Article

Higher regularity, inverse and polyadic semigroups

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In this note we generalize the regularity concept for semigroups in two ways simultaneously: to higher regularity and to higher arity. We show that the one-relational and multi-relational formulations of higher regularity do not coincide, and each element has several inverses. The higher idempotents are introduced, and their commutation leads to unique inverses in the multi-relational formulation, and then further to the higher inverse semigroups. For polyadic semigroups we introduce several types of higher regularity which satisfy the arity invariance principle as introduced: the expressions should not depend of the numerical arity values, which allows us to provide natural and correct binary limits. In the first definition no idempotents can be defined, analogously to the binary semigroups, and therefore the uniqueness of inverses can be governed by shifts. In the second definition called sandwich higher regularity, we are able to introduce the higher polyadic idempotents, but their commutation does not provide uniqueness of inverses, because of the middle terms in the higher polyadic regularity conditions.

1. Introduction

The concept of regularity was introduced in [1] and then widely used in the construction of regular and inverse semigroups (see, e.g., [2–5] and [6,7], and refs therein).

In this note we propose to generalize the concept of regularity for semigroups in two different aspects simultaneously: 1) higher regularity, which can be informally interpreted that each element has several inverse elements; 2) higher arity, which extends the binary multiplication to that of arbitrary arity, i.e. the consideration of polyadic semigroups.

The higher regularity concept was introduced in semisupermanifold theory [8] for generalized transition functions, which then gave rise to the development of a new kind of so called regular obstructed category [9] and to their application to Topological Quantum Field Theory [10],the Yang-Baxter equation [11] and statistics with a doubly regular R-matrix [12]. Moreover, it was shown that the higher regular semigroups naturally appear in the framework of the polyadic-binary correspondence principle [13]. Polyadic semigroups were introduced in [14], and regular and inverse polyadic semigroups were investigated in [15–17].

Here we will show that even for binary semigroups the one-relational and multi-relational formulations of higher regularity are different. The higher regular idempotents can be introduced, and their commuting leads to the higher inverse semigroups. In the case of polyadic semigroups, several definitions of regularity and higher regularity can be introduced. We do not apply reduction in the number of multiplications as in [17] which can be done in the one-relational approach only, but in this way we also do not receive polyadic idempotents. Therefore the exchange of the commuting of idempotents by commuting of shifts [17] is still valid.

To get polyadic idempotents (by analogy with the ordinary regularity for semigroups) we introduce the so called sandwich regularity for polyadic semigroups (which differs from [15]). In trying to connect the commutation of idempotents with uniqueness of inverse elements (as in the standard regularity) in order to obtain inverse polyadic semigroups, we observe that this is prevented by the middle elements in the sandwich polyadic regularity

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conditions. Thus, further investigations are needed to develop the higher regular and inverse polyadic semigroups.

2. Generalized n-regular elements in semigroups

Here we formulate some basic notions in terms of tuples and the "quantity/number of multiplications", in such a way that they can be naturally generalized from the binary to polyadic case [18].

2.1. Binary n-regular single elements

Indeed, if S is a set, its ℓth Cartesian product $S^{\times\ell}$ consists of all ℓ-tuples $\mathfrak{x}^{(\ell)} = (x_1, x_2, \ldots, x_\ell)$, $x_i \in S$ (we omit powers (ℓ) in the obvious cases $\mathfrak{x}^{(\ell)} \equiv \mathfrak{x}$, $\ell \in \mathbb{N}$). Let $\mathcal{S}_2 \equiv \mathcal{S}_{k=2} = \langle S \mid \mu_2, assoc_2 \rangle$ be a binary (having arity $k = 2$) semigroup with the underlying set S, and the (binary $k = 2$) multiplication $\mu_2 \equiv \mu_{k=2} : S \times S \to S$ (sometimes we will denote $\mu_2[x_1, x_2] = x_1 x_2$). The (binary) associativity $assoc_2 : (x_1 x_2) x_3 = x_1 (x_2 x_3)$, $\forall x_i \in S$, allows us to omit brackets, and in the language of tuples $x_1 x_2 \ldots x_\ell = \mu_2^{[\ell-1]}\left[\mathfrak{x}^{(\ell)}\right]$ $(= \boldsymbol{x}^{(\ell)} \equiv \boldsymbol{x}$, if the number of elements is not important or evident, and also distinguish between the product $\boldsymbol{x}^{(\ell)}$ of ℓ elements and the ℓ-tuples $\mathfrak{x}^{(\ell)}$), i.e. the product of ℓ elements contains $\ell - 1$ (binary) multiplications (whereas in the polyadic case this is different [19]). If all elements in a tuple coincide, we write $x^\ell = \mu_2^{[\ell-1]}\left[x^{(\ell)}\right]$ (this form will be important in the polyadic generalization below). An element $x_i \in S$ satisfying $x^\ell = x$ (or $\mu_2^{[\ell-1]}\left[x^{(\ell)}\right] = x$) is called (binary) ℓ-idempotent.

An element x of the semigroup $\mathcal{S}_2$ is called (von Neumann) regular, if the equation

$$xyx = x, \quad \text{or} \tag{2.1}$$

$$\mu_{k=2}^{[\ell-1=2]}[x, y, x] = x, \quad x, y \in S, \tag{2.2}$$

has a solution y in $\mathcal{S}_2$, which need not be unique. An element y is called a regular element conjugated to x [3] (also, an inverse element [2]). The set of the regular elements conjugated to x is denoted by V_x [4].

In search of generalizing the regularity condition (2.1) in $\mathcal{S}_2$ for the fixed element $x \in S$, we arrive at the following possibilities:

1. *Higher regularity generalization*: instead of one element $y \in S$, use an n-tuple $\mathfrak{y}^{(n)} \in S^{\times n}$ and the corresponding product $\boldsymbol{y}^{(n)}$.
2. *Higher arity generalization*: instead of the binary product μ_2, consider the polyadic (or k-ary semigroup) product with the multiplication μ_k.

It follows from the generalization **(i)** that we can write (for the binary case)

Definition 1. *An element x of the (binary) semigroup $\mathcal{S}_2$ is called higher n-regular, if there exists (at least one, not necessarily unique) $(n-1)$-element solution $\mathfrak{y}^{(n-1)}$ of*

$$x\boldsymbol{y}^{(n-1)}x = x, \quad \textit{or} \tag{2.3}$$

$$\mu_{k=2}^{[n]}\left[x, \mathfrak{y}^{(n-1)}, x\right] = x, \quad x, y_1, \ldots, y_{n-1} \in S. \tag{2.4}$$

Definition 2. *The tuple $\mathfrak{y}^{(n-1)}$ from (2.4) is called higher n-regular conjugated to x, or $\mathfrak{y}^{(n-1)}$ is an n-inverse (tuple) for x.*

Without any additional conditions, for instance splitting the tuple $\mathfrak{y}^{(n-1)}$ for some reason (as below), the definition (2.3) reduces to the ordinary regularity (2.1), because the binary semigroup is closed with respect to the multiplication of any number of elements, and so there exists $z \in S$, such that $\boldsymbol{y}^{(n-1)} = z$ for any $y_1, \ldots, y_{n-1} \in S$. Therefore, we have the following reduction

Assertion 1. *Any higher n-regular element in a binary semigroup is 2-regular (or regular in the ordinary sense (2.1)).*

Indeed, if we consider the higher 3-regularity condition $x_1 x_2 x_3 x_1 = x_1$ for a single element x_1, and $x_2, x_3 \in S$, it obviously coincides with the ordinary regularity condition for the single element x (2.1). However, if we cycle both the regularity conditions, they will not necessarily coincide, and new structures can appear. Therefore, using several different mutually consistent n-regularity conditions (2.3) we will be able to construct a corresponding (binary) semigroup which will not be reduced to an ordinary regular semigroup (see below).

2.2. Polyadic n-regular single elements

Following the higher arity generalization **(ii)**, we introduce higher n-regular single elements in polyadic (k-ary) semigroups and show that the definition of regularity for higher arities cannot be done similarly to ordinary regularity, but requires a polyadic analog of n-regularity for a single element of a polyadic semigroup. Before proceeding we introduce

Definition 3 (Arity invariance principle)**.** *In an algebraic universal system with operations of different arities the general form of expressions should not depend on numerical arity values.*

This gives the following prescription for how to generalize expressions from the binary shape to a polyadic shape:

1. Write down an expression using the operations manifestly.
2. Change arities from the binary values to the needed higher values.
3. Take into account the corresponding changes of tuple lengths according to the concrete argument numbers of operations.

Example 1. *If on a set S one defines a binary operation (multiplication) $\mu_2 : S \times S \to S$, then one (left) neutral element $e \in S$ for $x \in S$ is defined by $\mu_2[e, x] = x$ (usually $ex = x$). But for 4-ary operation $\mu_4 : S^{\times 4} \to S$ it becomes $\mu_4[e_1, e_2, e_3, x] = x$, where $e_1, e_2, e_3 \in S$, so we can have a neutral sequence $\mathfrak{e} = \mathfrak{e}^{(3)} = (e_1, e_2, e_3)$ (a tuple $\mathfrak{e}$ of the length 3), and only in the particular case $e_1 = e_2 = e_3 = e$, do we obtain one neutral element e. Obviously, the composition of two ($n = 2$) 4-ary ($k = 4$) operations, denoted by $\mu_4^{[2]} \equiv \mu_{k=4}^{[\ell-1=2]}$ (as in (2.2)) should act on a word of a length 7, since $\mu_4[\mu_4[x_1, x_2, x_3, x_4], x_5, x_6, x_7]$, $x_i \in S$.*

In general, if a polyadic system $\langle S \mid \mu_k \rangle$ has only one operation (k-ary multiplication), the length of words is "quantized"

$$L_k^{(\ell)} = \ell(k-1) + 1, \tag{2.5}$$

where ℓ is a "quantity of multiplications".

Let $\mathcal{S}_k = \langle S \mid \mu_k, assoc_k \rangle$ be a *polyadic* (k-*ary*) *semigroup* with the underlying set S, and the k-ary multiplication $\mu_k : S^{\times k} \to S$ [19]. The k-ary (*total*) *associativity* $assoc_k$ can be treated as invariance of two k-ary product composition $\mu_k^{[2]}$ with respect to the brackets [18]

$$assoc_k : \mu_k^{[2]}[\mathfrak{x}, \mathfrak{y}, \mathfrak{z}] = \mu_k[\mathfrak{x}, \mu_k[\mathfrak{y}], \mathfrak{z}] = invariant, \quad \forall x_i, y_i, z_i \in S, \tag{2.6}$$

for all placements of the inner multiplication μ_k in the r.h.s. of (2.6) with the fixed ordering of $(2k-1)$ elements (see (2.5)), and $\mathfrak{x}, \mathfrak{y}, \mathfrak{z}$ are tuples of the needed length. The total associativity (2.6) allows us to write a product containing ℓ multiplications using external brackets only $\mu_k^{[\ell]}[\mathfrak{x}]$ (the *long product* [20]), where $\mathfrak{x}$ is a tuple of length $\ell(k-1)+1$, because of (2.5). If all elements of the tuple $\mathfrak{x} = \mathfrak{x}^{(\ell)}$ are the same we write it as $x^{(\ell)}$. A polyadic power (reflecting the "number of multiplications", but not of the number of elements, as in the binary case) becomes

$$x^{\langle \ell \rangle} = \mu_k^{[\ell]} \left[x^{\ell(k-1)+1} \right], \tag{2.7}$$

because of (2.5). So the ordinary (binary) power p (as a number of elements in a product x^p, which is different from the tuple $x^{(\ell)}$) is $p = \ell + 1$. An element x from a polyadic semigroup $\mathcal{S}_k$ is called *polyadic* $\langle \ell \rangle$*-idempotent*, if $x^{\langle \ell \rangle} = x$, and it is *polyadic idempotent*, if [19]

$$x^{\langle \ell = 1 \rangle} \equiv \mu_k \left[x^k \right] = x. \tag{2.8}$$

Recall also that a *left polyadic identity* $e \in S$ is defined by

$$\mu_k \left[e^{(k-1)}, x \right] = x, \quad \forall x \in S, \tag{2.9}$$

and e is a *polyadic identity*, if x can be on any place. If $e = e(x)$ depends on x, then we call it a *local polyadic identity* (see [21] for the binary case). It follows from (2.9) that all polyadic identities (or neutral elements) are idempotents.

An element x of the polyadic semigroup $\mathcal{S}_k$ is called *regular* [17], if the equation

$$\mu_k^{[2]} \left[x, \mathfrak{y}^{(2k-3)}, x \right] = x, \quad x \in S, \tag{2.10}$$

has $2k-3$ solutions $y_1, \ldots, y_{2k-3} \in S$, which need not be unique (since $L_k^{(2)} - 2 = 2k - 3$, see (2.5)). If we have only the relation (2.10), then polyadic associativity allows us to apply one (internal) polyadic multiplication and remove $k-1$ elements to obtain [17]

$$\mu_k \left[x, \mathfrak{z}^{(k-2)}, x \right] = x, \quad x \in S, \tag{2.11}$$

and we call (2.11) the *reduced polyadic regularity* of a single element, while (2.10) will be called the *full regularity* of a single element. Thus, for a single element $x \in S$, by analogy with **Assertion 1**, we have

Assertion 2. *In a polyadic semigroup $\mathcal{S}_k$, if any single element $x \in S$ satisfies regularity (2.10), it also satisfies reduced regularity (2.11).*

From (2.11) it follows

Corollary 1. *Reduced regularity exists, if and only if the arity of multiplication has $k > 2$.*

Example 2. *In the minimal ternary case $k = 3$ we have the regularity $\mu_3^{[2]}[x_1, x_2, x_3, x_4, x_1] = x_1, \quad x_i \in S$, and putting $z = \mu_3[x_2, x_3, x_4]$ gives the reduced regularity*

$$\mu_3[x_1, z, x_1] = x_1. \tag{2.12}$$

But these regularities will give different cycles: one is of length 5, and the other is of length 3.

We now construct a polyadic analog of the higher n-regularity condition (2.4) for a single element.

Definition 4. *An element x of a polyadic semigroup $\mathcal{S}_k$ is called full higher n-regular, if there exists (at least one, not necessarily unique) $(n(k-1)-1)$-element solution (tuple) $\mathfrak{y}^{(n(k-1)-1)}$ of*

$$\mu_k^{[n]} \left[x, \mathfrak{y}^{(n(k-1)-1)}, x \right] = x, \quad x, y_1, \ldots, y_{n(k-1)-1} \in S, \quad n \geq 2. \tag{2.13}$$

Definition 5. *The tuple $\mathfrak{y}^{(n(k-1)-1)}$ from (2.13) is called higher n-regular k-ary conjugated to x,* *or* $\mathfrak{y}^{(n(k-1)-1)}$ *is n-inverse k-ary tuple for x of the length* $(n(k-1)-1)$.

Example 3. *In the full higher* 3*-regular ternary semigroup* $\mathcal{S}_3 = \langle S \mid \mu_3 \rangle$ *we have for each element* $x \in S$ *the condition*

$$\mu_3^{[2]}[x, y_1, y_2, y_3, y_4, y_5, x] = x, \tag{2.14}$$

and so each element $x \in S$ *has a* 3*-inverse ternary* 5*-tuple* $\mathfrak{y}^{(5)} = (y_1, y_2, y_3, y_4, y_5)$, *or the* 5*-tuple* $\mathfrak{y}^{(5)}$ *is the higher* 3*-regular ternary conjugated to x.*

Comparing (2.13) with the full regularity condition (2.10) and its reduction (2.11), we observe that the full n-regularity condition can be reduced several times to get different versions of reduced regularity for a single element.

Definition 6. *An element x of a polyadic semigroup* $\mathcal{S}_k$ *is called m-reduced higher n-regular, if there exists a (not necessarily unique)* $((n-m)(k-1)-1)$*-element solution (tuple)* $\mathfrak{y}^{((n-m)(k-1)-1)}$ *of*

$$\mu_k^{[n]}\left[x, \mathfrak{y}^{((n-m)(k-1)-1)}, x\right] = x, \quad x, y_1, \ldots, y_{(n-m)(k-1)-1} \in S, \;\; 1 \leq m \leq n-1, \;\; n \geq 2. \tag{2.15}$$

Example 4. *In the ternary case* $k = 3$, *we have (for a single element* $x_1 \in S$*) the higher* 3*-regularity condition*

$$\mu_3^{[3]}[x_1, x_2, x_3, x_4, x_5, x_6, x_1] = x_1, \quad x_i \in S. \tag{2.16}$$

We can reduce this relation twice, for instance, as follows: put $t = \mu_3[x_2, x_3, x_4]$ *to get*

$$\mu_3^{[2]}[x_1, t, x_5, x_6, x_1] = x_1, \tag{2.17}$$

and then $z = \mu_3[t, x_5, x_6]$ *and obtain*

$$\mu_3[x_1, z, x_1] = x_1, \quad t, z \in S, \tag{2.18}$$

which coincides with (2.12). Again we observe that (2.16), (2.17) and (2.18) give different cycles of the length 7, 5 *and* 3*, respectively.*

3. Higher n-inverse semigroups

Here we introduce and study the higher n-regular and n-inverse (binary) semigroups. We recall the standard definitions to establish notations (see, e.g., [2,3,5]).

There are two different definitions of a regular semigroup:

1. *One-relation definition* [22].
2. *Multi-relation definition* [23,24].

According to the one-relation definition **(i)**: A semigroup $\mathcal{S} \equiv \mathcal{S}_2 = \langle S \mid \mu_2, assoc_2 \rangle$ is called regular if each element $x \in S$ has its regular conjugated element $y \in S$ defined by $xyx = x$ (2.1), and y need not be unique [22].

Elements x_1, x_2 of $\mathcal{S}$ are called inverse to each other (regular conjugated, mutually regular), if

$$x_1 x_2 x_1 = x_1, \quad x_2 x_1 x_2 = x_2, \tag{3.1}$$

$$\mu_2^{[2]}[x_1, x_2, x_1] = x_1, \quad \mu_2^{[2]}[x_2, x_1, x_2] = x_2, \quad x_1, x_2 \in S. \tag{3.2}$$

According to the multi-relation definition **(ii)**: A semigroup is called regular, if each element x_1 has its inverse x_2 (3.1), and x_2 can be not unique.

For a binary semigroup $\mathcal{S}_2$ and ordinary 2-regularity these definitions are equivalent [2,25]. Indeed, if x_1 has some regular conjugated element $y \in S$, and $x_1 y x_1 = x_1$, then choosing $x_2 = y x_1 y$, we immediately obtain (3.1).

Definition 7. *A 2-tuple $\mathfrak{X}^{(2)} = (x_1, x_2) \in S \times S$ is called 2-regular sequence of inverses, if it satisfies (3.1).*

Thus we can define regular semigroups in terms of regular sequences of inverses.

Definition 8. *A binary semigroup is called a regular (or 2-regular) semigroup, if each element belongs to a (not necessary unique) 2-regular sequence of inverses $\mathfrak{X}^{(2)} \in S \times S$.*

3.1. Higher n-regular semigroups

Following [15], in a regular semigroup $\mathcal{S}_2$, for each $x_1 \in S$ there exists $x_2 \in S$ such that $x_1 x_2 x_1 = x_1$, and for $x_2 \in S$ there exists $y \in S$ satisfying and $x_2 y x_2 = x_2$. Denoting $x_3 = y x_2$, then $x_2 = x_2 (y x_2) = x_2 x_3$, and we observe that $x_1 x_2 x_3 x_1 = x_1$, i.e. in a regular semigroup, from 2-regularity there follows 3-regularity of $\mathcal{S}_2$ in the one-relation definition **(i)** (cf. for a single element **Assertion** 1). By analogy, in a regular semigroup, 2-regularity of an element implies its n-regularity in the one-relation definition **(i)**. Indeed, in this point there is a difference between the definitions **(i)** and **(ii)**.

Let us introduce higher analogs of the regular sequences (3.1) (see **Definition** 7).

Definition 9. *A n-tuple $\mathfrak{X}^{(n)} = (x_1, \ldots, x_n) \in S^{\times n}$ is called an (ordered) n-regular sequence of inverses, if it satisfies the n relations*

$$x_1 x_2 x_3 \ldots x_n x_1 = x_1, \tag{3.3}$$
$$x_2 x_3 \ldots x_n x_1 x_2 = x_2, \tag{3.4}$$
$$\vdots$$
$$x_n x_1 x_2 \ldots x_{n-1} x_n = x_n, \tag{3.5}$$

or in the manifest form of the binary multiplication μ_2 (which is needed for the higher arity generalization)

$$\mu_2^{[n]}[x_1, x_2, x_3, \ldots, x_n, x_1] = x_1, \tag{3.6}$$
$$\mu_2^{[n]}[x_2, x_3, \ldots, x_n, x_1, x_2] = x_2, \tag{3.7}$$
$$\vdots$$
$$\mu_2^{[n]}[x_n, x_1, x_2, \ldots, x_{n-1}, x_n] = x_n. \tag{3.8}$$

Denote for each element $x_i \in S$, $i = 1, \ldots, n$, its n-inverse $(n-1)$-tuple (see Definition 2) by $\bar{\mathfrak{x}}_i \in S^{\times(n-1)}$, where

$$\bar{\mathfrak{x}}_1 = (x_2, x_3, \ldots, x_{n-1}, x_n), \tag{3.9}$$
$$\bar{\mathfrak{x}}_2 = (x_3, x_4 \ldots, x_{n-1}, x_n, x_1), \tag{3.10}$$
$$\vdots$$
$$\bar{\mathfrak{x}}_n = (x_1, x_2, \ldots, x_{n-2}, x_{n-1}). \tag{3.11}$$

Then the definition of an n-regular sequence of inverses (3.3)–(3.8) will take the concise form (cf. (2.1) and (2.3))

$$x_i \bar{\mathfrak{x}}_i x_i = x_i, \tag{3.12}$$
$$\mu_2^{[n]}[x_i, \bar{\mathfrak{x}}_i, x_i] = x_i, \quad x_i \in S, \quad i = 1, \ldots, n. \tag{3.13}$$

Now by analogy with **Definition** 8 we have the following (multi-relation)

Definition 10. *A binary semigroup $\mathcal{S}_2$ is called a (cycled) n-regular semigroup, if each its element belongs to a (not necessary unique) n-regular sequence of inverses $\mathfrak{X}^{(n)} \in S^{\times n}$ satisfying (3.3)–(3.5).*

This definition incorporates the ordinary regular semigroups (3.1) by putting $n = 2$.

Remark 1. *The concept of n-regularity is strongly multi-relational, in that all relations in (3.3)–(3.5) or (3.12) should hold, which leads us to consider all the elements in the sequence of inverses on a par and cycled, analogously to the ordinary case (3.1). This can be expressed informally as "one element of a n-regular semigroup has $n-1$ inverses".*

Example 5. *In the 3-regularity case we have three relations defining the regularity*

$$x_1x_2x_3x_1 = x_1, \tag{3.14}$$
$$x_2x_3x_1x_2 = x_2, \tag{3.15}$$
$$x_3x_1x_2x_3 = x_3. \tag{3.16}$$

Now the sequence of inverses is $\mathfrak{X}^{(3)} = (x_1, x_2, x_3)$, and we can say in a symmetrical way: all of them are "mutually inverses one to another", iff all three relations (3.14)–(3.15) hold simultaneously. Also, using (3.9)–(3.11) we see that each element x_1, x_2, x_3 has its 3-inverse 2-tuple (or, informally, the pair of inverses)

$$\bar{\mathfrak{x}}_1 = (x_2, x_3), \tag{3.17}$$
$$\bar{\mathfrak{x}}_2 = (x_3, x_1), \tag{3.18}$$
$$\bar{\mathfrak{x}}_3 = (x_1, x_2). \tag{3.19}$$

Remark 2. *In what follows, we will use this 3-regularity example, when providing derivations and proofs, for clarity and conciseness. This is worthwhile, because the general n-regularity case does not differ from $n = 3$ in principle, and only forces us to consider cumbersome computations with many indices and variables without enhancing our understanding of the structures.*

Let us formulate the Thierrin theorem [25] in the n-regular setting, which connects one-element regularity and multi-element regularity.

Lemma 1. *Every n-regular element in a semigroup has its n-inverse tuple with $n-1$ elements.*

Proof. Let x_1 be a 3-regular element x_1 of a semigroup $\mathcal{S}_2$, then we can write

$$x_1 = x_1y_2y_3x_1, \qquad x_1, y_2, y_3 \in S. \tag{3.20}$$

We are to find elements x_2, x_3 which satisfy (3.14)–(3.16). Put

$$x_2 = y_2y_3x_1y_2, \tag{3.21}$$
$$x_3 = y_3x_1y_2y_3. \tag{3.22}$$

Then for the l.h.s. of (3.14)–(3.16) we derive

$$
\begin{aligned}
x_1x_2x_3x_1 &= x_1y_2y_3(x_1y_2y_3x_1)y_2y_3x_1 = x_1y_2y_3(x_1y_2y_3x_1) \\
&= x_1y_2y_3x_1 = x_1, \qquad (3.23)\\
x_2x_3x_1x_2 &= y_2(y_3x_1y_2y_3)(x_1y_2y_3x_1)y_2y_3(x_1y_2y_3x_1)y_2 \\
&= y_2x_3(x_1x_2y_3x_1)y_2 = y_2x_3x_1y_2 = x_2, \qquad (3.24)\\
x_3x_1x_2x_3 &= y_3x_1y_2y_3(x_1y_2y_3x_1)y_2y_3(x_1y_2y_3x_1)y_2y_3 \\
&= y_3(x_1y_2y_3x_1)y_2y_3 = y_3x_1y_2y_3 = x_3. \qquad (3.25)
\end{aligned}
$$

Therefore, if we have any 3-regular element (3.20), the relations (3.14)-(3.19) hold. □

We now show that in semigroups the higher n-regularity (in the multi-relation formulation) is wider than for 2-regularity.

Lemma 2. *If a semigroup $\mathcal{S}_2$ is n-regular, such that the n relations (3.12) are valid, then $\mathcal{S}_2$ is 2-regular as well.*

Proof. In the case $n = 3$ we use (3.14)-(3.16) and denote $x_2x_3 = y \in S$. Then $x_1yx_1 = x_1$, and multiplying (3.15) and (3.16) by x_3 from the right and x_2 from the left, respectively, we obtain

$$(x_2x_3x_1x_2)x_3 = (x_2)x_3 \Longrightarrow (x_2x_3)x_1(x_2x_3) = (x_2x_3) \Longrightarrow yx_1y = y, \qquad (3.26)$$
$$x_2(x_3x_1x_2x_3) = x_2(x_3) \Longrightarrow (x_2x_3)x_1(x_2x_3) = (x_2x_3) \Longrightarrow yx_1y = y. \qquad (3.27)$$

This means that x_1 and y are mutually 2-inverse (or 2-regularly conjugated), and so $\mathcal{S}_2$ is 2-regular. □

The converse requires some conditions on the semigroup.

Theorem 1. *The following statements are equivalent for a regular (2-regular) semigroup $\mathcal{S}_2$:*

1. The 2-regular semigroup $\mathcal{S}_2$ is n-regular.
2. The regular semigroup $\mathcal{S}_2$ is cancellative.

Proof. **(i)⇒(ii)** Suppose $\mathcal{S}_2$ is 2-regular, which means that there are tuples satisfying (3.1), or in our notation here for each $x_1 \in S$ there exists $y \in S$ such that $x_1yx_1 = x_1$ and $yx_1y = y$. Let y be presented as a product of two elements $y = x_2x_3$, $x_2, x_3 \in S$, which is possible since the underlying set S of the semigroup is closed with respect to multiplication. After substitution y into the 2-regularity conditions, we obtain

$$x_1yx_1 = x_1 \Longrightarrow x_1(x_2x_3)x_1 = x_1 \Longrightarrow x_1x_2x_3x_1 = x_1, \qquad (3.28)$$
$$yx_1y = y \Longrightarrow (x_2x_3)x_1(x_2x_3) = (x_2x_3) \Longrightarrow (x_2x_3x_1x_2)x_3 = (x_2)x_3 \qquad (3.29)$$
$$\Longrightarrow x_2(x_3x_1x_2x_3) = x_2(x_3). \qquad (3.30)$$

We observe that the first line coincides with (3.14), but obtaining (3.15) and (3.16) from the second and third lines here requires right and left cancellativity, respectively.

(ii)⇒(i) After applying cancellativity to (3.29) and (3.30), then equating expressions in brackets one obtains (3.15) and (3.16), correspondingly. The first line (3.28) coincides with (3.14) in any case. □

It is known that a cancellative regular semigroup is a group [2,7].

Corollary 2. *If a regular semigroup $\mathcal{S}_2$ is n-regular, it is a group.*

3.2. Idempotents and higher n-inverse semigroups

Lemma 3. *In an n-regular semigroup there are at least n (binary higher) idempotents (no summation)*

$$e_i = x_i \bar{x}_i, \quad i = 1, \dots, n. \tag{3.31}$$

Proof. Multiply (3.12) by $\bar{x}_i$ from the right to get $x_i \bar{x}_i x_i \bar{x}_i = x_i \bar{x}_i$. □

In the case $n = 3$ the (higher) idempotents (3.31) become

$$e_1 = x_1 x_2 x_3, \;\; e_2 = x_2 x_3 x_1, \;\; e_3 = x_3 x_1 x_2, \tag{3.32}$$

having the following "chain" commutation relations with elements

$$e_1 x_1 = x_1 e_2 = x_1, \tag{3.33}$$
$$e_2 x_2 = x_2 e_3 = x_2, \tag{3.34}$$
$$e_3 x_3 = x_3 e_1 = x_3. \tag{3.35}$$

Corollary 3. *Each higher idempotent e_i is a left unit (neutral element) for x_i and a right unit (neutral element) for x_{i-1}.*

Lemma 4. *If in an n-regular semigroup (higher) idempotents commute, then each element has a unique n-regular sequence of inverses.*

Proof. Suppose in a 3-regular semigroup defined by (3.14)–(3.16) the element x_1 has two pairs of inverses (3.9) $\bar{\mathfrak{x}}_1 = (x_2, x_3)$ and $\bar{\mathfrak{x}}'_1 = (x'_2, x_3)$. Then, in addition to (3.14)–(3.16) and the triple of idempotents (3.32) we have

$$x_1 x'_2 x_3 x_1 = x_1, \tag{3.36}$$
$$x'_2 x_3 x_1 x'_2 = x'_2, \tag{3.37}$$
$$x_3 x_1 x'_2 x_3 = x_3. \tag{3.38}$$

and

$$e'_1 = x_1 x'_2 x_3, \;\; e'_2 = x'_2 x_3 x_1, \;\; e'_3 = x_3 x_1 x'_2, \tag{3.39}$$

and also the idempotents (3.32) and (3.39) commute. We derive

$$\begin{aligned} x_2 &= x_2 x_3 x_1 x_2 = x_2 x_3 (x_1 x'_2 x_3 x_1) x_1 x_2 = x_2 x_3 (x_1 x'_2 x_3 (x_1 x'_2 x_3 x_1)) x_2 \\ &= x_2 x_3 x_1 x'_2 (x_3 x_1 x'_2)(x_3 x_1 x_2) = x_2 x_3 x_1 x'_2 (x_3 x_1 x_2)(x_3 x_1 x'_2) \\ &= (x_2 x_3 x_1)(x'_2 x_3 x_1) x_2 x_3 x_1 x'_2 = (x'_2 x_3 x_1)(x_2 x_3 x_1) x_2 x_3 x_1 x'_2 \\ &= x'_2 x_3 (x_1 x_2 x_3 x_1) x_2 x_3 x_1 x'_2 = x'_2 x_3 (x_1 x_2 x_3 x_1) x'_2 = x'_2 x_3 x_1 x'_2 = x'_2. \end{aligned} \tag{3.40}$$

A similar derivation can be done for $\bar{\mathfrak{x}}''_1 = (x_2, x''_3)$ and other indices. □

Definition 11. *A higher n-regular semigroup is called a higher n-inverse semigroup, if each element $x_i \in S$ has its unique n-inverse $(n-1)$-tuple $\bar{\mathfrak{x}}_i \in S^{\times(n-1)}$ (3.9)–(3.11).*

Thus, as in the regular (multi-relation 2-regular) semigroups [26–28], we arrive at a higher regular generalization of the characterization of the inverse semigroups by

Theorem 2. *A higher n-regular semigroup is an inverse semigroup, if their idempotents commute.*

Further properties of n-inverse semigroups will be studied elsewhere using the standard methods (see, e.g. [2,6,7,29]).

4. Higher n-inverse polyadic semigroups

We now introduce a polyadic version of the higher n-regular and n-inverse semigroups defined in the previous section. For this we will use the relations with the manifest form of the binary multiplication to follow **(ii)-(iii)** of the arity invariance principle (see **Definition** 3).

4.1. Higher n-regular polyadic semigroups

First, define the polyadic version of the higher n-regular sequences (3.6)–(3.8) in the framework of the arity invariance principle by substituting $\mu_2^{[n]} \mapsto \mu_k^{[n]}$ and then changing sizes of tuples, and generalizing to higher n (2.10).

Definition 12. *In a polyadic (k-ary) semigroup $\mathcal{S}_k = \langle S \mid \mu_k \rangle$ (see* ***Subsection*** *2.2) a $n(k-1)$-tuple $\mathfrak{X}^{(n(k-1))} = \left(x_1, \ldots, x_{n(k-1)}\right) \in S^{\times n(k-1)}$ is called a polyadic n-regular sequence of inverses, if it satisfies the $n(k-1)$ relations*

$$\mu_k^{[n]}\left[x_1, x_2, x_3, \ldots, x_{n(k-1)}, x_1\right] = x_1, \tag{4.1}$$

$$\mu_k^{[n]}\left[x_2, x_3, \ldots, x_{n(k-1)}, x_1, x_2\right] = x_2, \tag{4.2}$$

$$\vdots$$

$$\mu_k^{[n]}\left[x_{n(k-1)}, x_1, x_2, \ldots, x_{n(k-1)-1}, x_{n(k-1)}\right] = x_{n(k-1)}. \tag{4.3}$$

Definition 13. *A polyadic semigroup $\mathcal{S}_k$ is called an n-regular semigroup, if each element belongs to a (not necessary unique) n-regular sequence of inverses $\mathfrak{X}^{(n(k-1))} \in S^{\times n(k-1)}$.*

In a polyadic n-regular semigroup, each element instead of having one inverse (regular conjugated) element will have a tuple, and we now, instead of (2.1)–(2.2), have

Definition 14. *In a polyadic (k-ary) n-regular semigroup each element $x_i \in S$, where $i = 1, \ldots, n(k-1)$, has a polyadic n-inverse, as the $(n(k-1)-1)$-tuple $\bar{\mathfrak{x}}_i^{[k]} \in S^{\times(n(k-1)-1)}$ (cf. the binary case* ***Definition*** *2 and (3.9)–(3.11))*

$$\bar{\mathfrak{x}}_1^{[k]} = \left(x_2, x_3, \ldots, x_{n(k-1)-1}, x_{n(k-1)}\right), \tag{4.4}$$

$$\bar{\mathfrak{x}}_2^{[k]} = \left(x_3, x_4 \ldots, x_{n(k-1)-1}, x_{n(k-1)}, x_1\right), \tag{4.5}$$

$$\vdots$$

$$\bar{\mathfrak{x}}_{n(k-1)}^{[k]} = \left(x_1, x_2, \ldots, x_{n(k-1)-2}, x_{n(k-1)-1}\right). \tag{4.6}$$

In this notation the polyadic (k-ary) n-regular sequence of inverses (4.1)–(4.3) can be written in the customary concise form (see (3.13) for a binary semigroup)

$$\mu_k^{[n]}\left[x_i, \bar{\mathfrak{x}}_i^{[k]}, x_i\right] = x_i, \quad x_i \in S, \quad i = 1, \ldots, n(k-1). \tag{4.7}$$

Note that at first sight one could consider here also the procedure of reducing the number of multiplications from any number to one [17], as in the single element regularity case (see **Assertion 2** and **Example 2**, **4**). However, if we consider the whole polyadic n-regular sequence of inverses (4.1)–(4.3) consisting of $n(k-1)$ relations (4.1)–(4.3), it will be not possible in general to reduce arity in all of them simultaneously.

Example 6. *If we consider the ordinary regularity (2-regularity) for a ternary semigroup $\mathcal{S}_3$, we obtain the ternary regular sequence of $n(k-1)=4$ elements and four regularity relations*

$$\mu_3^{[2]}[x_1, x_2, x_3, x_4, x_1] = x_1, \tag{4.8}$$
$$\mu_3^{[2]}[x_2, x_3, x_4, x_1, x_2] = x_2, \tag{4.9}$$
$$\mu_3^{[2]}[x_3, x_4, x_1, x_2, x_3] = x_3, \tag{4.10}$$
$$\mu_3^{[2]}[x_4, x_1, x_2, x_3, x_4] = x_4, \quad x_i \in S. \tag{4.11}$$

Elements x_1, x_2, x_3, x_4 have these ternary inverses (cf. the binary 3-regularity (3.17)–(3.19))

$$\bar{\mathfrak{x}}_1^{[3]} = (x_2, x_3, x_4), \tag{4.12}$$
$$\bar{\mathfrak{x}}_2^{[3]} = (x_3, x_4, x_1), \tag{4.13}$$
$$\bar{\mathfrak{x}}_3^{[3]} = (x_4, x_1, x_2), \tag{4.14}$$
$$\bar{\mathfrak{x}}_4^{[3]} = (x_1, x_2, x_3). \tag{4.15}$$

Remark 3. *This definition of regularity in ternary semigroups is in full agreement with the arity invariance principle (**Definition 3**): it has (the minimum) two (k-ary) multiplications as in the binary case (2.2). Although we can reduce the number of multiplications to one, as in (2.12) for a single relation, e.g. by the substitution $z = \mu_3[x_2, x_3, x_4]$, this cannot be done in all the cycled relations (4.8)–(4.11): the third relations (4.10) cannot be presented in terms of z as well as the right hand sides.*

Remark 4. *If we take the definition of regularity for a single relation, reduce it to one (k-ary) multiplication and then "artificially" cycle it (see [17] and references citing it), we find a conflict with the arity invariance principle. Indeed, in the ternary case we obtain $\mu_3[x, y, x] = x$, $\mu_3[y, x, y] = y$. Despite the similarity to the standard binary regularity (3.1)-(3.2), it contains only one multiplication, and therefore it could be better treated as a symmetry property of the ternary product μ_3, rather than as a relation between variables, e.g., as a regularity which needs at least two products. Also, the length of the sequence is two, as in the binary case, but it should be $n(k-1)=4$, as in the ternary 2-regular sequence (4.8)–(4.11) according to the arity invariance principle.*

Example 7. *For a 3-regular ternary semigroup $\mathcal{S}_3 = \langle S \mid \mu_3 \rangle$ with $n=3$ and $k=3$ we have a 3-regular sequence of 6 (mutual) inverses $\mathfrak{X}^{(6)} = (x_1, x_2, x_3, x_4, x_5, x_6) \in S^{\times 6}$, which satisfy the following six 3-regularity conditions*

$$\mu_3^{[3]}[x_1, x_2, x_3, x_4, x_5, x_6, x_1] = x_1, \tag{4.16}$$
$$\mu_3^{[3]}[x_2, x_3, x_4, x_5, x_6, x_1, x_2] = x_2, \tag{4.17}$$
$$\mu_3^{[3]}[x_3, x_4, x_5, x_6, x_1, x_2, x_3] = x_3, \tag{4.18}$$
$$\mu_3^{[3]}[x_4, x_5, x_6, x_1, x_2, x_3, x_4] = x_4, \tag{4.19}$$
$$\mu_3^{[3]}[x_5, x_6, x_1, x_2, x_3, x_4, x_5] = x_5, \tag{4.20}$$
$$\mu_3^{[3]}[x_6, x_1, x_2, x_3, x_4, x_5, x_6] = x_6, \quad x_i \in S. \tag{4.21}$$

*This is the first nontrivial case in both arity $k \neq 2$ and regularity $n \neq 2$ (see **Remark 2**).*

Remark 5. *In this example we can reduce the number of multiplications, as in **Example 4**, to one single relation, but not to all of them. For instance, we can put in the first relations (4.16) $t = \mu_3[x_2, x_3, x_4]$ and further $z = \mu_3[t, x_5, x_6]$, but, for instance, the third relation (4.18) cannot be presented in terms of t, z, as in the first one in (4.16)–(4.21), because of the splitting of variables in t, z, as well as in the right hand sides.*

The polyadic analog of the Thierrin theorem [25] in n-regular setting is given by:

Lemma 5. *Every n-regular element in a polyadic (k-ary) semigroup has a polyadic n-inverse tuple with $(n(k-1)-1)$ elements.*

Its proof literally repeats that of **Lemma 1**, but with different lengths of sequences. The same is true for the **Lemma 2** and **Theorem 1** by exchanging $\mathcal{S}_2 \to \mathcal{S}_k$.

Definition 15. *A higher n-regular polyadic semigroup $\mathcal{S}_k$ is called a higher n-inverse polyadic semigroup, if each element $x_i \in S$ has a unique n-inverse $(n(k-1)-1)$-tuple $\bar{\mathfrak{x}}_i \in S^{\times(n(k-1)-1)}$ (4.4)–(4.6).*

In searching for polyadic idempotents we observe that the binary regularity (3.13) and polyadic regularity (4.7) differ considerably in lengths of tuples. In the binary case any length is allowed, and one can define idempotents e_i by (3.31), as a left neutral element for x_i, such that (no summation) $e_i x_i = x_i$, $x_i \in \mathcal{S}_2$ (see (3.33)–(3.35)). At first sight, for polyadic n-regularity (4.7), we could proceed in a similar way. However we have

Proposition 1. *In the n-regular polyadic (k-ary) semigroup the length of the tuple $(x_i, \bar{\mathfrak{x}}_i)$ is allowed to give an idempotent, only if $k = 2$, i.e. the semigroup is binary.*

Proof. The allowed length of the tuple $(x_i, \bar{\mathfrak{x}}_i)$ (to give one element, an idempotent, see (2.5)), where $\bar{\mathfrak{x}}_i$ are in (4.4)–(4.6), is $\ell(k-1)+1$. While the tuple $(x_i, \bar{\mathfrak{x}}_i, x_i)$ in (4.7) has the given length $n(k-1)+1$, and we obtain the equation for ℓ as

$$\ell(k-1)+1 = n(k-1) \Longrightarrow \ell = n - \frac{1}{k-1} \tag{4.22}$$

The equation (4.22) has only one solution over $\mathbb{N}$ namely $\ell = n-1$, iff $k = 2$. □

This means that to go beyond the binary semigroups $k > 2$ and have idempotents, one needs to introduce a different regularity condition to (2.10) and (4.7), which we will do below.

4.2. Idempotents and sandwich polyadic regularity

Here we go in the opposite direction to that above: we will define the idempotents and then construct the needed regularity conditions using them, which in the limit $k = 2$ and $n = 2$ will give the ordinary binary regularity (3.1)–(3.2).

Let us formulate the binary higher n-regularity (3.13) in terms of the local polyadic identities (2.9). We write the idempotents (3.31) in the form

$$e_i = \mu_2[x_i, \bar{\mathfrak{x}}_i], \tag{4.23}$$

where $\bar{\mathfrak{x}}_i$ are n-inverses being $(n-1)$-tuples (3.9)–(3.11).

In terms of the idempotents e_i (4.23) the higher n-regularity conditions (3.13) become

$$\mu_2[e_i, x_i] = x_i. \tag{4.24}$$

Assertion 3. *The binary n-regularity conditions coincide with the definition of the local left identities.*

Proof. Compare (4.24) and the definition (2.9) with $k = 2$ (see [21]). □

Thus, the main idea is to generalize (using the arity invariance principle) to the polyadic case the binary n-regularity in the form (4.24) and also idempotents (4.23), but not (3.12)–(3.13), as it was done in (4.7).

By analogy with (4.23) let us introduce

$$e_i' = \mu_k\left[x_i, \overbrace{\bar{\mathfrak{x}}_i', \ldots, \bar{\mathfrak{x}}_i'}^{n-1}\right], \quad x_i \in S, \quad i = 1, \ldots, k, \tag{4.25}$$

where $\bar{\mathfrak{x}}_i'$ is the polyadic $(k-1)$-tuple for x_i (which differs from (4.4)–(4.6))

$$\bar{\mathfrak{x}}_1' = (x_2, x_3, \ldots, x_k), \tag{4.26}$$
$$\bar{\mathfrak{x}}_2' = (x_3, x_4, \ldots, x_k, x_1), \tag{4.27}$$
$$\vdots$$
$$\bar{\mathfrak{x}}_k' = (x_1, x_2, \ldots, x_{k-2}, x_{k-1}). \tag{4.28}$$

Definition 16. *Sandwich polyadic n-regularity conditions are defined as*

$$\mu_k\left[\overbrace{e_i' \ldots e_i'}^{k-1}, x_i\right] = x_i. \tag{4.29}$$

It follows form (4.29), that e_i' are polyadic idempotents (see (2.8) and (2.9)). It is seen that in the limit $k = 2$ and $n = 2$ (4.25)–(4.29) give the ordinary regularity (3.1).

Now, by analogy with the binary case **Assertion 3**, we have

Assertion 4. *The sandwich polyadic n-regularity conditions (4.29) coincide with the definition of the local left polyadic identities.*

Proof. Compare (4.29) and the definition (2.9). □

By analogy with **Definition 13** we have

Definition 17. *A k-ary semigroup $\mathcal{S}_k$ is called a sandwich polyadic n-regular semigroup, if each element has a (not necessary unique) k-tuple of inverses satisfying (4.25)–(4.29), or an element $x_i \in S$ has its local left polyadic identity e_i' (4.25).*

Now instead of (4.4)–(4.6) we have for the inverses

Definition 18. *Each element of a sandwich polyadic (k-ary) n-regular semigroup $x_i \in S$, $i = 1, \ldots, k-1$ has its (sandwich) polyadic k-inverse, as the $(k-1)$-tuple $\bar{\mathfrak{x}}_i'^{[k]} \in S^{\times(k-1)}$ (see (4.26)–(4.28)).*

Example 8. *In the ternary case $k = 3$ and $n = 2$ we have the following 3 ternary idempotents*

$$e_1' = \mu_3[x_1, x_2, x_3], \tag{4.30}$$
$$e_2' = \mu_3[x_2, x_3, x_1], \tag{4.31}$$
$$e_3' = \mu_3[x_3, x_1, x_2], \quad x_i \in S, \tag{4.32}$$

such that

$$\mu_3\left[e_i', e_i', e_i'\right] = e_i', \quad i = 1, 2, 3. \tag{4.33}$$

The sandwich ternary regularity conditions become (cf. (4.8)–(4.11))

$$\mu_3^{[3]}[x_1, x_2, x_3, x_1, x_2, x_3, x_1] = x_1, \tag{4.34}$$
$$\mu_3^{[3]}[x_2, x_3, x_1, x_2, x_3, x_1, x_2] = x_2, \tag{4.35}$$
$$\mu_3^{[3]}[x_3, x_1, x_2, x_3, x_1, x_2, x_3] = x_3. \tag{4.36}$$

It can be seen from (4.34)–(4.36) why we call such regularity "sandwich": each x_i appears in the l.h.s. not only 2 times, on the first and the last places, as in the previous definitions, but also in the middle, which gives the possibility for us to define idempotents. In general, the ith condition of sandwich regularity (4.29) will contain $k-2$ middle elements x_i.

Each of the elements $x_1, x_2, x_3 \in S$ has a 2-tuple of the ternary inverses (cf. the inverses for the binary 3-regularity (3.17)–(3.19) and the ternary regularity (4.12)–(4.15)) as

$$\bar{\mathfrak{x}}'_1 = (x_2, x_3), \tag{4.37}$$
$$\bar{\mathfrak{x}}'_2 = (x_3, x_1), \tag{4.38}$$
$$\bar{\mathfrak{x}}'_3 = (x_1, x_2). \tag{4.39}$$

Remark 6. *In (4.34) we can also reduce the number of multiplications to one (see [17]), but only in a single relation (see **Example 4** and Remark 5). However, in this case the whole system (4.34)–(4.36) will lose its self-consistency, since we are using the multi-relation definition of the sandwich regular semigroup (see (3.1) for the ordinary regularity).*

Example 9. *The non-trivial case in both higher arity $k \neq 2$ and higher sandwich regularity $n \neq 2$ is the sandwich 3-regular 4-ary semigroup $\mathcal{S}_4$ in which there exist 4 elements satisfying the higher sandwich 3-regularity relations*

$$\mu_4^{[4]}[x_1, x_2, x_3, x_4, x_1, x_2, x_3, x_4, x_1, x_2, x_3, x_4, x_1] = x_1, \tag{4.40}$$
$$\mu_4^{[4]}[x_2, x_3, x_4, x_1, x_2, x_3, x_4, x_1, x_2, x_3, x_4, x_1, x_2] = x_2, \tag{4.41}$$
$$\mu_4^{[4]}[x_3, x_4, x_1, x_2, x_3, x_4, x_1, x_2, x_3, x_4, x_1, x_2, x_3] = x_3, \tag{4.42}$$
$$\mu_4^{[4]}[x_4, x_1, x_2, x_3, x_4, x_1, x_2, x_3, x_4, x_1, x_2, x_3, x_4] = x_4, \quad x_i \in S. \tag{4.43}$$

The 4-ary idempotents are

$$e'_1 = \mu_4[x_1, x_2, x_3, x_4], \tag{4.44}$$
$$e'_2 = \mu_4[x_2, x_3, x_4, x_1],$$
$$e'_3 = \mu_4[x_3, x_4, x_1, x_2], \tag{4.45}$$
$$e'_3 = \mu_4[x_4, x_1, x_2, x_3], \tag{4.46}$$

and they obey the following commutation relations with elements (cf. (3.33)–(3.35))

$$\mu_4\left[e'_1, e'_1, e'_1, x_1\right] = \mu_4\left[x_1, e'_2, e'_2, e'_2\right] = x_1, \tag{4.47}$$
$$\mu_4\left[e'_2, e'_2, e'_2, x_2\right] = \mu_4\left[x_2, e'_3, e'_3, e'_3\right] = x_2, \tag{4.48}$$
$$\mu_4\left[e'_3, e'_3, e'_3, x_3\right] = \mu_4\left[x_3, e'_4, e'_4, e'_4\right] = x_3, \tag{4.49}$$
$$\mu_4\left[e'_4, e'_4, e'_4, x_4\right] = \mu_4\left[x_4, e'_1, e'_1, e'_1\right] = x_4. \tag{4.50}$$

Each of the elements x_1, x_2, x_3, x_4 has its triple of 4-ary inverses

$$\bar{\mathfrak{x}}'_1 = (x_2, x_3, x_4), \tag{4.51}$$
$$\bar{\mathfrak{x}}'_2 = (x_3, x_4, x_1), \tag{4.52}$$
$$\bar{\mathfrak{x}}'_3 = (x_4, x_1, x_2), \tag{4.53}$$
$$\bar{\mathfrak{x}}'_4 = (x_1, x_2, x_3). \tag{4.54}$$

By analogy with **Corollary 3** we now have

Corollary 4. *Each polyadic idempotent e'_i of the sandwich higher n-regular k-ary semigroup $\mathcal{S}_k$ is a local left polyadic identity (neutral element) for x_i and a local right polyadic identity for x_{i-1}.*

Definition 19. *A sandwich higher n-regular polyadic semigroup $\mathcal{S}_k$ is called a sandwich n-inverse polyadic semigroup, if each element $x_i \in S$ has a unique n-inverse $(k-1)$-tuple $\bar{\mathfrak{x}}'_i \in S^{\times(k-1)}$ (4.26)–(4.28).*

In search of a polyadic analog of the **Lemma 4**, we have found that the commutation of polyadic idempotents does not lead to uniqueness of the n-inverse elements (4.26)–(4.28). The problem appears because of the presence of the middle elements in (4.25) and, e.g., in (4.34) (see the discussion after (4.36)), while the middle elements do not exist in the previous formulations of regularity.

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