spin case. Thus the point is that the Bargmann Superselection rule does hold good even with no spin.

Note that if two different particles have the same mass, then these two should be identical particles like a proton-proton pair or a neutron-neutron pair. However if these are different like neutron-proton pair, then their masses are necessarily different. This is true of all fermionic pairs. However for the case of bosons we know that two different particles π^+ *and* π^-, do have the same mass of magnitude 140 $\frac{MeV}{c^2}$. Hence these two may be treated as identical particles. This suggests that if in any scenario proton and neutron may be treated as bosons, then these should have the same mass and be treated as identical particles. Note this discussion of identity of particles, would be relevant for the Bargmann Superselection rule in quantum mechanics.

However the identity of π^+ *and* π^- with the same mass, at present is understood as a manifestation of the nuclear $SU(2)_I$ isospin, a new internal degree of freedom proposed to underatand such identical particles. These two pions correspond to a representation of isospin I=1, with $I_3 = +1 \ , \ and \ -1$ respectively. However, the $T_3 = 0$ component of pion π^0, has a smaller mass of 135 $\frac{MeV}{c^2}$. This difference in mass is explained by the fact that the $SU(2)_I$ isospin symmetry is slightly broken by the electromagnetic interaction [15]. Consistency is ensured, as the extra contribution from Coulomb interaction for charged particles, makes it actually heavier than the uncharged one. Thus isospin symmetry breaking due to Coulomb interaction works fine for the case of the pion masses.

It appears that the $SU(2)_I$ isospin symmetry holds reasonably well also for the neutron-proton pair, due to the fact that mass difference between them in units of neutron mass $\frac{m_n - m_p}{m_n} \sim \frac{939.57-938.28}{939.57} \sim 0.00138$, is pretty small. The small difference between the masses of neutron and proton may be assumed to be due to the electromagnetic interaction. But the fact that charged proton is lighter than the uncharged neutron is disconcerting. One may transfer the mass issue to the level of quarks. But still even today, the issue is not resolved. Gasser, Leutwyler and Rusetsky in a recent paper entitled, "On the mass difference between proton and neutron" [12], are still struggling to resolve this issue. One is hoping that in future lattice QCD calculations may yield more conclusive results [12]. Hence the proton-neutron mass difference issue, is very much still an open problem in particle physics.

Our work here points to a novel resolution of this problem. As per above discussion, if in any scenario proton and neutron may be treated as bosons (i.e. the spin is altogether ignored), then these could have the same mass, a la Bargmann Superselection rule in quantum mechanics. Below we shall show that indeed, in a well known framework - that of Majorana interaction between a neutron-proton pair in a nucleus, this property of spinlessness is prominently there [13], [14]. To understand this, let us first pin down the basis of our present understanding of the nuclear phenomenon.

4. Generalized Pauli Exclusion Principle:

At present the most successful model of particle physics is the quantum field theoretical Standard Model with group theoretical structure $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ [15]. The masses of the matter particles as arising in the Standard Model belong to the corresponding representation of the Poincare group. Next is the successful $SU(3)_F$, the three flavour quark model at low energies. This contains the $SU(2)_I$ isospin structure with proton and neutron forming its fundamental representation. This leads to isospin group forming the basis of our successful model of the nucleus, in terms of the shell structure obtained in the so called Independent Particle Model (IPM) [15]. This low energy quantum mechanical model of nucleus has the group structure $SO(3)_l \otimes SU(2)_I \otimes SU(2)_S$. The Pauli Principle, invoked for spin 1/2 objects like electron, in this model, becomes the Generalised Pauli Exclusion

Principle (GPEP), with exchanges of the coordinates of any pair of particles,

$$\mathbb{P}^M \, \mathbb{P}^\sigma \, \mathbb{P}^\tau = -1 \tag{16}$$

with space-exchange Majorana, spin-exchange Blatt, and isospsin-exchange Heisenberg operators respectively.

One knows that light nuclei are composed of equal number of neutrons and protons. This appears to means that the nuclear force is acting only between neutrons and protons, with no nuclear force acting between two protons and two neutrons. Many in the early days of nuclear physics, accepted this so called "drastic" conclusion. However, the nucleon-nucleon scattering data showed that all, proton-proton, neutron-neutron and neutron-proton forces were actually of equal strength [15]. This latter view also allows one to bypass the above "drastic" conclusion. This is made possible in terms of the IPM, the most successful model of nuclear physics. In the IPM view, any one particle may occupy one of the levels which are the elementary states in the common central potential. Given isospin and spin, each level may be occupied by four particles. Schematically, below the states in the nuclei ${}^{16}_{8}\mathrm{O}_8$ *and* ${}^{16}_{7}\mathrm{N}_9$ are,

$$\begin{array}{cc}
- - - - - - & -n^\uparrow - - - - \\
- \; n^\uparrow \; n^\downarrow \; p^\uparrow \; p^\downarrow \; - & -n^\uparrow \; n^\downarrow \; p^\uparrow - - \\
-n^\uparrow \; n^\downarrow \; p^\uparrow \; p^\downarrow \; - & -n^\uparrow \; n^\downarrow \; p^\uparrow \; p^\downarrow - \\
-n^\uparrow \; n^\downarrow \; p^\uparrow \; p^\downarrow \; - & -n^\uparrow \; n^\downarrow \; p^\uparrow \; p^\downarrow - \\
-n^\uparrow \; n^\downarrow \; p^\uparrow \; p^\downarrow \; - & -n^\uparrow \; n^\downarrow \; p^\uparrow \; p^\downarrow - \\
{}^{16}_{8}\mathrm{O}_8 & {}^{16}_{7}\mathrm{N}_9
\end{array} \tag{17}$$

Because of the GPEP, one of the neutrons in ${}^{16}_{7}\mathrm{N}_9$ is forced into a higher level, thus making the wave function to become less symmetric, making ${}^{16}_{8}\mathrm{O}_8$ as the more stable nucleus. In addition the main argument comes from the exchange character of the nuclear forces as in eqn. (16). Next we focus on the Majorana space exchange between a pair of particles as used in eqn. (16). What does Majorana space exchange really mean in quantum mechanics?

Let $\psi(q_i)$ define the spatial coordinates of a pair of particles. Write the symmetric and antisymmetric spatial wave function as,

$$\psi_S(q_i)/\psi_A(q_i) = \frac{1}{\sqrt{2}}[\phi_a(1)\phi_b(2) \pm \phi_a(2)\phi_b(1))] \tag{18}$$

where $\phi_m(i)$ is normalized space wave function of particle i. Let us consider the expectation value of the square of separation between the two particles,

$$< (\vec{r_2} - \vec{r_1})^2 >=< r^2 >_a + < r^2 >_b -2 < \vec{r} >_a \cdot < \vec{r} >_b \mp 2 \, |< a \mid \vec{r} \mid b >|^2 \quad (19)$$

$$< r^2 >_i= \int \phi_i{}^* r^2 \phi_i \; dxdydz \; ; \;\; < \vec{r} >_i= \int \phi_i{}^* \vec{r} \phi_i \; dxdydz \; ;$$

$$< a \mid r \mid b >= \int \phi_a{}^* \vec{r} \phi_b \; dxdydz \; ; \quad (20)$$

Given a distinguishable pair of particles, the wave function is $\psi(1,2) = \phi_a(1)\phi_b(2)$; and this gives only the first three terms of eqn. (19). The last term of the square of the separation arises for the demand of the state $\psi(1,2)$ to be either symmetric or antisymmetric with respect to the exchange of particles.

Therefore, in eqn. (19), we see that if the spatial wave function of the two particles is symmetric, the particles would attract each other, as the saparation of the two particles would be smaller than what it would be if the particles were distinguishable. In the same manner, the two particles would repel each othrer, if their spatial wave function was antisymmetric.

Thus exchange forces are attractive between particles in the same level, and less attractive or even repulsive between particles in different levels. This explains the more stable nature of ${}^{16}_{8}O_8$ compared to that of ${}^{16}_{7}N_9$ in eqn. (17) above. This simple picture ultimately has finally led to the presently successful IPM of the nucleus. With appropriate interactions, this model, explains practically "all" of nuclear physics. Hence it is a common feeling that except for minor phenomenological refinenements here and there, theoretical structure of nuclear physics is as complete as that of atomic physics and molecular physics.

However one important issue is that of saturation of nuclear forces. The nucleus is defined in the above IPM in terms of various potentials of quite different kinds. However what is common in practically all of them, is that to ensure saturation of the nuclear force (i.e. for large A, $\frac{BindingEnergy}{A} \propto\sim 8MeV$) all these have a short range repulsion (or a hard core) of size $\sim 0.5fm$. Although short range ($\sim 1fm$) attractive potential, and the GPEP play significant roles in the overall understanding of this saturation phenomenon, existence of an ad hoc hard core, may well be considered as the fundamantal defining characteristic of the isospin based IPM. However, let us point out right away that this property of saturation due to a hard core in a nucleus, has nothing to do with the corresponding saturation phenomenon as manifested in atomic and molecular physics, which is actually quantum mechanical in nature.

After the discovery of neutron in 1932, Heisenberg suggested [17] the concept of isospin. But his model could not explain the strong binding of the α particle, and for saturation he suggested the above kind of hard core in his interaction [13], [14], and thus his paper may be considered as the very first paper of the IPM kind and hence acted as precursor of the present successful nuclear models like the IPM etc. Note that due to the basic significance of the $SU(2)_I$ isospin group, the necleon-nucleon force is charge independent, and thus the neutron-neutron, proton-proton and neutron-proton forces are all equal to each other, and succces of the IPM kind of models confirms the validity of this assumption.

And therefore, the above discussed "drastic" model assumption of neutron-proton forces being strong and with the proton-proton and neutron-neutron forces put to zero, has practically been forgotten. Here however, we shall show that in the form of a Majorana interaction, it plays a very significant and basic role.

5. Majorana Neutron-Proton Interaction:

Majorana in 1933, gave a model opposite to that of Heisenberg's model. In fact he actually took a pure neutron-proton interaction model, with neutron-neutron and proton-proton interaction terms put to zero, exactly as per the demands of the "drastic" model above. As such he discarded the isospin symmetry. In fact Majorana was "gleeful" in avoiding these "troublesome ρ-coordinates" [18] (note: ρ was what Heisenberg had used for isospin). Not only that, he ignored the spin degree of freedom also in his interaction between neutron and proton [13], [14] . Thus both the spin and the isospin exchange terms in the GPEP in eqn. (16) are absent and what is left is just the space exchange. He also ignored the electromagnetic interaction [18], [13]. Majorana main motivation was to explain the reason for the strong binding of α particle, which Heisenberg's isospin model was unable to provide; and also not to have an ad hoc short range hard core, which was also demanded by Heisenberg [17], [13].

Now define a new Majorana pure space exchange interaction [18],

$$V = -\sum_{i>j} J(r_{ij})\,\mathbb{P}_{ij} \tag{21}$$

where i and j label the protons and neutrons respectively. Here $\mathbb{P}_{ij}$ is the Majorana exchange operator (called $\mathbb{P}^M$ in eqn. (16)), which interchanges the space coordinates x_1^i, x_2^i, x_3^i of proton i with space coordinates $\xi_1^i, \xi_2^i, \xi_3^i$ of neutron j; and r_{ij} is the distance between i and j.

The Majorana interaction in nuclear physics has always been somewhat "puzzling", as stated by Sachs ([16], p. 60,61), "Although a potential of this form has no meaning in classical mechanics, it causes no difficulty in quantum mechanics, since the interaction term in the Schroedinger equation contains the product $V\psi$". From the well known example of the helium atom, we know that the exchange force has real physical implications and is of pure quantum mechanical nature. Interestingly here, the Majorana interaction too, seems to have pure quantum mechanics intrinsically built into it, and which has no classical analogue.

As per eqn. (19), the ground state for this interaction would favour the orbital symmetric state. Hence each neutron-proton pair interaction is attractive in symmetrical state. In addition to ignoring isospin and spin, Majorana also ignored Coulomb repulsion [13], [14], [18]. If one used Heisenberg's total exchange operator in eqn. (16), then it can be shown that in alpha there are only two attractive bonds - thus alpha in this model is not bound too strongly. But with eqn. (21) Majorana-interaction, it can be shown that each proton in the alpha particle, interacts with both the neutrons, instead of only one and the same vice versa. And thus there are four attractive bonds compared to only one in deuteron (see Appendix for details). This explains the very strong binding per nucleon of 7.07 MeV in alpha, and thus explaining the cause of its saturation. When going to heavier nuclei, the binding energy per nucleon in heavier nuclei would not be considerably bigger than that of the alpha particle. All alpha-clusters in a nucleus, would sit in contact with each other, "with the same properties of size and impenetrability as macroscopic matter" [18]. What does it mean for the phenomenon of saturation?

Now as stated above, the property of saturation in IPM is due to a hard core, and which has nothing to do with the corresponding saturation phenomena manifested in atomic and molecular physics, which is pure quantum mechanical in nature. We discuss this pure quantum mechanical saturation phenomenon now. For example in a drop of liquid hydrogen, one finds a strong homopolar binding between pairs of hydrogen atoms. This leads to formation of H_2 molecules. One finds that there is no substantial attraction for a third hydrogen atom. One therefore says that the hydrogen molecule is saturated ([19],p.300). The total binding energy of the drop is approximately equal to the total energies of the individual hydrogen atom pairs. Thus it is equal to the total number of atoms in the liquid. The forces between the molecules may lead to a slight increase in the total energy. The homopolar binding is mathematically represented as that of the exchange forces. Physically this corresponds to the exchanging of electrons from one atom with the atom in a molecule in a continued manner ([19],p.300).

Above we have already discussed the pure quantum mechanical nature of the Majorana interaction. Now in our Majorana-interaction model, deuteron is bound due to a single bond between the neutron-proton pair. Akin to the homopolar bonds in hydrogen pairs above, we have now four strong n-p bonds in alpha particle and which leads to it becoming a very strongly saturated nucleus. The saturation character of alpha is confirmed by the fact that nuclei with A=5 and 8 are nonexistent. The fully saturated alpha refuses to accept additional proton and neutron so that ${}^{5}_{2}\text{He}_3$ and ${}^{5}_{3}\text{Li}_2$ have no bound levels. Even ${}^{8}_{4}\text{Be}_4$ has insufficient energy to form a stable nucleus. One has to go as high as ${}^{12}_{6}\text{C}_6$ nucleus to be able to have the next stable even-even nucleus. This provides a strong empirical evidence, that given Majorana exchange interaction as in eqn. (21), one gets a saturated alpha, and the nature of this saturation is of quantum mechanical exchange character as in the atoms and the molecules ([19],p.300).

Thus this model is completely independent of saturation as per the IPM above, and which in itself is not rich enough to give a complete description of the nucleus. Clearly these two models are needed simultaneously to be able to provide a more complete picture of the nucleus. This point is obvious by the fact that as per IPM, ${}^{17}_{8}\text{O}_9$ is a good bound state of a single neutron sitting on top of a doubly magic nucleus ${}^{16}_{8}\text{O}_8$. Thus the same IPM will predict that the lighter nucleus ${}^{5}_{2}\text{He}_3$ should be a bound state of a single neutron sitting on top of a doubly magic nucleus ${}^{4}_{2}\text{He}_2$. But this does not hold true physically. Thus this nucleus's lack of stability cannot be explained by the IPM, and is dictated by the Majorana-interaction model as discussed above. Therefore, indeed these two models are needed simultaneously to provide a complete description of the nuclear phenomenon. Hence as per the duality of these two models, even as one has good IPM description of heavy nuclei, our new model continues to hold good in the backgroud.

The well known binding energy per nucleon plot displays a broad bump at $A \sim 60$ with a value of ~ 8.7 MeV [19]. Above this, the value drops monotonically for heavier nuclei, to a low value of 7.3 MeV for $^{238}_{92}U_{146}$; close to 7.07 MeV of the alphas. As per the model proposed here, this value pre-exists in the background, due to our co-existing alpha-saturated model. There exist no heavier nuclei, as the heavy nucleus would be energetically unstable to break up into a swarm of alphas [19].

We know that, to be able to explain the spontaneous α-decay of heavy nuclei, as per Gamow formalism, we need pre-formed α's to exist inside the nucleus, and which may be able to quantum mechanically tunnel through the corresponding Coulomb barrier. But even at present, it is a theoretically challenging issue, so as to be able to comprehend the existence of this α to pre-exist, to keep on going back and forth, and to keep on banging against the potential barrier.

This puzzle is naturally explained in our new model. As the α's are there in the background; thus these are the source of α's which are needed in the Gamow model to explain the α decay in a simple and consistent manner.

We need to clarify a source of potential confusion between Majorana and Heisenberg interactions. This has to do with the fact that though Majorana interaction ignored isospin and spin both, Heisenberg in Solvay conference in October 1933 [[13], p.15-21] showed that the isospin formalism was equivalent to a formalism that treats neutrons and protons as different and distinguishable particles (so called "ordinary method" [[13], p. 18]. The latter case was claimed to be true for Majorana interaction model without the "troublesome" isospin coordinates). Heisenberg claimed to have thereby incorporated Majorana interaction as well in his isospin formalism as in eqn. (16). This is now a well accepted logic in nuclear physics, and has led to the abandonement of Majorana interaction as an independent reality. But note that in this claimed equivalence, in the "ordinary method", space and spin coordinates were treated as one single coordinate. This led him to establish the equivalence $\mathbb{P}^{\tau} = -\mathbb{P}^{\sigma}\mathbb{P}^{M}$, as in eqn. (16). But the fact that Majorana interaction actually ignores spin altogether, was unfortunately considered as insignificant. But it is basic that pure Majorana interaction does ignore spin completely, and has been often emphasized [13], [14].

6. A New Nonrelativistic Quantum Mechanics Sans Relativity :

Above we saw that the Generalized Pauli Exclusion Principle, based on spin-isospin-space exchange mechanism as given in eqn. (16), to describe saturation demands and is based on, a short range repulsion - the so called hard core. This requires an IPM kind of machinery based on the language of a conventional nonrelativistic quantum mechanics. This constitutes our most successful model understanding of the nucleus [15]. Clearly this model, because of the relativistic Lorentz transformation and nonrelativistic transformation connection due to $v \ll c$ holding true, the relativistic quantum mechanics and the corresponding nonrelativistic quantum mechnaics are related to each other. To belabour the point, the nonrelativistic quantum mechanics exists entirely because there pre-exists a relativistic quantum mechanical structure.

In contrast to the above, we found that a pure Majorana interaction between neutron-proton pairs, using Shroedinger equation, leads to a force which ignores completely both the spin and the isospin internal degrees of freedom in nucleus. This leads to an independent structure to understand saturation, and which is of pure quantum mechanical exchange character. Thus this is another pure nonrelativistic quantum mechanical language, independent of the above mentioned conventional nonrelativistic quantum mechanics. This new quantum mechanics does not arise in terms of some limiting velocity, like the velocity of light, to compare with. Remember that in Majorana interaction, the electromagnetic force is neglected. Hence there does not exist any photon to mediate any eletromagnetic force, and hence there exists no velocity of light in this particular quantum mechanical setup.

In this context it is important to go back and note an important fact. Given the Poincare Lie algebra, to be able to get the Lie algebra of the Galilean group, we need to use the group theoretical technique of "contraction" [9]. Here contraction, a la Inonu and Wigner technique [2], takes the speed of light c to infinity, $c \to \infty$. It is this Galilei group irreducible representation which behaves so differently with respect to the irreducilble representations of the Poincare, the Lorentz and the rotation groups.

Let us point out once again that in quantum mechanics one deals with the unitary projective representations of the relevant group. In most of the cases like, the Poincare, the Lorentz, and the rotation groups the study of unitary projective representations of the group is reducible to the study of the true unitary representations of the corresponding universal covering group. But the Galilei group's behaviour is very different. Amazingly, the physical representations of the Galilei group are nontrivial projective representations [3], the true (vector) representations are devoid of any physical meaning. A

Bargmann's Superselection rule comes into play, which shows that, the intrinsic role that the mass plays in the relativistic and the nonrelativistic quantum mechanics, is fundamentally different in the two cases [3], [6], [7]. Obviously therefore these two masses correspond to two different languages. This is unequivocal support of the above claim of there being two different and simultaneously existing nonrelativistic quantum mechanical languages ([15],p.11-16).

Because of the significance of what we have said above, we state it diffrently as follows. The conventional nonrelativistic quantum mechanics, because of its intrinsic connection arising due to the limit $v \ll c$, with the Lorentz based relativistic quantum mechanics; should be completely different from the new nonrelativistic quantum mechanics a la Galilei symmetry arising due to $c \rightarrow \infty$ limit of the Poincare group. In the latter case, due to the fact that $c \rightarrow \infty$, there is no limiting velocity left to compare the nonrelativistic velocity "v" with. Hence these two different and simultaneously existing nonrelativistic quantum mechanical structures would ensure, that this nonrelativistic velocity in the Galilei group case, be the same as that in the Lorentz nonrelativistic limit case of $v \ll c$.

7. Galilei group and Majorana Interaction :

We know that the Standard Model in particle physics and the IPM in nuclear physics, deal with particles which are eigenstates of the Poincare representation [15]. How about Majorana interaction? If it is, as claimed above, a genuine non-relativistic quantum mechanics, its manifestation should be through the Galilei group, which is the group defining the Newtonian mechanics.

As emphasized by Blatt and Weisskopf ([20], p. 138), "The exchange forces should be considered as a first, crude approximation to a theory of the nuclear forces which takes the "structure" of the neutron and proton into account. Breit and Wigner have given an argument in favor of this point of view: if the neutron and proton really change places during their interaction, the center of gravity of the two particles also moves slightly because their masses are not quite equal. It would then be impossible to separate off the center-of-mass motion in neutron-proton scattering."

This should be contrasted with the GPEP in eqn. (16), as used in the standard and successful models of the nucleus like IPM etc. In these models, neutron-proton exchange force, does not lead to masses of proton and neutron themselves, being moved around in their exchanges. However, e.g. in the

Yukawa One Pion Exchange Potential (OPEP), the mass of pion determines the range of the nuclear force in this virtual exchange [15]. However, in Majorana interaction of eqn. (21) the exchange of actual neutron and proton masses, appears to make sense. And it is through group theory that we may understand the exchange mechanism in pure Majorana interaction eqn. (21), as done by Breit and Wigner [21].

The validity and goodness of the Majorana interaction operator V given in eqn. (21), arises from the fact that it is Hermitian and that it commutes with all the three components of the total momentum operator.

$$(\frac{\hbar}{i})(\Sigma_i \frac{\partial}{\partial x_k^i} + \Sigma_j \frac{\partial}{\partial \xi_k^j}) = P_k \tag{22}$$

Majorana interaction is satisfactory in this regard [21] . However as pointed out by Breit and Wigner [21], there is an "undesirable feature" of Majorana interaction, which arises from the difference in the mass m_p of proton and mass m_n of neutron. Though this difference is small, it will show up as an improver behaviour of the centre of mass and also leading to a lack of invariance to Galilean transformations. Our interest here is in the Galilei group. Let us look at deuteron. As shown by Breit and Wigner [21], momentum P and energy E is obtained on substituting $\psi = \phi(y)\exp[\frac{i}{2\hbar}\Sigma(x_s + \xi_s)P_s]$; $(y = \xi - x)$, and using eqn. (21),

$$-\frac{\hbar^2}{2}(\frac{1}{m_n} + \frac{1}{m_p})\Sigma_s \frac{\partial^2 \phi(y)}{\partial {y_s}^2} + \frac{\hbar}{2i}\frac{m_n - m_p}{m_n m_p}\Sigma_s P_s \frac{\partial \phi(y)}{\partial y_s} + J(y)\phi(-y)$$

$$= [E - \frac{P^2}{8}(\frac{1}{m_n} + \frac{1}{m_p})]\phi(y) \tag{23}$$

The right hand side is real eigenvalue of the Hermitain operator on the left. If ordinary potential replaces the Majorana opertor above, the term with $(m_n - m_p)$ would be absent, and on the right $\frac{P^2}{8}(\frac{1}{m_n} + \frac{1}{m_p})$ would be changed to $\frac{P^2}{2(m_n+m_p)}$. One thus finds that with Majorana interaction, it is not possible to interpret the eigenvalue of E as the sum of $\frac{P^2}{2(m_n+m_p)}$ and an internal energy independent of P. If we take $m_n - m_p \neq 0$, then Galilean invariance would be violated. So for Majorana interaction to hold good, Galilei symmetry should be exact. And thus we shoul necessarily have $m_n = m_p$ for Galilei symmetry to hold good a la Majorana interaction.

Note the remarkable parallel of the Breit-Wigner description of Majorana interaction exchange a la Galilei group, and the same via the Bargmann Superselection rule to ensure the goodness of the Galilei group. Clearly these two, originating from two different realities; the second one based on pure mathematical reasons of the Galilei symmetry demands implemented through the Bargmann Superselection rule, and the first one of pure physical reasons, based on the significance of the Majorana interaction between neutron-proton pairs in nuclei. Thus both these require the neutron and proton masses to be exactly equal to each other. Both, the pure mathematical picture and the independent but simultaneous pure physical picture, depending upon each other, and thus also supporting each other in an inseparable manner, form the basis of the language of nuclear reality([15],p.11-16).

8. Galilei Group, Majorana Interaction and Dark Matter :

Above we have seen that there are two independent nonrelativistic quantum mechanical structures which satisfy the same Schroedinger equation. The potentials however would differ as per whether there is a relativistic quantum mechanical connection or not. Clearly the corresponding masses would differ from each other too. Let us look into the mass issue now.

The conventional nonrelativistic quantum mechanics gives mass $m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}$, in the limit $v \ll c$. The particle with this mass corresponds to the Poincare irreducible representation, and partakes in all three forces - the strong, the electromagnetic and the weak forces. And of courese this is the mass $\frac{m_n - m_p}{m_n} \sim \frac{939.57-938.28}{939.57} \sim 0.00138$ which breaks the $SU(2)_I$ isospin symmetry, and of course as we saw above, in addition it also breaks the Galilei symmetry. These are the masses which go into an intrinsically broken $SU(2)_I$ isospin symmetry, which forms the basis of the IPM.

In contrast, the mass, a la Bargmann Superselection rule arise as being an invariant of the Galilei symmetry. In magnitude these are unconnected with the physical inertial masses of proton and neutron shown above. Clearly these are completely different from each other. Thus better to say that these new masses are $M_n = M_p$, signifyng the intrinsic difference in the notation itself.

Note that Majorana ignores electrodynamics in his interaction in eqn. (21). In addition Majorana interaction ignores spin as well as isopin too [18] [13], [14]. Ignoring isospin means that the strong interaction is also ignored. As parity is good in eqn. (21), so the weak interaction is also ignored here. Thus the Majorana interaction between neutron-proton pair in nucleus, ignores all the three forces - the strong, the electromangnetic and the weak. Now as spin is also ignored, so the quantum mechanical exchange force due to spin, is missing here too. Hence amazingly, the only force left, and the only one present, in the Majorana interaction between neutron-proton pair in nucleus, in eqn. (21), is the space exchange force (as discussed above). This is a pure quantum mechanical force.

Now as out of the four forces known in nature, three (strong, electromagnetic, and weak) forces are missing, for our neutron-proton pair, what is left is the gravitational force. Hence the mass in $M_n = M_p$, a la Bargmann Superselection rule, should be gravitaional mass in nature. There being no other forces left, it is pure gravitational mass, and which is pure quantum nechanical in nature too.

This remarkable fact emerges from the Majorana interaction - Galilei group connection above. What we are seeing is a completely unexpected connection between quantum mechanics and the gravitational force.

Thus given our model here, each proton and neutron has two different masses; one m_n *and* m_p a la isospin symmetry partaking in all the three forces, and immune to these forces, another pure gravitational masses M_n *and* M_p. Let us write it explicitly for the total masses in each case of a neutron-proton pair in a nucleus,

$$ {M_n}^T = m_n + M_n \ ; \quad {M_p}^T = m_p + M_p \tag{24} $$

Without much ado, right away we have an explanation of the Dark Matter problem of the universe. We saw that in each α there are four such pairs creating quantum mechanical bonds. The universe has about one-fourth of the total matter of the universe in the form of α's. This would be expected to hold roughly for each galaxy too. Therefore the galaxies would weigh much more than expected, and this will manifest itself as Dark Matter through the velocity curve of spiral galaxies, and be the source of 90% as Dark Matter. Hence our model offers the most basic reason of the existence of Dark Matter. One may use this extra pure gravitational masses M_n *and* M_p as parameters to model any Dark Matter scenario,

9. Conclusions :

First we found that the Galilei symmetry demands that in nonrelativistic quantum mechanics, we should have Bargmann Superselection rule which demands that it will only hold good, if these two independent particles are having the same mass. This is true for two identical particles like a pair of protons and a pair of neutrons, but would not hold for a pair of neutron and protron as their masses differ, albeit slightly. Thus this neutron-proton pair does break the Bargmann Superselection rule. These masses, we know do correspond to the slighly broken isospin symmetry, which is part of the conventional nonrelativistic quantum mechanical structure used in nuclear physics. Hence we found that the Bargmann kind of masses can not be the mass of particles as obtained in the nonrelativistic limiting case of the Lorentz transformation scenario. Not only are the masses completely independent, but the very norelativistic quantum mechanical structures are diffrent too; and with the Galilei group - Bargmann Superselection rule case, having no relativistic counterpart as well. These mathematical conclusions are consolidated by the fact that there exists a physical Majorana interaction between each pair of neutrons-protons in nuclei.

Now this Majorana interaction between a pair of neutron and proton, to be good for the Galilei transformation, demands that these masses be equal $M_n = M_p$. This picture, a la Majorana interaction, ignores the force of electromagnetism. It also ignores spin and the isospin degrees of freedom too. This leads to confirming the goodness of the Bargmann Supersspelction rule in nucleus through the existence of the Majorana interaction. Hence there do exist two indepndent and simulatineous nonrelativistic quantum mechanical structures in nature.

Next we point out that the Majorana interaction neglects all the three forces - the strong, the electromagnetic and the weak. And also having neglected the spin as well, only space coordinates are left, and whose exchange makes it a pure quantum mechanical force. And thus the mass here is pure gravitational mass and which is immune to the other three forces. This makes an amazing connection between the graviational force and quantum mechanics.

The above thus leads to generatig independent gravitational masses for each neutron-proton pair in nucleus. Subject only to the gravitational force, this explains the Dark Matter problem of the universe.

10. Appendix :

Heiesenberg-interaction [17], [13], exchanges isospin coordinates of a pair of neutron and proton (zero for p-p and n-n pairs) as per $J(r)P^{\tau}$. Heiesenberg showed that owing to eqn. (16), this can be written as $J(r)P^{\sigma}P^{M}$.

Heisenberg took [17] $J(r) > 0$, and then the interaction energy would be positive if the wave function was symmteric under the exchange of position and spin, and negative for it were antisymmetric. The ground state of the nucleus would then have as many antisymmetric neutron-proton pairs as possible [13]. Heiesenberg's model did not provide saturation, and thus needed a short range hard core in the nucleon-interaction to obtain saturation.

Majorana opposed both these features of Heisenberg's model. He took $J(r) < 0$ [18], [13], [14]. The ground state in that case would have the greatest possible numbers of symmetric neutron-proton pairs. The interaction $J(r)P^{\sigma}P^{M}$ with $J(r) < 0$ is attractive, giving complete symmetry of the protopn-neutron wave function under an exchange of both the spin and the space coordinates. In deuteron one expects ground state to be in symmetric orbital s-state, with spin symmetric S=1 state. Majorana assumed that the ground state of alpha should be completely symmetric in the position coordinates of all protons and neutrons. Then Heisenberg interaction with $J(r) < 0$ will give net attraction for neutron and proton with parallel spin (S=1), but no force with those with opposite spins. This provides only two attractive bonds in alpha and with a potential energy about twice that of deuteron. This is unsatisfactory [13].

Majorana thus predicted [18] the interaction in eqn. (21) with $J(r) > 0$. This is independent of both isospin and spin degrees of freedom [18], [13], [14]. Thus Majorana interaction is attractive in symmetric orbtal state and antisymmetric in it. Thus each proton in alpha particle interacts with two neutrons instead of only one, and vice versa. This explains the stability of alpha-particle [18], [13].

But note that deuteron in Majorana-interaction model, having no spin and no isospin degree of freedom, and favouring a space symmetric space in the ground state, is demanding in total, a symmetric state [18], [13] [14]. It has no Coulomb interaction also. This conventional undrestanding of Majorana interaction is unfoprtunstely incomplete [13] [14], [16], [18], [20]. In this paper we have shown that in reality, Majorana interaction has much more to it.

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