Article

The Expansion Theorems for Sturm-Liouville Operators with an Involution Perturbation

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Abstract: In this work, we studied the Green's functions of the second order differential operators with involution. Uniform equiconvergence of spectral expansions related to the second-order differential operators with involution is obtained. Basicity of eigenfunctions of the second-order differential operator operator with complex-valued coefficient is established.

Keywords: differential equations; involution; boundary value problems; Green's function; eigenfunction expansions; equiconvergence; Riesz basis; spectral properties

1. Introduction

In this paper we consider in the Hilbert space $L_2(-1,1)$ a second-order differential operator L_0 defined by

$$L_0 y = -y''(x) + \alpha y''(-x), \tag{1}$$

with domain $D(L_0) \subset L_2(-1,1)$, where $-1 < \alpha < 1$, $\alpha \neq 0$. We denoted by $AC[-1,1]$ the space of absolute continuous functions on $[-1,1]$ and denoted

$$AC^1[-1,1] = \left\{ y(x) \in C^1[-1,1] \mid y'(x) \in AC[-1,1] \right\}.$$

The functions $y(x) \in D(L_0)$ satisfy the conditions: $y(x)$ belongs to $AC^1[-1,1]$ and

$$y(-1) = y(1), \;\; y'(-1) = y'(1). \tag{2}$$

Along with operator L_0 we also consider an operator L defined by

$$Ly = -y''(x) + \alpha y''(-x) + q(x)y(x), \tag{3}$$

with domain $D(L) = D(L_0) \subset L_2(-1,1)$, where $q(x) \in L_1[-1,1]$ is complex-valued function.

Uniform equiconvergence of spectral expansions related to the operators L_0 and L, given by (1), (3) respectively, is studied.

Differential equations with involution form a special class of linear functional-differential equations, with their theory having been developed since the middle of the last century. Among a variety of studies in this direction, one can mention the books [1–3]. The existence of a solution of the partial differential equation with involution has been studied in [2] by the separation of variables method. As in the case of classical equations, applying the Fourier method to partial differential equations with involution leads to the related spectral problems for differential operators with involution. The study of spectral problems for differential operators with involution started relatively recently. In [4–7] the spectral problems for the first-order differential operators with an involution have been studied. In [8] (see also references therein), [9] the spectral problems for differential operators with involution in the lower terms have been considered. The

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spectral problems related to the second-order differential operators with involution have been studied in [10–16]. Qualitative analysis of the solutions (Green's functions) to boundary value problems for the differential equations with involution is available in [2, 3,17–19]. In [14,15,20,21] the Green's functions of the second-order differential operators with involution have been investigated and theorems on basicity of eigenfunctions are proved. Theorems on basicity of eigenfunctions of the second order differential operators with involution [16] have been used to solving inverse problems in [22–24]. Solvability of problems for partial differential equations with involution is discussed in [25–30].

In this paper the integral Cauchy method [31] (well-known in the spectral theory of ordinary differential operators) is modified for the case of differential operators with involution. The method is based on proving the equiconvergence of the known expansion with the eigenfunction expansion of the considered problem.

2. Auxiliary Statements

In this section the boundary value problem

$$Ly = -y''(x) + \alpha y''(-x) + q(x)y(x) = \lambda y(x),$$

$$U_i(y) = a_{i1}y'(-1) + a_{i2}y(-1) + a_{i3}y'(1) + a_{i4}y(1) = 0, (i = 1, 2)$$

will be first considered. Here λ is a complex parameter.

We introduce the definition of the Green's function. Let the boundary value problem not has a nontrivial solution. But can be exists a function $G_q(x, t, \lambda)$ such that:

1) $G_q(x, t, \lambda)$ is continuous on the rectangle $-1 \le x, t \le 1$;

2) the function $G_q(x, t, \lambda)$ has the continuous derivative $\left(G_q(x, t, \lambda)\right)'_x$ for $x \ne \mp t$ and satisfies the conditions:

$$\left(G_q(x,t,\lambda)\right)'_x\Big|_{t=-x-0} - \left(G_q(x,t,\lambda)\right)'_x\Big|_{t=-x+0} = \frac{\alpha}{\sqrt{1-\alpha^2}},$$

$$\left(G_q(x,t,\lambda)\right)'_x\Big|_{t=x-0} - \left(G_q(x,t,\lambda)\right)'_x\Big|_{t=x+0} = \frac{-1}{\sqrt{1-\alpha^2}};$$

3) the function $G_q(x, t, \lambda)$ has the derivative $\left(G_q(x, t, \lambda)\right)''_{xx}$, satisfies $Ly = \lambda y$ (except at $x \ne \mp t$) and $U_i(y) = 0, (i = 1, 2)$.

The function $G_q(x, t, \lambda)$ is called the Green's function of the considered boundary value problem (of the operator $L - \lambda I$, where L defined by the rule (3) on the functions for which $U_i(y) = 0, (i = 1, 2)$, I is identity operator).

If the function $G_q(x, t, \lambda)$ is the Green's function of the operator $L - \lambda I$, then the function

$$y(x) = \int_{-1}^{1} G_q(x, t, \lambda) f(t) dt$$

gives the solution to the problem

$$-y''(x) + \alpha y''(-x) + q(x)y(x) = \lambda y(x) + f(x), \ -1 < x < 1,$$

with boundary conditions $U_i(y) = 0, (i = 1, 2)$), for any function $f(x) \in C[-1, 1]$ (this statement, existence and uniqueness of the Green's function can be proved by standard methods [32](see chapter 1).

Now write homogeneous equation $L_0 y = \lambda y(x)$ according to (1). Let us denote by $y_1(x) = \cos\alpha_0\rho x$, $y_2(x) = sin\alpha_1\rho x$, where $\sqrt{\lambda} = \rho$, $\arg\rho \in (-\pi/2, \pi/2)$, $\alpha_0 = \sqrt{(1-\alpha)^{-1}}$, $\alpha_1 = \sqrt{(1+\alpha)^{-1}}$, the linearly independent solutions of this homogeneous equation. Let $\rho_0 = Im\rho$ be the imaginary part of the complex number ρ. Let

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a function $G(x,t,\lambda)$ is the Green's function of the operator $L_0 - \lambda I$ and the function $y(x) = \int\limits_{-1}^{1} G(x,t,\lambda) f(t) dt$ is the solution of problem

$$L_0 y = \lambda y(x) + f(x), \tag{4}$$

with boundary conditions (2), for any function $f(x) \in C[-1,1]$.

Lemma 1. *If λ is not an eigenvalue of the operator L_0, then a function*

$$y(x) = -\frac{\alpha_0}{2\rho}\frac{\cos\alpha_0\rho}{\sin\alpha_0\rho}\cos(\alpha_0\rho x)\int\limits_{-1}^{1}\cos(\alpha_0\rho t) f(t) dt$$

$$-\frac{\alpha_1}{2\rho}\frac{\cos\alpha_1\rho}{\sin\alpha_1\rho}\sin(\alpha_1\rho x)\int\limits_{-1}^{1}\sin(\alpha_1\rho t) f(t) dt + g_0(x)$$

is the solution of non-homogeneous problem (4), (2) for any continuous function $f(x)$, where

$$g_0(x) = \frac{1}{2\rho}\int\limits_{-1}^{-x}[\alpha_0\cos(\alpha_0\rho x)\sin(\alpha_0\rho t) - \alpha_1\sin(\alpha_1\rho x)\cos(\alpha_1\rho t)] f(t) dt$$

$$+\frac{1}{2\rho}\int\limits_{-x}^{x}[-\alpha_0\cos(\alpha_0\rho t)\sin(\alpha_0\rho x) + \alpha_1\sin(\alpha_1\rho t)\cos(\alpha_1\rho x)] f(t) dt$$

$$+\frac{1}{2\rho}\int\limits_{x}^{1}[-\alpha_0\cos(\alpha_0\rho x)\sin(\alpha_0\rho t) + \alpha_1\sin(\alpha_1\rho x)\cos(\alpha_1\rho t)] f(t) dt.$$

This Lemma 1 can be proved by direct calculations. From Lemma 1, we get the following

Corollary 1. *The Green's function of the operator $L_0 - \lambda I$ can be represented in the form*

$$G(x,t,\lambda) = -\frac{\alpha_0}{2\rho}\frac{\cos\alpha_0\rho}{\sin\alpha_0\rho}\cos(\alpha_0\rho x)\cos(\alpha_0\rho t) - \frac{\alpha_1}{2\rho}\frac{\cos\alpha_1\rho}{\sin\alpha_1\rho}\sin(\alpha_1\rho x)\sin(\alpha_1\rho t)$$

$$+\frac{1}{2\rho}\begin{cases} \alpha_0\cos(\alpha_0\rho x)\sin(\alpha_0\rho t) - \alpha_1\sin(\alpha_1\rho x)\cos(\alpha_1\rho t), t < -x, \\ -\alpha_0\cos(\alpha_0\rho x)\sin(\alpha_0\rho t) + \alpha_1\sin(\alpha_1\rho x)\cos(\alpha_1\rho t), t > x, \\ -\alpha_0\cos(\alpha_0\rho t)\sin(\alpha_0\rho x) + \alpha_1\sin(\alpha_1\rho t)\cos(\alpha_1\rho x), -x < t < x. \end{cases}$$

Green's function of the operator $L_0 - \lambda I$ has the following properties:
1) $G(x,t,\lambda)$ is the symmetric: $G(x,t,\lambda) = G(t,x,\lambda)$, for all $-1 \le x,t \le 1$;
2) $G(x,t,\lambda)$ is continuous on the rectangle$-1 \le x,t \le 1$;
3) the function $G(x,t,\lambda)$ has the continuous derivative $G'{}_x(x,t,\lambda)$ for $x = \mp t$, and satisfies the conditions:

$$G'_x(x,t,\lambda)|_{t=-x-0} - G'_x(x,t,\lambda)|_{t=-x+0} = \frac{\alpha}{\sqrt{1-\alpha^2}},$$

$$G'_x(x,t,\lambda)|_{t=x-0} - G'_x(x,t,\lambda)|_{t=x+0} = \frac{-1}{\sqrt{1-\alpha^2}};$$

4) the function $G(x,t,\lambda)$ has the derivative$G''{}_{xx}(x,t,\lambda)$, satisfies $L_0 y = \lambda y$ (except at$x = \mp t$) and (2).

The operator L_0 of the form (1) with periodic boundary conditions (2) has the eigenvalues $\lambda_{k1} = (1-\alpha)(k\pi)^2,\ k = 0,1,2,\ldots;\ \lambda_{k2} = (1+\alpha)(k\pi)^2,\ k = 1,2,\ldots.$ The system of eigenfunctions $\{y_{k1} = cosk\pi x, y_{k2} = sink\pi x\}$ of the operator L_0 is complete and orthogonal in $L_2(-1,1)$. Denote $\rho_{k1} = \sqrt{(1-\alpha)}k\pi, k = 0,1,2,\ldots,$ $\rho_{k2} = \sqrt{(1+\alpha)}k\pi, k = 1,2,\ldots.$

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Since $\rho_{k+1,1} - \rho_{k1} = \sqrt{(1-\alpha)}\pi$; $\rho_{k+1,2} - \rho_{k2} = \sqrt{(1+\alpha)}\pi$, we denote by $O_\varepsilon(\rho_{01}) = \{\rho : |\rho - \rho_{01}| < \varepsilon,\}$,
$O_\varepsilon(\rho_{kl}) = \{\rho : |\rho - \rho_{kl}| < \varepsilon, k = 1, 2, \ldots; l = 1, 2,\}$ a circle $C_{01}, C_{kl}, k = 1, 2, \ldots; l = 1, 2$, of radius $\varepsilon = \frac{1}{4}min((1-\alpha)\pi, (1+\alpha)\pi)$. Then the circles $C_{01}, C_{kl}, k = 1, 2, \ldots; l = 1, 2$, with equations $\rho = \frac{\varepsilon}{2}, \rho = \rho_{kl} + \frac{\varepsilon}{2}$ do not intersect the circles $O_\varepsilon(\rho_{01}), O_\varepsilon(\rho_{kl})$ for large k .

Further we need an estimate of the Green's function of the operator $L_0 - \lambda I$.

Lemma 2. *Let $\rho \notin O_\varepsilon(\rho_{kl})$ and $|\rho| > 1$. Then the Green's function $G(x, t, \lambda)$ of the operator $L_0 - \lambda I$ satisfies the uniform with respect to $-1 \le x, t \le 1$ estimate*

$$|G(x, t, \lambda)| \le c_0(\alpha, \varepsilon)|\rho|^{-1} r(x, t, \rho)$$

where
$r(x, t, \rho) = \left(e^{-\alpha_2|\rho_0|(2-|x|-|t|)} + e^{-\alpha_2|\rho_0|||x|-|t||}\right), \rho_0 = Im\rho, \ \alpha_2 = min\{\alpha_1, \alpha_0\}.$

Proof of Lemma 2. We have to examine three cases: $1 \le t < -x, -x < t < x, x < t \le 1$. Let $t > x$. Then the Green's function $G(x, t, \lambda)$ can be rewritten in the form

$$G(x, t, \lambda) = \frac{\alpha_0}{4i\rho}\left\{\frac{e^{-i\alpha_0\rho}}{e^{i\alpha_0\rho} - e^{-i\alpha_0\rho}}\left[e^{i\alpha_0\rho(x+t)} + e^{i\alpha_0\rho(t-x)}\right]\right.$$
$$\left. + \frac{e^{i\alpha_0\rho}}{e^{i\alpha_0\rho} - e^{-i\alpha_0\rho}} + \left[e^{i\alpha_0\rho(x-t)} + e^{i\alpha_0\rho(-x-t)}\right]\right\}$$

$$+ \frac{\alpha_1}{4i\rho}\frac{e^{i\alpha_1\rho}}{e^{i\alpha_1\rho} - e^{-i\alpha_1\rho}}\left[-e^{i\alpha_1\rho(x+t)} + e^{i\alpha_1\rho(t-x)}\right] + \frac{e^{i\alpha_1\rho}}{e^{i\alpha_1\rho} - e^{-i\alpha_1\rho}}\left[e^{i\alpha_1\rho(x-t)} - e^{i\alpha_1\rho(-x-t)}\right].$$

From this and $\rho_0 = Im\rho$ we obtain the inequality

$$|G(x, t, \lambda)| \le \frac{\alpha_0}{4|\rho|}\frac{e^{\alpha_0\varrho_0}}{|e^{-\alpha_0\varrho_0} - e^{\alpha_0\varrho_0}|}\left[e^{-\alpha_0\varrho_0(x+t)} + e^{-\alpha_0\varrho_0(t-x)}\right]$$
$$+ \frac{\alpha_0}{4|\rho|}\frac{e^{-\alpha_0\varrho_0}}{|e^{-\alpha_0\varrho_0} - e^{\alpha_0\varrho_0}|}\left[e^{-\alpha_0\varrho_0(x-t)} + e^{-\alpha_0\varrho_0(-x-t)}\right]$$
$$+ \frac{\alpha_1}{4|\rho|}\frac{e^{\alpha_1\varrho_0}}{|e^{-\alpha_1\varrho_0} - e^{\alpha_1\varrho_0}|}\left[e^{-\alpha_1\varrho_0(x+t)} + e^{-\alpha_1\varrho_0(t-x)}\right]$$
$$+ \frac{\alpha_1}{4|\rho|}\frac{e^{-\alpha_1\varrho_0}}{|e^{-\alpha_1\varrho_0} - e^{\alpha_1\varrho_0}|}\left[e^{-\alpha_1\varrho_0(x-t)} + e^{-\alpha_1\varrho_0(-x-t)}\right]. \quad (5)$$

Let $\rho_0 > 0$ and γ is arbitrary positive number. For sufficiently large $\rho_0 > 0$, the estimates

$$\frac{e^{\gamma\rho_0}}{|e^{-\gamma\rho_0} - e^{\gamma\rho_0}|} \sim 1, \quad \frac{e^{-\gamma\rho_0}}{|e^{-\gamma\rho_0} - e^{\gamma\rho_0}|} \sim e^{-2\gamma\rho_0} \quad (6)$$

holds. Applying these inequalities (6) to (5) we get

$$|G(x, t, \lambda)| \le \frac{\alpha_0}{4|\rho|}\left[\left(e^{-\alpha_0\rho_0(x+t)} + e^{-\alpha_0\rho_0(t-x)}\right) + \left(e^{-\alpha_0\rho_0(2+x-t)} + e^{-\alpha_0\rho_0(2-t-x)}\right)\right]$$

$$+ \frac{\alpha_1}{4|\rho|}\left[\left(e^{-\alpha_1\rho_0(x+t)} + e^{-\alpha_1\rho_0(t-x)}\right) + \left(e^{-\alpha_1\rho_0(2+x-t)} + e^{-\alpha_1\rho_0(2-t-x)}\right)\right].$$

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If $t > x > 0$, it is obviously that

$$t + x > t - x > 0, 2 + x - t > 2 - x - t > 0. \tag{7}$$

Therefore we obtain the relation

$$|G(x,t,\lambda)| \leq \frac{\alpha_0}{4|\rho|}\left[e^{-\alpha_0\rho_0(2-x-t)} + e^{-\alpha_0\rho_0(t-x)}\right] + \frac{\alpha_1}{4|\rho|}\left[e^{-\alpha_1\rho_0(2-x-t)} + e^{-\alpha_1\rho_0(t-x)}\right].$$

Hence,

$$|G(x,t,\lambda)| \leq \frac{\alpha_0+\alpha_1}{4|\rho|}\left(e^{-\alpha_2|\rho_0|(2-x-t)} + e^{-\alpha_2|\rho_0|(t-x)}\right), \quad \alpha_2 = \min(\alpha_0, \alpha_1). \tag{8}$$

Let $\rho_0 < 0$ and γ is any positive number. For sufficiently large $|\rho_0|$ the estimates hold:

$$\frac{e^{\gamma\rho_0}}{|e^{-\gamma\rho_0} - e^{\gamma\rho_0}|} \sim e^{-2\gamma\rho_0}, \frac{e^{-\gamma\rho_0}}{|e^{-\gamma\rho_0} - e^{\gamma\rho_0}|} \sim 1. \tag{9}$$

Applying these inequalities (9) to (5)

$$|G(x,t,\lambda)| \leq \frac{\alpha_0}{4|\rho|}\left[\left(e^{\alpha_0\rho_0(2-x-t)} + e^{\alpha_0\rho_0(t-x)}\right) + \left(e^{\alpha_0\rho_0(t-x)} + e^{\alpha_0\rho_0(t+x)}\right)\right]$$

$$+\frac{\alpha_1}{4|\rho|}\left[\left(e^{\alpha_1\rho_0(2-x-t)} + e^{\alpha_1\rho_0(t-x)}\right) + \left(e^{\alpha_1\rho_0(t-x)} + e^{\alpha_1\rho_0(t+x)}\right)\right].$$

Now if we recall (7), we get

$$|G(x,t,\lambda)| \leq \frac{\alpha_0}{4|\rho|}\left[e^{\alpha_0\rho_0(2-x-t)} + e^{\alpha_0\rho_0(t-x)}\right] + \frac{\alpha_1}{4|\rho|}\left[e^{\alpha_1\rho_0(2-x-t)} + e^{\alpha_1\rho_0(t-x)}\right].$$

From this we obtain the inequality

$$|G(x,t,\lambda)| \leq \frac{\alpha_0+\alpha_1}{4|\rho|}\left(e^{\alpha_2\rho_0(2-x-t)} + e^{\alpha_2\rho_0(t-x)}\right), \ \alpha_2 = \min\{\alpha_0, \alpha_1\}. \tag{10}$$

The estimates (8), (10) implies the inequality

$$|G(x,t,\lambda)| \leq \frac{\alpha_0+\alpha_1}{4|\rho|}\left(e^{-\alpha_2|\rho_0|(2-x-t)} + e^{-\alpha_2|\rho_0|(t-x)}\right), \quad \alpha_2 = \min(\alpha_0, \alpha_1). \tag{11}$$

Thus, for $t > x > 0$ the Green's function satisfies the estimate (11). In the case $-x < t < x$ the Green's function $G(x,t,\lambda)$ can be rewritten appropriately:

$$G(x,t,\lambda) = \frac{\alpha_0}{4i\rho}\left\{\frac{e^{-i\alpha_0\rho}}{e^{i\alpha_0\rho} - e^{-i\alpha_0\rho}}\left[e^{i\alpha_0\rho(x+t)} + e^{i\alpha_0\rho(x-t)}\right]\right.$$

$$\left. + \frac{e^{i\alpha_0\rho}}{e^{i\alpha_0\rho} - e^{-i\alpha_0\rho}} + \left[e^{i\alpha_0\rho(t-x)} + e^{i\alpha_0\rho(-x-t)}\right]\right\}$$

$$+\frac{\alpha_1}{4i\rho}\left\{\frac{e^{-i\alpha_1\rho}}{e^{i\alpha_1\rho} - e^{-i\alpha_1\rho}}\left[-e^{i\alpha_1\rho(x+t)} + e^{i\alpha_1\rho(x-t)}\right] + \frac{e^{i\alpha_1\rho}}{e^{i\alpha_1\rho} - e^{-i\alpha_1\rho}}\left[e^{i\alpha_1\rho(t-x)} - e^{i\alpha_1\rho(-x-t)}\right]\right\}.$$

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From this we obtain the inequality

$$G(x,t,\lambda)=\frac{\alpha_0}{4i\rho}\left\{\frac{e^{-i\alpha_0\rho}}{e^{i\alpha_0\rho}-e^{-i\alpha_0\rho}}\left[e^{i\alpha_0\rho(x+t)}+e^{i\alpha_0\rho(x-t)}\right]\right.$$
$$\left.+\frac{e^{i\alpha_0\rho}}{e^{i\alpha_0\rho}-e^{-i\alpha_0\rho}}\left[e^{i\alpha_0\rho(t-x)}+e^{i\alpha_0\rho(-x-t)}\right]\right\}$$
$$+\frac{\alpha_1}{4i\rho}\left\{\frac{e^{-i\alpha_1\rho}}{e^{i\alpha_1\rho}-e^{-i\alpha_1\rho}}\left[-e^{i\alpha_1\rho(x+t)}+e^{i\alpha_1\rho(x-t)}\right]\right.$$
$$\left.+\frac{e^{i\alpha_1\rho}}{e^{i\alpha_1\rho}-e^{-i\alpha_1\rho}}\left[e^{i\alpha_1\rho(t-x)}-e^{i\alpha_1\rho(-x-t)}\right]\right\}. \quad (12)$$

Let $\rho_0>0$. Applying (6) to (12) we get

$$|G(x,t,\lambda)|\leq\frac{\alpha_0}{4|\rho|}\left[\left(e^{-\alpha_0\rho_0(x+t)}+e^{-\alpha_0\rho_0(x-t)}\right)+\left(e^{-\alpha_0\rho_0(2+t-x)}+e^{-\alpha_0\rho_0(2-x-t)}\right)\right]$$

$$+\frac{\alpha_1}{4|\rho|}\left[\left(e^{-\alpha_1\rho_0(x+t)}+e^{-\alpha_1\rho_0(x-t)}\right)+\left(e^{-\alpha_1\rho_0(2+t-x)}+e^{-\alpha_1\rho_0(2-x-t)}\right)\right].$$

In fact, $0<t+x<x-t, 2+t-x<2-t-x$ for $t>0$, and
$0<t+x<x-t, 2+t-x<2-t-x$ for $t<0$. From this and the last inequality we obtain

$$|G(x,t,\lambda)|\leq\frac{\alpha_0}{4|\rho|}\left(e^{-\alpha_0\rho_0(x-|t|)}+e^{-\alpha_0\rho_0(2-x-|t|)}\right)+\frac{\alpha_1}{4|\rho|}\left(e^{-\alpha_1\rho_0(x-|t|)}+e^{-\alpha_1\rho_0(2-x-|t|)}\right).$$

This implies the estimate

$$|G(x,t,\lambda)|\leq\frac{\alpha_0+\alpha_1}{4|\rho|}\left(e^{-\alpha_2\rho_0(x-|t|)}+e^{-\alpha_2\rho_0(2-x-|t|)}\right),\quad \alpha_2=\min(\alpha_0,\alpha_1). \quad (13)$$

Let $\rho_0<0$. Next, using (9) in (12) we get

$$|G(x,t,\lambda)|\leq\frac{\alpha_0+\alpha_1}{4|\rho|}\left(e^{\alpha_2\rho_0(x-|t|)}+e^{\alpha_2\rho_0(2-x-|t|)}\right),\quad \alpha_2=\min(\alpha_0,\alpha_1). \quad (14)$$

From inequalities (13) and (14) it follows that the estimate

$$|G(x,t,\lambda)|\leq\frac{c_1}{|\rho|}\left(e^{-\alpha_2|\rho_0|(2-|x|-|t|)}+e^{-\alpha_2|\rho_0|(x-|t|)}\right),\ \alpha_2=\min\{\alpha_0,\alpha_1\}, \quad (15)$$

holds. Let us consider the case $t<-x$. Then the Green's function $G(x,t,\lambda)$ can be rewritten in the form

$$G(x,t,\lambda)=\frac{\alpha_0}{4i\rho}\left\{\frac{e^{i\alpha_0\rho}}{e^{i\alpha_0\rho}-e^{-i\alpha_0\rho}}\left[e^{i\alpha_0\rho(x+t)}+e^{i\alpha_0\rho(t-x)}\right]\right.$$

$$\left.+\frac{e^{-i\alpha_0\rho}}{e^{i\alpha_0\rho}-e^{-i\alpha_0\rho}}\left[e^{i\alpha_0\rho(x-t)}+e^{i\alpha_0\rho(-x-t)}\right]\right\}$$

$$+\frac{\alpha_1}{4i\rho}\left\{\frac{e^{i\alpha_1\rho}}{e^{i\alpha_1\rho}-e^{-i\alpha_1\rho}}\left[-e^{i\alpha_1\rho(x+t)}+e^{i\alpha_1\rho(t-x)}\right]+\frac{e^{-i\alpha_1\rho}}{e^{i\alpha_1\rho}-e^{-i\alpha_1\rho}}\left[e^{i\alpha_1\rho(x-t)}-e^{i\alpha_1\rho(-x-t)}\right]\right\}.$$

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From this it follows the estimate

$$G(x,t,\lambda) \le \frac{\alpha_0}{4|\rho|}\left\{\frac{e^{-\alpha_0\rho_0}}{|e^{-\alpha_0\rho_0}-e^{\alpha_0\rho_0}|}\left[e^{-\alpha_0\rho_0(x+t)}+e^{-\alpha_0\rho_0(t-x)}\right]\right.$$
$$\left.+\frac{e^{\alpha_0\rho_0}}{|e^{-\alpha_0\rho_0}-e^{\alpha_0\rho_0}|}\left[e^{-\alpha_0\rho_0(x-t)}+e^{-\alpha_0\rho_0(-x-t)}\right]\right\}$$
$$+\frac{\alpha_1}{4|\rho|}\frac{e^{-\alpha_1\rho_0}}{e^{-\alpha_1\rho_0}+e^{\alpha_1\rho_0}}\left[e^{-\alpha_1\rho_0(x+t)}+e^{-\alpha_1\rho_0(t-x)}\right].$$
$$+\frac{\alpha_1}{4|\rho|}\frac{e^{\alpha_1\rho_0}}{e^{-\alpha_1\rho_0}+e^{\alpha_1\rho_0}}\left[e^{-\alpha_1\rho_0(x-t)}+e^{-\alpha_1\rho_0(-x-t)}\right] \tag{16}$$

Let $\rho_0 > 0$. Using (6) in (16) we get

$$|G(x,t,\lambda)| \le \frac{\alpha_0}{4|\rho|}\left[\left(e^{-\alpha_0\rho_0(2+x+t)}+e^{-\alpha_0\rho_0(2+t-x)}\right)+\left(e^{-\alpha_0\rho_0(x-t)}+e^{-\alpha_0\rho_0(-x-t)}\right)\right]$$

$$+\frac{\alpha_1}{4|\rho|}\left[\left(e^{-\alpha_1\rho_0(2+x+t)}+e^{-\alpha_1\rho_0(2+t-x)}\right)+\left(e^{-\alpha_1\rho_0(x-t)}+e^{-\alpha_1\rho_0(-x-t)}\right)\right].$$

This implies the following inequality

$$|G(x,t,\lambda)| \le \frac{\alpha_0}{4|\rho|}\left(e^{-\alpha_0\rho_0(2+t-x)}+e^{-\alpha_0\rho_0(-x-t)}\right)+\frac{\alpha_1}{4|\rho|}\left(e^{-\alpha_1\rho_0(2+t-x)}+e^{-\alpha_1\rho_0(-x-t)}\right).$$

Thus, we have established the estimate

$$|G(x,t,\lambda)| \le \frac{\alpha_0+\alpha_1}{4|\rho|}\left(e^{-\alpha_2\rho_0(2+t-x)}+e^{-\alpha_2\rho_0(-x-t)}\right), \quad \alpha_2=\min\{\alpha_0,\alpha_1\}, \tag{17}$$

for $t < -x$ and $\rho_0 > 0$. If $\rho_0 < 0$ then using (9) in (16) we get the inequality

$$|G(x,t,\lambda)| \le \frac{\alpha_0+\alpha_1}{4|\rho|}\left(e^{\alpha_2\rho_0(2+t-x)}+e^{\alpha_2\rho_0(-x-t)}\right), \quad \alpha_2=\min\{\alpha_0,\alpha_1\}. \tag{18}$$

From inequalities (17) and (18) it follows that the estimate

$$|G(x,t,\lambda)| \le \frac{\alpha_0+\alpha_1}{4|\rho|}\left(e^{-\alpha_2|\rho_0|(2-|t|-x)}+e^{-\alpha_2|\rho_0|(|t|-x)}\right), \tag{19}$$

holds. Combining (11), (15) and (19), we get

$$|G(x,t,\lambda)| \le \frac{\alpha_0+\alpha_1}{4|\rho|}\left[e^{-\alpha_2|\rho_0|(2-|x|-|t|)}+e^{-\alpha_2|\rho_0|||x|-|t||}\right].$$

Lemma 2 is proved. □

Let L_0 be the differential operator defined by (1), (2) and let L be the differential operator defined by (3)–(2). Let $G_q(x,t,\lambda)$ denote the Green's function of the operator $L-\lambda I$, where I is identity operator. Denote by $G(x,t,\lambda)$ the Green's function of the operator $L_0-\lambda I$. It is easy to see that the function $G_q(x,t,\lambda)$ satisfies

$$LG_q(x,t,\lambda) = \lambda G_q(x,t,\lambda)$$

and the function $G(x,t,\lambda)$ satisfies the equation

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$$L_0 G(x,t,\lambda) = \lambda G(x,t,\lambda).$$

Then the difference $G_q(x,t,\lambda) - G(x,t,\lambda)$ satisfies the equation

$$L_0[G_q(x,t,\lambda) - G(x,t,\lambda)] = \lambda[G_q(x,t,\lambda) - G(x,t,\lambda)] - q(x)G_q(x,t,\lambda), x \neq \mp t,$$

and boundary conditions (2). From this it follows that, except at the poles of functions $G(x,t,\lambda)$ and $G_q(x,t,\lambda)$

$$G_q(x,t,\lambda) - G(x,t,\lambda) = -\int_{-1}^{1} G(x,s,\lambda)q(s)G_q(s,t,\lambda)ds. \tag{20}$$

Then a solution of the equation (20) is the Green's function of the operator $L - \lambda I$. Let us assume that all eigenvalues of the operator L are simple. Further we need the following

Lemma 3. *Suppose all assumptions of Lemma 2 hold true. Then for all sufficiently large ρ, $\rho \notin O_\varepsilon(\rho_{kl}) = \{\rho : |\rho - \rho_{kl}| < \varepsilon\}$, there exists the solution of the integral equation (20).*

Proof of Lemma 3. We apply the method of successive approximations. Let us introduce the following functions

$$G_{q0}(x,t,\lambda) \equiv 0, \quad G_{q,n+1}(x,t,\lambda) = G(x,t,\lambda) - \int_{-1}^{1} G(x,s,\lambda)q(s)G_{q,n}(s,t,\lambda)ds.$$

By Lemma 2

$$|G(x,t,\lambda)| \leq \frac{C}{|\rho|}r(x,t), \;\; r(x,t,\rho) = \left(e^{-\alpha_2|\rho_0|(2-|x|-|t|)} + e^{-\alpha_2|\rho_0|||x|-|t||}\right), \;\; \alpha_2 = \min\{\alpha_1, \alpha_0\}.$$

For brevity, let us introduce the notations

$$\max_{-1\leq x\leq 1} \left|G_{q1}(x,t,\lambda)\right||\rho|r^{-1}(x,t) = C_0,$$
$$\max_{-1\leq x\leq 1} \left|G_{q,n+1}(x,t,\lambda) - G_{q,n}(x,t,\lambda)\right||\rho|r^{-1}(x,t) = C_n, \tag{21}$$

for fixed t and sufficiently large ρ, $\rho \notin O_\epsilon(\rho_{kl})$. Now let us show that

$$C_j \leq \frac{C}{2^j}, j = 0,1,2,\ldots,n. \tag{22}$$

For $j = 0$ the estimate (22) directly follows from Lemma 2 and the relation (21). Let us assume that the assertion is true for $j = n$ and prove its validity for $j = n+1$. Using the relations (20), (21) and Lemma 2 we obtain

$$C_{n+1} \leq C \cdot C_n|\rho|^{-1} \max \int_{-1}^{1} r(x,s)r(s,t)r^{-1}(x,t)|q(s)|ds. \tag{23}$$

In fact,

$$r(x,s)\cdot r(s,t) = e^{-\alpha_0|\rho_0|(4-|x|-2|s|-|t|)} + e^{-\alpha_0|\rho_0|(2-|x|-|s|+||s|-|t||)}$$

$$+e^{-\alpha_0|\rho_0|(2-|s|-|t|+||x|-|s||)} + e^{-\alpha_0|\rho_0|(||x|-|s||+||s|-|t||)}.$$

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Since $||x|-|t|| \leq ||x|-|s||+||s|-|t||$, the relation $|t| = |t|+|s|-|s| \geq |s|-||t|-|s||$ implies $|x|+|t| \geq |x|+|s|-||t|-|s||$. The inequality $|x| \geq |s|-||x|-|s||$ implies $|x|+|t| \geq |t|+|s|-||x|-|s||$.

In fact

$||x|-|t|| = ||x|-1+1-|t|| < 1-|x|+1-|t| < 1-|x|+1-|t|+2-2|s| = 4-|x|-|t|-2|s|$.

From these relations it follows that $r(x,s)\cdot r(s,t) \leq 2r(x,t)$. Therefore, from (23) follows the estimate

$$C_{n+1} \leq 2 \leq CC_n|\rho|^{-1}\int_{-1}^{1}|q(s)|ds. \tag{24}$$

For sufficiently large $|\rho|$, the inequality

$$2C|\rho|^{-1}\int_{-1}^{1}|q(s)|ds \leq \frac{1}{2}$$

holds true. Applying this inequality in (24) we conclude that (22) remains valid for all integer n. From the estimate (22) it follows that the series

$$\sum_{1}^{\infty}(G_{q,n+1}(x,t,\lambda)-G_{qn}(x,t,\lambda))$$

converges uniformly. The partial sums of this series is

$$\chi_m(x) = G_{q,n+m}(x,t,\lambda)-G_{q1}(x,t,\lambda)$$

and the sequence $G_{q,n}(x,t,\lambda)$ converges uniformly. Its limit $G_q(x,t,\lambda)$ satisfies the equation (20). Lemma 3 is proved. □

3. Main Results

Now, we are ready to prove our main result. Denote by

$$S_m(f) = -\frac{1}{2\pi i}\int_{-1}^{1}\left[\int_{C_{ml}} G_q(x,t,\lambda)2\rho d\rho\right]f(t)dt,$$

$$s_m(f) = -\frac{1}{2\pi i}\int_{C_{ml}}\left(\int_{-1}^{1} G(x,t,\lambda)]f(t)dt\right)d\lambda.$$

the partial sums of eigenfunction expansions for the operators L and L_0 respectively, $f(x) \in L_1(-1,1)$. This representation holds true in the case when all eigenvalues of the operator L and L_0 are simple. Note that all eigenvalues of the operator L_0 are simple if $\frac{\alpha_0}{\alpha_1} \neq \nu_0$ and $\frac{\alpha_1}{\alpha_0} \neq \nu_1$ for any integer ν_0, ν_1.

We say that the sequence $S_m(f)$ equiconverges with $s_m(f)$ on the interval $-1 \leq x \leq 1$ if $S_m - s_m \to 0$ uniformly on this interval as $m \to \infty$. Now, we are ready to state our main results.

Theorem 1. *Let all eigenvalues of operator L are simple, $\frac{\alpha_0}{\alpha_1} \neq \nu_0$ and $\frac{\alpha_1}{\alpha_0} \neq \nu_1$ for any integer ν_0, ν_1. Then for any function $f(x) \in L_1(-1,1)$ the sequence $S_m(f)$ equiconverges with $s_m(f)$ on the interval $-1 \leq x \leq 1$.*

Proof of Theorem 1. To prove Theorem 1, we consider the difference

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$$S_m(f)-s_m(f)=-\frac{1}{2\pi i}\int\limits_{C_{ml}}\left\{\int\limits_{-1}^{1}[G_q(x,t,\lambda)-G(x,t,\lambda)]f(t)dt\right\}2\rho d\rho.$$

It follows from Lemma 3 that

$$|G_q(x,t,\lambda)|\leq\frac{2C}{|\rho|}r(x,t).$$

This inequality and expression (20) yields the estimate

$$|G_q(x,t,\lambda)-G(x,t,\lambda)|\leq 2C^2|\rho|^{-2}\int\limits_{-1}^{1}r(x,s)|q(s)|r(s,t)ds.$$

Since $r(x,s)\cdot r(s,t)\leq 2r(x,t)$, the latter inequality can be rewritten in the form

$$|G_q(x,t,\lambda)-G(x,t,\lambda)|\leq 4C^2|\rho|^{-2}r(x,t)\int\limits_{-1}^{1}|q(s)|ds.$$

From this inequality and relation (20) we obtain the estimate

$$|S_m(f)-s_m(f)|\leq\frac{2C^2}{\pi}\int\limits_{C_{ml}}\left[\int\limits_{-1}^{1}r(x,t)|f(t)|dt\right]\frac{2|\rho|}{|\rho|^2}d\rho\cdot\int\limits_{-1}^{1}|q(s)|ds$$

$$=\frac{4C^2}{\pi}\int\limits_{-1}^{1}|q(s)|ds\int\limits_{C_{ml}}\left[\int\limits_{-1}^{1}r(x,t)|f(t)|dt\right]\left|\frac{d\rho}{\rho}\right|.$$

Let us denote $C_1=\frac{4C^2}{\pi}\int\limits_{-1}^{1}|q(s)|ds$. Then we have

$$|S_m(f)-s_m(f)|\leq C_1\int\limits_{C_{ml}}\left[\int\limits_{-1}^{1}r(x,t)|f(t)|dt\right]\left|\frac{d\rho}{\rho}\right|. \tag{25}$$

Further we split the interval of integration into two parts:

$$\Delta_1=(-1+\delta,-x-\delta)\cup(-x+\delta,x-\delta)\cup(x+\delta,1-\delta),$$

$$\Delta_2=(-1,-1+\delta)\cup(-x-\delta,-x+\delta)\cup(x-\delta,x+\delta)\cup(1-\delta,1),$$

thus $(-1,1)=\Delta_1+\Delta_2$ and $\delta>0$ is a sufficiently small number. Now the inequality (25) takes the form

$$|S_m(f)-s_m(f)|\leq C_1\int\limits_{C_{ml}}\int\limits_{\Delta_1}\left(e^{-\alpha_0|\rho_0|||x|-|t||}+e^{-\alpha_0|\rho_0|(2-||x|-|t||)}\right)|f(t)|\,dt\left|\frac{d\rho}{\rho}\right|$$

$$+2C_1\pi\int\limits_{\Delta_2}|f(t)|dt. \tag{26}$$

The identity

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$$\int_{\Delta_2} |f(t)|dt = \int_{-1}^{-1+\delta} |f(t)|dt + \int_{-x-\delta}^{-x+\delta} |f(t)|dt + \int_{x-\delta}^{x+\delta} |f(t)|dt + \int_{1-\delta}^{1} |f(t)|dt$$

yields the inequality

$$2C_1\pi \int_{\Delta_2} |f(t)|dt < \frac{\varepsilon}{2}.$$

Let r_{ml} be the radius of the circle C_{ml}. Then the relation

$$\int_{C_{ml}} e^{-\alpha_0|\rho_0|\delta}|\frac{d\rho}{\rho}| = \int_0^{\frac{\pi}{4}} e^{-\alpha_0\delta\rho_{ml}|\sin t|}dt + \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} e^{-\alpha_0\delta\rho_{ml}|\cos t|}dt + \int_{\frac{3\pi}{4}}^{\frac{5\pi}{4}} e^{-\alpha_0\delta\rho_{ml}|\sin t|}dt$$

$$+ \int_{\frac{5\pi}{4}}^{\frac{7\pi}{4}} e^{-\alpha_0\delta\rho_{ml}|\cos t|}dt + \int_{\frac{7\pi}{4}}^{2\pi} e^{-\alpha_0\delta\rho_{ml}|\sin t|}dt$$

gives the inequality

$$\int_{C_{ml}} e^{-\alpha_0|\rho_0|\delta}\left|\frac{d\rho}{\rho}\right| < \frac{C_2}{|r_{ml}|\delta}.$$

With sufficiently large value of m , the first term in (26) can be made less than $\frac{\varepsilon}{2}$. The Theorem 1 is proved. □

From Theorem 1 we derive the following assertion.

Theorem 2. *Suppose all assumptions of Theorem 1 hold true. Then the system of eigenfunctions of the operator L forms the basis in $L_2(-1,1)$.*

Proof of Theorem 2. To prove Theorem 2 it suffices to show that $\|f - S_m\|_{L_2} < \varepsilon$ for any function $f(x) \in L_2(-1,1)$. But it easily follows from the basicity of the system of eigenfunctions for operator L_0 and the equiconvergence result of Theorem 1. The Theorem 2 is proved. □

There arises the question of whether this basis is also unconditional basis. The answer to this question is given by the following theorems.

Theorem 3. *Suppose all assumptions of Theorem 1 hold true. If the coefficient $q(x) \in L_1(-1,1)$ in (3) is the real-valued function, then the system of eigenfunctions of the operator L forms orthonormal basis in $L_2(-1,1)$.*

Since the operator L is self-adjoint, Theorem 3 follows directly from Theorem 2.

Theorem 4. *Suppose all assumptions of Theorem 1 hold true. Then the system of eigenfunctions for the operator L forms the Riesz basis in $L_2(-1,1)$.*

From the estimate for the Green's function $G_q(x,t,\lambda)$ of the operator L (see proof of Theorem 1), results of Theorem 2, and the theorems in [11] Theorem 4 follows.

Remark 1. *Note that in the case $\alpha = 0$ the question of the basis property of eigenfunctions of the classical periodic problem is open problem.*

4. Conclusions

Summarizing the investigation carried out, we note that the Green's function of the second order differential operator with involution was constructed. Theorems on the basicity of eigenfunctions to the problems under consideration were proven. These theorems might be useful in the theory of solvability mixed problems for partial differential equations with involution. In the future, we plan to investigate these operators with the general boundary conditions.

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