Article

The Language of "Rate of Change"

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Abstract: Language is an essential aspect of teaching and learning mathematics. It is necessary for communicating, transmission of concepts and ideas, and formation of meaning of mathematical concepts. In mathematics, besides symbols, which are usually common in different languages, words and expressions are used, which may invoke different concept images to students in various languages. Some words are used in mathematics and in everyday language with different meanings, while others are used only in mathematics or in mathematics and other disciplines in similar but non-identical ways. In Mathematical Analysis, the used vocabulary is gradually enhanced, and the concepts are defined in a more formal way. In the current study, the language used regarding mathematics of change is examined, focusing on rate of change and in relation to misconceptions of students.

Keywords: calculus; rate of change; language of mathematics; mathematics education

1. Introduction

Learning of the language of mathematics is part of learning mathematics [1,2]. The language of mathematics may be considered a precise, global language with no ambiguities. Mathematical terms are defined rigorously, and the symbols are mostly common in different languages.

Nevertheless, the way the language is used in mathematics differs from the everyday language and may cause challenges to learners [2]. Ambiguities regarding technical vocabulary as also other linguistic challenges, including grammar, may confuse students and impact their understanding of mathematics.

In a social activity as education, the language is not always strictly used. Sometimes words are omitted, phrases are shortened, and gestures and pronouns are used instead of the mathematical terms. Teachers try to help students understand the new mathematical concepts and move from informal, everyday language to formal language of mathematics [1, 2].

Use of known words in a more strictly way in mathematics or ambiguities should not be always considered a problem. In the social discourse of a classroom, formal and informal uses of language are combined, and ambiguity provides opportunities for teachers and students to extend their thinking and understanding of the language of mathematics [3].

Rate of change is an important concept of mathematics, used in everyday life and applied in a variety of disciplines. It is studied in school curriculum and in more advanced courses in university. The concepts related to rate of change are expressed with different terms and expressions, some of which are not clearly defined in school textbooks, while others are partially defined and enhanced later.

Research on students' understanding of rate of change has revealed difficulties in conceptualization of rate and its interpretation in physical phenomena [4, 5, 6, 7]. Some of the studies provide indications that some difficulties are related to the used language [5, 8].

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As language is important in teaching and infers the knowledge and the meaning of concepts that students construct, the vocabulary used for rate of change and related concepts is examined in the current study. Change and rate of change are studied as fundamental topics of Mathematical Analysis and the linguistic challenge associated with them is addressed. Specifically, misconceptions of rate, slope, tangent, derivative, and velocity are examined regarding the language used to express them in school mathematics. The vocabulary used regarding rate of change is very rich with complicated expressions and some terms are not clearly defined in school textbooks. A new term as an alternative to rate of change is proposed for use in mathematics, sciences, and economics, which should be clearly defined and used throughout the disciplines.

2. The Role of Language in Mathematics Education

Mathematics is considered as a global language, which students must master to be able to understand the concepts and communicate their ideas fluently. Mathematics language consists of words, numbers, symbols, and diagrams and is the key to access mathematical concepts [1]. In mathematics education, it involves the ability to use words to explain concepts, justify procedures and communicate mathematically [1].

The term concept image as expressed by Tall and Vinner [9] is used to describe "the total cognitive structure that is associated with the concept, which includes all the mental pictures, associated properties and processes". Concept image may be a visual representation of a concept or a collection of impressions or experiences [10]. Students construct a concept image of a mathematical concept according to their personal experiences, which may diverge from the mathematical concept as mathematicians or teachers understand it [11]. The term concept definition is used as a form of words used by the learner to define the concept [9].

Students have spontaneous conceptions for mathematical concepts before the formal teaching [12]. Part of their conceptions is the meaning of mathematical terms according to their experiences and the use of everyday language [12]. Informal language and terms of everyday language known to students are often used by educators to explain new mathematical concepts.

The meaning of terms and expressions that students already know may differ from the formal mathematical language and result to misconceptions. Moreover, the spontaneous conceptions remain even at an advanced level of learning [12] and are difficult to change especially when the mathematical terms are used in everyday language and their mathematical definition is not clear [10].

Sometimes students have not the same concept image for a mathematical concept as the mathematicians, even if they use the same language to describe it [13]. The definition of a mathematical concept is not always understood by students the way teachers believe and the concept definition students form may differ from the formal definition [9]. There are indications that students use their concept image instead of the concept definition to solve tasks [10]. Moreover, students tend to use informal language to express themselves and may use mathematical terms in contexts where they have a colloquial meaning [14].

For students to form a concept image that is similar to the image mathematicians and teachers hold, students and teachers should share a common code of communication. To help students learn and use the language of mathematics, educators should be aware of the difficulties related to terminology [1]. The formation of a productive concept image is not achieved by just stating the concept definition and should be related to everyday life with examples and non-examples [10].

Lexical ambiguity is the result of assigning different meanings in a word or phrase and may appear when a word of everyday language is used as a technical term in a discipline [3]. Rubenstein & Thompson [15] have reported 11 categories of difficulties related to the learning of mathematical language: (a) words common in mathematics and everyday language with different meanings in the two contexts, (b) words common in mathematics and everyday language with comparable meanings, but mathematical meaning is more precise, (c) terms used only in mathematics, (d) words used in mathematics with

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more than one meaning, (e) terms used in mathematics and other disciplines with different technical meanings, (f) mathematical words homonyms with everyday language words, (g) words that are related, but students confuse their distinct meanings, (h) English words that are translated in different ways in another language, and the opposite (i) English spelling and usage irregularities, (j) mathematical concepts that are verbalized in more than one way, (k) use of informal term as if it was a mathematical term.

Linguistic challenges and lexical ambiguities are apparent in many fields of mathematics. There is some recent research on lexical ambiguity regarding mathematical terms in statistics [16. 17] and algebra [18, 19].

Regarding Mathematical Analysis, there are references in research to students' misconceptions related to language [12, 10]. There are indications that more emphasis in Calculus teaching is put on procedures instead of the concepts [20]. Students can solve problems of Calculus algorithmically by memorizing standard procedures, but they have not acquired the meaning of the concepts [11]. Some of the misconceptions in concepts of Calculus seem to be related to the language used, as for example regarding the terms limit or tangent [12, 10]. Moreover, in Calculus, long phrases are often used to correctly express a concept, and a shortening of a phrase may result in completely different meaning.

Language skills are essential in word problem solving, firstly to understand the problem and then to express the solution [21]. There are indications that students' language proficiency is related to their proficiency in solving mathematics word problems [22]. Calculus is widely used to model and solve problems of sciences. Language is reported by students as one of their main difficulties in solving applications problems [23].

The role of school textbooks is vital in education. Teachers use them to decide what to teach and they are students' main study tool. The language used in textbooks affects students' understanding and may result to misconceptions [24]. Research on the language used in mathematics and the possible misconceptions it may cause, would help improving school textbooks and teaching.

Ambiguous meanings are not necessary obstacles in education and particularly mathematics. If teachers are aware of possible meanings that are recalled by new terminology in mathematics, they could take advantage of the language used by students and connect it to mathematical concepts [25].

3. Method

In the current study, the language used in relation to change and rate of change is addressed. Aspects of rate of change and other concepts related to it are discussed and presented in a concept map. Literature is reviewed regarding the language used in rate of change and possible misconceptions it may cause to students. The language used in textbooks and mathematics education to describe rate of change is examined. In addition, concepts related to rate of change as ratio, slope, tangent, derivative, and applications of rate of change in contexts are addressed regarding linguistic challenges.

4. Results

4.1. Rate of change aspects

Mathematical Analysis expresses the mathematics of change. Mathematical Analysis, as a name of an area of mathematics, is not clear as Geometry or Algebra, and may be used in different contexts with different meanings.

A part of the vocabulary used in Mathematical Analysis is not new to students. There are words already known and used in everyday language. "Limit" for example, as also the expressions 'tends towards' have many different meanings for students [12]. These ambiguities result in a concept image that may conflict with the formal definition of the concept [12].

Continuity suffers from lexical ambiguity as well. Expressions from everyday life as "it rained continuously all day" or "the railway line is continuously welded" are used by

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teachers as an introduction to continuity [9]. However, such expressions may result to the idea that the graph of a continuous function has "no gaps" [9].

Rate of change as a fundamental concept of Mathematical Analysis, is rich of meanings and complex for most students. To describe a dynamically changing phenomenon, besides the study of the change of one quantity, the way this change happens is also required.

Rate of change is taught in different levels of education, from school to university and in various disciplines. Besides mathematics it is used in other sciences and has applications in physics, economics, biology, mechanics, and electronics. Some well-known and frequently used rates of change are velocity, acceleration, power, rate of change of kinetic energy, of dynamic energy and of momentum, as also economical concepts as marginal cost and profit.

The language used to describe dynamically changing phenomena indicates the understanding and the image one has for the concepts. Students may form different images of rate of change, according to their age, grade, and previous experiences. Some misconceptions and errors may result from the language used in school textbooks and by educators [6].

Figure 1 stands as a concept map of rate of change and presents some of the concepts related to rate. For some of the concepts, more than one word is needed to precisely express their meaning and if they are omitted, the concepts are not clear.

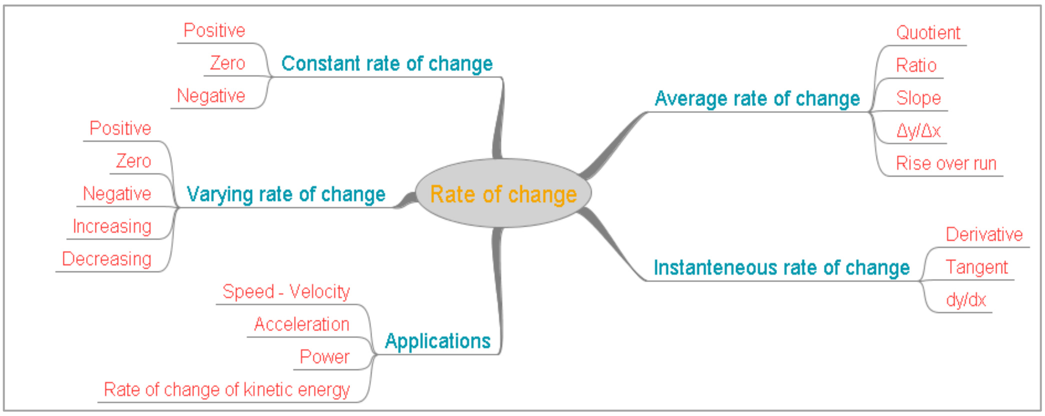

Figure 1. A concept map of rate of change

The English word "rate" has the meaning of value or ratio. In Greek, the term is used in music, art, architecture, mathematics, physical sciences, economics, biology, and everyday life. It usually has the meaning of a periodical motion.

Thompson [26] refers to the terms "ratio" and "rate", noticing that the use of two different terms implies that there are two different concepts but in practice they are used as interchangeable. According to the researcher, the two terms are used in education and research without definition and without clear distinction [26].

In mathematics, "rate" is used both in Statistics, Mathematical Analysis and Algebra (for simplicity in the sequel we use it without quotation marks). Students are asked to express and explain rates in mathematics areas with precise language and distinguish it from the terms used in everyday language.

Rate of change can be average or instantaneous. Average rate of change is related to the ratio of differences or the slope, while instantaneous rate of change is the derivative. It may be constant, in which case the instantaneous and the average rate of change are the same, or varying. The value of rate of change may be positive, negative or zero and moreover it may be increasing or decreasing. Moreover, the rate of change of a function has a rate of change itself. All these aspects, with the used words and adverbs, form a complicated image of rate of change.

One of the linguistic challenges of mathematics are the dense noun phrases that participate in relational processes [2]. Speaking of rate of change includes such phrases, that are difficult to interpret. When students describe dynamically changing phenomena, they must combine nouns, as "derivative", "rate of change", "slope", with adverbs, such as "at a point", "on an interval" and verbs, as "increase", "decrease" or adjectives, as "positive/negative", "general/specific" [27].

The rate of change is usually related to specific contexts, where the double or triple use of the proposition "of" presents another challenging factor. A not unusual expression of a rate of change problem may be "The rate of change of volume of water in the tank is increasing at a decreasing rate", which requires a familiarity of the student with the language to be able to interpret. In some tasks, shortened expressions as "rate of increase/decrease" are used.

For the description of the behavior of function phrases as "increasing at a decreasing (or an increasing) rate" or "decreasing at an increasing (or a decreasing) rate" are used, which are quite confusing [28]. The contrasting adjectives are not easy to interpret. The difficulties many students face with such expressions is apparent in studies, in which students fail to distinguish the behavior of the function and the behavior of its derivative [29]. This is mostly observed in studies with tasks of rate of change in a specific context, in which students express the behavior of the function in that context but have difficulties to use more formal mathematical language and confuse the behavior of the function with that of its rate of change [8].

Some researchers have proposed to split these long phrases in two, one describing the behavior of the function and one for the derivative, as "The function is increasing (decreasing) and its rate of change is increasing (decreasing)" [29].

Zandieh & Knapp [30] examined the role of metonymy in students reasoning about derivative, pointed out that students may believe that a shorten phrase is accurate and true or that both the extended and shorten phrase are true. Metonymy is often used in Calculus and it is accepted in education, but the excessive use of shortened expressions may hide information that is not trivial and may be related to a lack of understanding [30].

Some rates of change, as velocity, acceleration, power, and electric current in physics, express another physical magnitude. In these cases, a new term is used instead of rate of change. Even if it seems easier for students to understand these concepts, the different terms that are used may cause misunderstandings and it is more difficult to relate the concepts to rate of change and its properties. It is simpler to speak about acceleration instead of rate of change of velocity, but it is harder to conceptualize the connection between the two magnitudes. Students seem to conceptualize the concepts as new entities and not as the measure of two covarying quantities [31].

A common misconception described in literature regarding rate of change is to express the magnitude but not the sign of rate [8]. Negative rates of change that increase are difficult to express, because students confuse the absolute value which decreases with the signed value which increases. This is more intense when describing phenomena for which the everyday language refers to the magnitude of the quantity [8, 29].

The term "constant" for describing rate of change and function may cause misconceptions. A rate of change that is "increasing constantly" may be conceptualized as constantly increasing (increasing all the time), while in mathematical terms it means that it is increasing at a constant rate [8].

Another mathematical term that may cause misconceptions due to lexical ambiguity is average [13]. Students experiences of the term average in everyday language conflict with its use in Statistics and Calculus [13]. Average may refer to arithmetic mean, median or mode in statistics or may express something normal or usual in everyday language [13]. In Bezuidenhout's [11] research, some university students added up a number of g(x) or g'(x) values and divided by the number of instances to find the average rate of change. Average in Calculus is confused with the arithmetic mean in statistics. Average rate of change, average value of a function and arithmetic mean are confused and cause misconceptions for many students [11], as also average rate of change and instantaneous rate of change [5]. Students' concept images of average prevent students from thinking about average rate of change as a quantity changing at a constant rate with respect to another quantity [13].

4.2. Figures, Tables and Schemes

4.2.1. Slope

Average rate of change of a function between two points is the slope of the secant line in these points, while instantaneous rate of change in a point is the slope of the tangent line at this point. In linear functions, rate of change equals the slope of the line. Understanding of slope of linear functions is crucial for the conceptualization of derivative and rate of change [32].

The terms and symbolism used for slope in school textbooks are ambiguous and may cause misconceptions [33]. The term "slope" is associated to the words "steep", "elevation", "descent" and "inclined" from everyday language [34]. Students hold an image of slope, as a roof, a mountain, or another inclined surface [35]. Besides mathematics, slope is used in other scientific disciplines, as art, architecture, mechanics, and physical sciences [34]. Students have an intuitive understanding of slope from their experiences, which needs to be transformed to the formal definition in mathematics.

In mathematics, slope can be conceptualized geometrically, algebraically, trigonometrically and in Calculus [36]. The phrase "rise over run" is used as a mnemonic rule to remember how to compute slope and students who use it correctly seem to understand slope as a ratio or rate [35].

Slope, rate of change and steepness are sometimes used as having the same meaning in textbooks [6]. Coe [37], in a research on mathematics teachers, noted that even experienced teachers have difficulties to express the connection between division, rate and slope.

4.2.2. Tangent

Students encounter the notion of tangent in different contexts in mathematics, as a tangent to circle in Geometry, tangent line of conic sections in Analytic Geometry and tangent line to a graph in Calculus [38]. While the used term remains the same in the different contexts, the definition changes.

The word tangent stems from the Latin adjective tangents, which means touching [39]. Students learn first about tangent lines in the context of geometry, as a tangent to a circle. Teachers and some textbooks, attempting to clarify the concept of tangent line, use everyday language and experience of the students and define tangent as the line that touches the circle at exactly one point [24].

In Calculus the concept of tangent is extended and includes cases in which the tangent line may cross the curve at another point [39]. Moreover, the term tangent is used in trigonometry, as tangent of an angle. The same term is used because the tangent of the angle in a unit circle equals the segment of the tangent line to the circle [39]. Students may confuse the idea of tangent line with the function $y = \tan(x)$, and tangent of an angle [40].

Students tend to use the 'touching' cognitive model of a tangent in Calculus [24]. A common misconception of students is that a tangent line to a more general curve may only meet the curve at one point and may not cross the curve at that point [10, 24, 38].

4.2.3. Derivative

Tangent, slope, and derivative are confusing students, and it has been reported that some students talk about derivative as the tangent line instead of its slope [27, 30]. There are indications that students have a better understanding of the derivative if their concept image of tangent includes the limiting position of secant lines, instead of the touching cognitive model [41]. In a research, mechanical engineering students were found to think of derivative in terms of rate of change and mathematics students in terms of tangents [42].

Shortening of long, complicated phrases may result to misconceptions. In Amit and Vinner's [20] study, the research noticed that the definition of the derivative at a point as "The derivative of a function at a certain point is the slope of the tangent to the graph of the function at this point" may result in "The derivative is the tangent to the function at a certain point". Omitting slope in the definition results in different meaning and it is not clear if students understand the way tangent is used or see derivative as the equation of the tangent.

The word derivative refers both to derivative at a point and derivative of a function [27]. Category g) of Rubenstein & Thompson [15] refers to words that are related, but students confuse their distinct meanings. Moreover, the challenge is more intense when speaking about phrases that are similar but refer to different concepts, as "derivative of a function at a point" and "derivative function" or "average rate of change of a function" and "average rate of change function".

4.2.4. Speed-velocity

Speed is the rate of change of distance travelled over time. Students sometimes use the term speed to describe rate of change while an analogy of rate of change to speed in a problem may help students interpret the problem right [30]. As with constant and average rate of change, misconceptions about constant and average speed have been reported [5].

A confusion has been observed in literature when the same word is used for magnitude and the signed value of a quantity [29]. In such cases, attention should be paid to the language used to help students understand and communicate about phenomena with negative rates [29].

In English there are two different words, speed, which describes the magnitude, and velocity, which expresses the signed value. Contrary, in Greek language the same word is used for speed and velocity in everyday language and in textbooks it is not always clear which concept is referred.

Other concepts that are not clearly distinguished for many students are the quantity and the change in a quantity and change and rate of change [43]. In these cases, the same words are used in phrases that express different concepts.

5. Discussion

Rate of change is a general concept that involves many aspects and can be conceptualize in different ways and in various contexts. It is not emphasized in primary and secondary education, and it seems to cause difficulties to students. Most of the terms discussed and used when speaking about rate are words common in mathematics and everyday language with comparable meanings, as for example rate, slope, constant, average. This category of ambiguity may cause significant misconceptions to students as they already hold a concept image for them.

The literature review on the effects of language to construction of concepts of Mathematical Analysis and especially rate of change implies that there are some difficulties related to language. Terminology regarding rate of change has long sentences, ambiguities, terms used in everyday language, terms used without definition or defined in different ways. Lack of familiarity with the vocabulary used in Mathematical Analysis results to errors and poor understanding of the concepts. Words used in colloquial language with different meaning need to be clarified. Nevertheless, students' experiences and words from everyday life may be carefully used to trigger the introduction of new concepts.

If teachers are aware of the potential problems used by mathematics language, they can find ways to overcome the difficulties and clarify statements that lead to misconceptions. Emphasis to the correct use of language should be placed in definition of terms and properties of concepts. Moreover, students should have opportunities to express orally and written their conceptions and participate in classroom discourse to enhance their mathematical linguistic abilities and as a result their understanding of concepts.

Modeling of dynamically changing events requires familiarity with the language of change and rate of change. Activities related to real-world problems, modelling and analysis of phenomena could help students enhance their linguistic abilities and mathematical terminology. Students should actively participate in class discourse to familiarize with the language of mathematics. Interpretation of phenomena in physics or other sciences may help students better understand the phenomena and the mathematics behind it. Concept maps, examples and non-examples can be used as tools to reveal similarities and differences between terms, expressions, and notations in mathematics.

Regarding rate of change, related concepts as slope, variation of quantities, constant and average rate of change could be taught in early secondary education as part of mathematics of change, extended later to derivative and instantaneous rate of change. A robust understanding of a concept image of rate could be generalized to derivative and more advanced topics of Calculus.

Instead of rate of change another term could be used in mathematics, sciences, and economics. Using one word would simplify some complicated expressions. Moreover, the phenomenon of omitting words to make the sentence smaller would be avoided, as also the double use of "of". One proposal for a one-word term that would express rate of change is the term "variationality". If a specific term is clearly defined and used throughout the disciplines, students could form a general, robust image of rate of change, integrating the various aspects and emphasizing the covariational nature of rate of change. As a result, they could conceptualize quantities of sciences as different aspects of the same concept and better understand their properties.

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References

1. Riccomini, P. J.; Smith, G. W.; Hughes, E. M.; Fries, K. M. The language of mathematics: The importance of teaching and learning mathematical vocabulary. *Reading & Writing Quarterly* 2015, 31(3), 235–252.
2. Schleppegrell, M. J. The linguistic challenges of mathematics teaching and learning: A research review. *Reading & writing quarterly* 2007, 23(2), 139-159.
3. Barwell, R. Ambiguity in the mathematics classroom. *Language and Education* 2005, 19(2), 117–125.
4. Orton, A. Students' understanding of differentiation. *Educational Studies in Mathematics* 1983, 14(3), 235–250.
5. Orton, A. Understanding rate of change. *Mathematics in School* 1984, 13(5), 23–26.
6. Teuscher, D.; Reys, R. E. Slope, Rate of Change, and Steepness: Do Students Understand These Concepts?. *Mathematics Teacher* 2010, 103(7), 519–524.
7. Ubuz, B. Interpreting a graph and constructing its derivative graph: stability and change in students' conceptions. *International Journal of Mathematical Education in Science and Technology* 2007, 38(5), 609–637.
8. Ärlebäck, J. B.; Doerr, H. M.; O'Neil, A. H. A modeling perspective on interpreting rates of change in context. *Mathematical thinking and learning* 2013, 15(4), 314–336.
9. Tall, D.; Vinner, S. Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational studies in mathematics* 1981, 12(2), 151–169.
10. Vinner, S. The role of definitions in the teaching and learning of mathematics. *Advanced mathematical thinking* 2002 (pp. 65–81). Springer.
11. Bezuidenhout, J. First-year university students' understanding of rate of change. *International Journal of Mathematical Education in Science and Technology* 1998, 29(3), 389–399.
12. Cornu B. Limits. In Advanced Mathematical Thinking Tall D. (eds); Mathematics Education Library, Springer, Dordrecht 2002; vol. 11.
13. Weber, E.; Dorko, A. Students' concept images of average rate of change. Proceedings of the 35th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education 2013 (pp. 386-393). Chicago, IL: University of Illinois at Chicago.
14. Tall, D. Inconsistencies in the learning of calculus and analysis. *Focus on Learning Problems in mathematics* 1990, 12(3), 49-63.
15. Rubenstein, R. N.; Thompson, D. R. Understanding and support children's mathematical vocabulary development. *Teaching Children Mathematics* 2002, 9(2), 107.
16. Dunn, P. K.; Carey, M. D.; Richardson, A. M.; McDonald, C. Learning the Language of Statistics: Challenges and Teaching Approaches. *Statistics Education Research Journal* 2016, 15(1).

17. Kaplan, J. J.; Fisher, D. G.; Rogness, N. T. Lexical ambiguity in statistics: What do students know about the words association, average, confidence, random and spread. *Journal of Statistics Education* 2009, 17(3), 1–19.
18. Adu, E. O.; Olaoye, O. An Investigation into Some Lexical Ambiguities in Algebra: South African Experience. *Mediterranean Journal of Social Sciences* 2014, 5(20), 1243.
19. Olaoye, O.; Adu, E. O.; Moyo, G. Lexical Ambiguity in Algebra, Method of Instruction as Determinant of Grade 9 Students' Academic Performance in East London District. *Mediterranean Journal of Social Sciences* 2014, 5(23), 897.
20. Amit, M.; Vinner, S. Some misconceptions in calculus – anecdotes or the tip of an iceberg? Proceedings of the Annual Conference of the International Group for the Psychology of Mathematics Education 1990, 3–10.
21. Adams, T. L. Reading mathematics: More than words can say. *The Reading Teacher* 2003, 56(8), 786-795.
22. Abedi, J.; Lord, C. The language factor in mathematics tests. *Applied measurement in education* 2001, 14(3), 219-234.
23. Klymchuk, S.; Zverkova, T.; Gruenwald, N.; Sauerbier, G. University students' difficulties in solving application problems in calculus: Student perspectives. *Mathematics Education Research Journal* 2010, 22(2), 81-91.
24. Kajander, A.; Lovric, M. Mathematics textbooks and their potential role in supporting misconceptions. *International Journal of Mathematical Education in Science and Technology* 2009, 40(2), 173–181.
25. Moschkovich, J. N. "I went by twos, he went by one": Multiple interpretations of inscriptions as resources for mathematical discussions. *The Journal of the Learning Sciences* 2008, 17(4), 551-587.
26. Thompson, P. W. The development of the concept of speed and its relationship to concepts of rate. The development of multiplicative reasoning in the learning of mathematics 1994b, 179–234.
27. Park, J. Is the derivative a function? If so, how do students talk about it?. *International Journal of Mathematical Education in Science and Technology* 2013, 44(5), 624-640, doi: 10.1080/0020739X.2013.795248.
28. Gough, J. Conceptual complexity and apparent contradictions in mathematics language. *The Australian Mathematics Teacher* 2007, 63(2), 8.
29. Doerr, H. M.; Bergman Ärlebäck, J.; O'Neil, A. H. Interpreting and communicating about phenomenon with negative rate of change. 120th ASEE (American Society for Engineering Education) annual conference & exposition 2013; Atlanta, Georgia, USA, June 23-26, 2013.
30. Zandieh, M. J.; Knapp, J. Exploring the role of metonymy in mathematical understanding and reasoning: The concept of derivative as an example. *The Journal of Mathematical Behavior* 2006, 25(1), 1–17.
31. Lobato, J.; Thanheiser, E. Re-thinking slope from quantitative and phenomenological perspectives. Proceedings of the 21st Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education 1999, vol. 1, pp. 291–297.
32. Nagle, C.; Moore-Russo, D.; Viglietti, J.; Martin, K. Calculus students' and instructors' conceptualizations of slope: A comparison across academic levels. *International Journal of Science & Mathematics Education* 2013, 11(6).
33. Confrey, J.; Smith, E. Exponential functions, rates of change, and the multiplicative unit. *Learning Mathematics* 1994, 31–60, Springer.
34. Deniz, O.; Kabael, T. Students' Mathematization Process of the Concept of Slope within the Realistic Mathematics Education. *Hacettepe University Journal of Education* 2017, 32(1), 123-142. doi: 10.16986/HUJE.2016018796.
35. Stump, S. L. High school precalculus students' understanding of slope as measure. *School Science and Mathematics* 2001a, 101(2), 81–89.
36. Stump, S. L. Developing preservice teachers' pedagogical content knowledge of slope. *The Journal of Mathematical Behavior* 2001b, 20(2), 207–227.
37. Coe, E. Modeling teachers' thinking about rate of change. (Unpublished PhD dissertation). Department of Mathematics and Statistics, Arizona State University, 2007.
38. Biza, I.; Christou, C.; Zachariades, T. Student perspectives on the relationship between a curve and its tangent in the transition from Euclidean Geometry to Analysis. *Research in Mathematics Education* 2008, 10(1), 53–70.
39. Schwartzman, S. The words of mathematics: An etymological dictionary of mathematical terms used in English. *Mathematical Association of America* 1994.
40. Vincent, B.; LaRue, R.; Sealey, V.; Engelke, N. Calculus students' early concept images of tangent lines. *International Journal of Mathematical Education in Science and Technology* 2015, 46(5), 641-657, doi: 10.1080/0020739X.2015.1005700.
41. Vincent, B.; Sealey, V. Students' concept image of tangent line compared to their understanding of the definition of the derivative. Proceedings of Conference on Research in Undergraduate Mathematics Education, Pittsburgh 2016, PA.
42. Bingolbali, E.; Monaghan, J. Concept image revisited. *Educational Studies in Mathematics* 2008, 68(1), 19–35.
43. Thompson, P. W. Images of rate and operational understanding of the fundamental theorem of calculus. *Learning Mathematics* 1994a (pp. 125–170). Springer.