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Proving Fixed Point Theorems Employing Fuzzy $(\sigma, \mathcal{Z})$-Contractive Type Mappings

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Abstract: In this article, the concept of fuzzy $(\sigma, \mathcal{Z})$-contractive mapping has been introduced in fuzzy metric spaces which is an improvement over the corresponding concept recently introduced by Shukla et al. [Fuzzy Sets and system. 350 (2018) 85–94]. Thereafter, we utilized our newly introduced concept to prove some existence and uniqueness theorems in $\mathcal{M}$-complete fuzzy metric spaces. Our results extend and generalize the corresponding results of Shukla et al.. Moreover, an example is adopted to exhibit the utility of newly obtained results.

Keywords: fuzzy $(\sigma, \mathcal{Z})$-contractive mappings; fuzzy metric spaces; fuzzy-$\mathcal{Z}$-contractive mappings

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Citation: Saleh, H.; Imdad, M.; Sessa, S.; Di Martino, F. Proving Fixed Point Theorems Employing Fuzzy $(\sigma, \mathcal{Z})$-Contractive Type Mappings. *Preprints* **2021**, *1*, 0. https://doi.org/

Received:
Accepted:
Published:

Publisher's Note: MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.

1. Introduction and Motivation

In the theory of fuzzy sets and systems, many researchers have attempted to formulate an appropriate definition of fuzzy metric space (e.g. [1–3]). The most natural and widely acceptable definition is essentially due to Kramosil and Michálek [4]. Grabiec [5] is one of the earliest mathematicians to study the theory of the fixed point in fuzzy metric spaces. In doing so, he introduced the notions of $\mathcal{G}$-Cauchy sequences and $\mathcal{G}$-completeness of fuzzy metric space and extended the fixed point theorems of Banach and Edelstein from metric spaces to fuzzy metric spaces introduced by Kramosil and Michálek. It has been observed that the notions of $\mathcal{G}$-Cauchy sequences and $\mathcal{G}$-completeness are relatively stronger. With a view to have a Hausdorff topology on fuzzy metric space, George and Veeramani [6] modified the definition of fuzzy metric space due to Kramosil and Michálek [4] and also established some valuable related results.

In 2002, Gregori and Sapena [7] initiated a class of mappings called fuzzy contractive mappings and proved the Banach contraction theorem in fuzzy metric spaces in the sense of George and Veeramani. Thereafter, employing a control function satisfying suitable properties, Miheţ [8], and Wardowski [9] generalized the class of fuzzy contractive mapping by introducing the concepts of fuzzy ψ-contractive mapping and fuzzy $\mathcal{H}$-contractive mapping, respectively. For such kind of work, we refer the reader to [10–24]. Very recently, Shukla et al. gave the concept of fuzzy-$\mathcal{Z}$-contractive, which unify all the classes of mappings mentioned earlier.

This article aims to enlarge the class of fuzzy $\mathcal{Z}$-contractive mappings by introducing the family of fuzzy $(\sigma, \mathcal{Z})$-contractive mappings to cover all of the concepts introduced in [7–9,25,26]. Our newly introduced notion has been utilized to prove some results in $\mathcal{M}$-complete fuzzy metric spaces. Finally, an example was adopted to demonstrate that our newly presented results are a proper extension of Shukla et al. [25] results.

2. Mathematical Preliminaries

In this section, we present some introductory material from the theory of fuzzy metric spaces needed to prove our results.

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Definition 1. *[27] Let $*:[0,1]\times[0,1]\to[0,1]$ be a binary operation. We say that $*$ is a continuous t-norm if the following assumptions are fulfilled:*
$(N1)$ $*$ is associative and commutative,
$(N2)$ $*$ is continuous,
$(N3)$ $r_1 * r_2 \leq r_3 * r_4$ whenever $r_1 \leq r_3$ and $r_2 \leq r_4$,
$(N4)$ $r_1 * 1 = r_1$,
for all $r_1, r_2, r_3, r_4 \in [0,1]$.

Three primary continuous t-norms examples are: $r_1 *_1 r_2 = r_1 \cdot r_2$, $r_1 *_2 r_2 = \min\{r_1, r_2\}$ and $r_1 *_3 r_2 = \max\{r_1 + r_2 - 1, 0\}$ which known as product, minimum and lukasiewicz t-norms respectively.

By modifying the concept of fuzzy metric space introduced in [4], George and Veeramani attempted the following definition:

Definition 2. *[6] Let $\mathcal{K}$ be a non-empty set and $\mathcal{M}:\mathcal{K}^2\times(0,\infty)\to[0,1]$ is a fuzzy set. The ordered triple $(\mathcal{K},\mathcal{M},*)$ is called a fuzzy metric space (in short, FMS), where $*$ is a continuous t-norm if the following assumptions are fulfilled (for all $\alpha,\beta,\gamma\in\mathcal{K}$ and $t,s>0$):*
$(G1)$ $\mathcal{M}(\alpha,\beta,t)>0$,
$(G2)$ $\mathcal{M}(\alpha,\beta,t)=1$ if and only if $\alpha=\beta$,
$(G3)$ $\mathcal{M}(\alpha,\beta,t)=\mathcal{M}(\beta,\alpha,t)$,
$(G4)$ $\mathcal{M}(\alpha,\gamma,t)*\mathcal{M}(\gamma,\beta,s)\leq\mathcal{M}(\alpha,\beta,t+s)$,
1. $\mathcal{M}(\alpha,\beta,.):(0,\infty)\to[0,1]$ is continuous.

Definition 3. *[5,6] A sequence $\{\alpha_n\}$ in a FMS, $(\mathcal{K},\mathcal{M},*)$ is called*
(a) Convergent and converges to $\alpha\in\mathcal{K}$, if $\lim_{n\to\infty}\mathcal{M}(\alpha_n,\alpha,t)=1$, for each $t>0$.
(b) $\mathcal{M}$-Cauchy, if for each $\epsilon\in(0,1)$ and $t>0$, there is $n_0\in\mathbb{N}$ such that $\mathcal{M}(\alpha_m,\alpha_n,t)>1-\epsilon$, for each $m,n\geq n_0$.
(c) $\mathcal{G}$-Cauchy, if $\mathcal{M}(\alpha_n,\alpha_{n+p},t)=1$, for each $t>0$ and $p\in\mathbb{N}$.

Lemma 1. *[5,6] In the fuzzy metric space $(\mathcal{K},\mathcal{M},*)$,*
(1) $\mathcal{M}$ is continuous on $\mathcal{K}^2\times(0,\infty)$.
(2) $\mathcal{M}(\alpha,\beta,.)$ is non-decreasing function on $(0,\infty)$, for each $\alpha,\beta\in\mathcal{K}$,
(3) the limit of a convergent sequence in $(\mathcal{K},\mathcal{M},*)$ is unique.

$(\mathcal{K},\mathcal{M},*)$ is called $\mathcal{M}$-complete (G-complete) FMS, if every $\mathcal{M}$-Cauchy ($\mathcal{G}$-Cauchy) sequence of $\mathcal{K}$ converges in $\mathcal{K}$.

Definition 4. *[7] Let S be a self-mapping of a FMS $(\mathcal{K},\mathcal{M},*)$. S is said to be a fuzzy contractive, if the following condition holds:*

$$\frac{1}{\mathcal{M}(S\alpha,S\beta,t)}-1\leq\lambda\left(\frac{1}{\mathcal{M}(\alpha,\beta,t)}-1\right),\ \forall\,\alpha,\beta\in\mathcal{K},t>0,$$

for some $\lambda\in(0,1)$.

Definition 5. *[26] Let $(\mathcal{K},\mathcal{M},*)$ be a FMS. A mapping $S:\mathcal{K}\to\mathcal{K}$ is called a Tirado contraction, if the following inequality holds:*

$$1-\mathcal{M}(S\alpha,S\beta,t)\leq k(1-\mathcal{M}(\alpha,\beta,t)),\ \forall\,\alpha,\beta\in\mathcal{K},t>0,$$

for some $k\in(0,1)$.

Let us denote by Ψ the set of all $\psi:(0,1]\to(0,1]$ which have the following properties:
$(\Psi1)$ ψ is a left continuous and non-decreasing function,

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$(\Psi 2)$ $\psi(r) > r$, for all $r \in (0,1)$.

Definition 6. *[8] Assume that $(\mathcal{K}, \mathcal{M}, *)$ be a FMS and $S : \mathcal{K} \to \mathcal{K}$. We say that S is a fuzzy ψ-contractive w.r.t $\psi \in \Psi$ if the following condition hold:*

$$\mathcal{M}(S\alpha, S\beta, t) \geq \psi(\mathcal{M}(\alpha, \beta, t)),\ \forall\ \alpha, \beta \in \mathcal{K}, t > 0,$$

We also denote by $\mathcal{H}$ the class of all $\eta : (0,1] \to [0,\infty)$ which satisfying the following properties:
$(\mathcal{H}1)$ η transforms $(0,1]$ onto $[0,\infty)$,
$(\mathcal{H}2)$ η is a strictly decreasing function.

Definition 7. *[9] Suppose that $(\mathcal{K}, \mathcal{M}, *)$ be a FMS and $S : \mathcal{K} \to \mathcal{K}$. We say that S is a fuzzy $\mathcal{H}$-contractive w.r.t $\eta \in \mathcal{H}$ if there is a constant $\lambda \in (0,1)$ such that*

$$\eta(\mathcal{M}(S\alpha, S\beta, t)) \leq \lambda \eta(\mathcal{M}(\alpha, \beta, t)),\ \forall\ \alpha, \beta \in \mathcal{K} \text{ and } t > 0.$$

Remark 1. *[28] In case $\eta \in \mathcal{H}$, the function η is bijective and continuous. Moreover, the mappings $k \cdot \eta : (0,1] \to [0,\infty)$ and $\eta^{-1} : [0,\infty) \to (0,1]$ are continuous, bijective and strictly decreasing.*

Let $\mathcal{Z}$ be the set of all $\xi : (0,1] \times (0,1] \to \mathbb{R}$ which satisfy the condition:

$$\xi(l,s) > s,\ \forall\ l,s \in (0,1).$$

Employing the function ξ which satisfy the above condition, Shukla et al. unified the above mentioned family of contractive type mappings by introducing the following interesting class of mappings:

Definition 8. *[25] Let S be a self-mapping of a FMS $(\mathcal{K}, \mathcal{M}, *)$. S is said to be a fuzzy $\mathcal{Z}$-contractive if there is $\xi \in \mathcal{Z}$ such that*

$$\mathcal{M}(S\alpha, S\beta, t) \geq \xi(\mathcal{M}(S\alpha, S\beta, t), \mathcal{M}(\alpha, \beta, t)),$$

for each $\alpha, \beta \in \mathcal{K}$ with $S\alpha \neq S\beta$ and $t > 0$.

Example 1. *[25] Consider the functions $\xi_i : (0,1] \times (0,1] \to \mathbb{R}$, $i = 1,2,3$ which defined as*

1. $\xi_1(l,s) = \frac{s}{l}$,
2. $\xi_2(l,s) = \frac{1}{s+l} + l$,
3. $\xi_3(l,s) = \begin{cases} l, & \text{if } l > s, \\ \sqrt{s} & \text{if } s \geq l, \end{cases}$

for all $l, s \in (0,1]$. Then, $\xi_i \in \mathcal{Z}$, $i = 1,2,3$.

Definition 9. *[25] Let S be a self-mapping of a FMS $(\mathcal{K}, \mathcal{M}, *)$ and $\xi \in \mathcal{Z}$. Assume that $\{\alpha_n\}$ be any Picard sequence for all $n \in \mathbb{N}$. The quadruple $(\mathcal{K}, \mathcal{M}, S, \xi)$ is said to have the property (S) if for each $n \in \mathbb{N}$ and $t > 0$ with*

$$\begin{aligned} & \inf_{m>n} \mathcal{M}(\alpha_n, \alpha_m, t) \leq \inf_{m>n} \mathcal{M}(\alpha_{n+1}, \alpha_{m+1}, t) \\ \text{implies} \quad & \lim_{n\to\infty} \inf_{m>n} \xi(\mathcal{M}(\alpha_{n+1}, \alpha_{m+1}, t), \mathcal{M}(\alpha_n, \alpha_m, t)) = 1. \end{aligned}$$

Inspired by the work of Samet et al. [29], Gopal and Vetro [30] employ a function $\sigma : \mathcal{K} \times \mathcal{K} \times (0,\infty) \to (0,\infty)$ instead of $\sigma : \mathcal{K} \times \mathcal{K} \to [0,\infty)$ and introduced the following definition under the same name as follows.

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Definition 10. *[30] Let S be a self-mapping of a FMS and $\sigma : \mathcal{K} \times \mathcal{K} \times (0, \infty) \to (0, \infty)$. A mapping S is said to be a σ-admissible if*

$$\forall\, \alpha, \beta \in \mathcal{K}, t > 0, \quad [\sigma(\alpha, \beta, t) \leq 1 \implies \sigma(S\alpha, S\beta, t) \leq 1]. \tag{1}$$

For σ-admissible mapping example, we refer the reader to [30]. Now, we add another examples of σ-admissible mapping.

Example 2. *Consider a mapping $S : \mathcal{K} \to \mathcal{K}$, where $\mathcal{K} = [0, \infty)$ and $\sigma : \mathcal{K} \times \mathcal{K} \times (0, \infty) \to (0, \infty)$ defined by*

$$S\alpha = \sqrt{\alpha}, \;\; \forall\, \alpha \in \mathcal{K} \qquad \text{and} \qquad \sigma(\alpha, \beta, t) = \begin{cases} e^{\frac{\beta-\alpha}{t}} & \alpha \geq \beta, \\ 2 & \alpha < \beta. \end{cases}$$

Then S is a σ-admissible mapping.

Example 3. *Assume that $S : \mathcal{K} \to \mathcal{K}$, where $\mathcal{K} = [0, \infty)$ and $\sigma : \mathcal{K} \times \mathcal{K} \times (0, \infty) \to (0, \infty)$ defined by*

$$S\alpha = \ln \alpha, \;\; \forall\, \alpha \in \mathcal{K} \qquad \text{and} \qquad \sigma(\alpha, \beta, t) = \begin{cases} \frac{1}{t} & \alpha \geq \beta, \\ 2 & \text{otherwise}. \end{cases}$$

Then S is a σ-admissible mapping.

3. Main Results

Throughout this article, $(\mathcal{K}, \mathcal{M}, *)$ is a fuzzy metric space in George and Veeramani sense. First of all, we start by introducing the notion of fuzzy $(\sigma, \mathcal{Z})$-contractive mappings and show that it is included many existing and familiar concepts as special cases.

Definition 11. *Let S be a self-mapping of a FMS $(\mathcal{K}, \mathcal{M}, *)$. We say that S is a fuzzy $(\sigma, \mathcal{Z})$-contractive w.r.t $\xi \in \mathcal{Z}$ if there is a $\sigma : \mathcal{K} \times \mathcal{K} \times (0, \infty) \to (0, \infty)$ such that*

$$\sigma(\alpha, \beta, t)\mathcal{M}(S\alpha, S\beta, t) \geq \xi(\mathcal{M}(S\alpha, S\beta, t), \mathcal{M}(\alpha, \beta, t)), \tag{2}$$

for all $\alpha, \beta \in \mathcal{K}, t > 0$ with $S\alpha \neq S\beta$.

Remark 2. *If $\sigma(\alpha, \beta, t) = 1$, for each $\alpha, \beta \in \mathcal{K}$ and $t > 0$, then the Definition 11 reduces to the Definition 8, that is, every fuzzy $\mathcal{Z}$-contractive is a fuzzy $(\sigma, \mathcal{Z})$-contractive mapping but the reverse is not in general true (see Example 4 given below)*

Remark 3. *By adopting the functions ξ and α suitably in Definition 11, we deduce some well known contractions as demonstrated under (for all $x, y \in X$ and $t > 0$).*

(a) Taking $\xi(l, s) = \psi(s)$, for each $l, s \in (0, 1]$ and $\psi \in \Psi$, we deduce Definition 6.
(b) Putting $\xi(l, s) = \eta^{-1}(\lambda\eta(s))$ and $\sigma(\alpha, \beta, t) = 1$ where $\eta \in \mathcal{H}$, $\lambda \in (0, 1)$ and for each $l, s \in (0, 1]$, we deduce Definition 7.
(c) Taking $\xi(l, s) = \frac{s}{k+(1-k)s}$ and $\sigma(\alpha, \beta, t) = 1$, for each $l, s \in (0, 1]$ and $k \in (0, 1)$, we obtain Definition 4.
(d) Setting $\sigma(\alpha, \beta, t) = 1$ and $\xi(l, s) = 1 + k(s - 1)$, for each $l, s \in (0, 1]$ and $k \in (0, 1)$, we obtain Definition 5.

Now, we able to formulate our first main result as follows:

Theorem 1. *Let $(\mathcal{K}, \mathcal{M}, *)$ be an $\mathcal{M}$-complete FMS and $\sigma : \mathcal{K} \times \mathcal{K} \times (0, \infty) \to (0, \infty)$. Assume that $S : \mathcal{K} \to \mathcal{K}$ is fuzzy $(\sigma, \mathcal{Z})$-contractive mapping and the following properties are hold:*

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(a) *S is σ-admissible,*
(b) *the quadruple $(\mathcal{K}, \mathcal{M}, S, \xi)$ owns the property (S),*
(c) *there exists $\alpha_0 \in \mathcal{K}$ with $\sigma(\alpha_0, S\alpha_0, t) \le 1$, for each $t > 0$,*
(d) *for each sequence $\{\alpha_n\}$ of $\mathcal{K}$ with the property that $\sigma(\alpha_n, \alpha_{n+1}, t) \le 1$, for each $t > 0$, there exists $k_0 \in \mathbb{N}$ such that $\sigma(\alpha_n, \alpha_m, t) \le 1$, for each $m, n \in \mathbb{N}$ with $m > n \ge k_0$, $t > 0$,*
(e) *S is continuous.*

Then T admits a fixed point.

Proof. Pick out an arbitrary point α_0 in $\mathcal{K}$ such that $\sigma(\alpha_0, S\alpha_0, t) \le 1$, for each $t > 0$ and consider a Picard sequence $\{\alpha_n\}$ in $\mathcal{K}$, that is,

$$\alpha_{n+1} = S\alpha_n, \text{ for all } n \in \mathbb{N}_0.$$

In case $\alpha_{n_0} = \alpha_{n_0+1}$, for some $n_0 \in \mathbb{N}_0$, then the fixed point of the mapping S is nothing but α_{n_0}. Assume that $\alpha_{n+1} \ne \alpha_n$, for each $n \in \mathbb{N}_0$. As S is σ-admissible, we have

$$\sigma(\alpha_0, \alpha_1, t) = \sigma(\alpha_0, S\alpha_0, t) \le 1 \implies \sigma(\alpha_1, \alpha_2, t) = \sigma(S\alpha_0, S\alpha_1, t) \le 1.$$

The induction on n, gives rise to

$$\sigma(\alpha_n, \alpha_{n+1}, t) \le 1, \text{ for each } n \in \mathbb{N}_0 \text{ and } t > 0. \tag{3}$$

Moreover, if for some $m > n$, $\alpha_n = \alpha_m$, then the contractive condition 2 and equation 3 imply that

$$\begin{aligned}\mathcal{M}(\alpha_{n+1}, \alpha_{n+2}, t) &\ge \sigma(\alpha_n, \alpha_{n+1}, t)\mathcal{M}(\alpha_{n+1}, \alpha_{n+2}, t) \\ &\ge \xi(\mathcal{M}(\alpha_{n+1}, \alpha_{n+2}, t), \mathcal{M}(\alpha_n, \alpha_{n+1}, t)) \\ &> \mathcal{M}(\alpha_n, \alpha_{n+1}, t),\end{aligned}$$

hence

$$\mathcal{M}(\alpha_{n+1}, \alpha_{n+2}, t) > \mathcal{M}(\alpha_n, \alpha_{n+1}, t).$$

Continuing in this way, one can show that

$$\mathcal{M}(\alpha_m, \alpha_{m+1}, t) > \mathcal{M}(\alpha_{m-1}, \alpha_m, t) > \cdots > \mathcal{M}(\alpha_{n+1}, \alpha_{n+2}, t) > \mathcal{M}(\alpha_n, \alpha_{n+1}, t).$$

Since $\alpha_n = \alpha_m$ for some $m > n$, we have $\alpha_{n+1} = \alpha_{m+1}$. This together with the above relation leads to a contradiction. Therefore, $\alpha_n \ne \alpha_m$ for each $m > n$.
In view of the condition (d), there exists $k_0 \in \mathbb{N}$ such that

$$\sigma(\alpha_n, \alpha_m, t) \le 1, \ \forall\, m, n \in \mathbb{N} \text{ with } m > n \ge k_0 \text{ and } t > 0.$$

Applying the contractive condition 2 and making the use of the above inequality, we get

$$\begin{aligned}\mathcal{M}(\alpha_{n+1}, \alpha_{m+1}, t) &\ge \sigma(\alpha_n, \alpha_m, t)\mathcal{M}(S\alpha_n, S\alpha_m, t) \\ &\ge \xi(\mathcal{M}(S\alpha_n, S\alpha_m, t), \mathcal{M}(\alpha_n, \alpha_m, t)) \\ &> \mathcal{M}(\alpha_n, \alpha_m, t),\end{aligned} \tag{4}$$

and hence

$$\forall\, m > n, \ \mathcal{M}(\alpha_{n+1}, \alpha_{m+1}, t) > \mathcal{M}(\alpha_n, \alpha_m, t).$$

In the above inequality, taking the infimun over $m(> n)$ and let $a_n(t) = \inf_{m>n} \mathcal{M}(\alpha_n, \alpha_m, t)$ we obtain that $a_n(t) \le a_{n+1}(t)$, for each $t > 0$, and hence $\{a_n(t)\}$ is a nondecreasing and bounded. Therefore, there exists $a(t)$ such that $\lim_{n\to\infty} a_n(t) = a(t)$. Our claim is to

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justify that $a(t) = 1$, for each $t > 0$. On contrary, we assume that $a(s) > 1$, for some $s > 0$. From the fact that the quadruple $(\mathcal{K}, \mathcal{M}, S, \xi)$ owns the property (S), we get

$$\lim_{n\to\infty} \inf_{m>n} \xi(\mathcal{M}(\alpha_n, \alpha_m, s), \mathcal{M}(\alpha_{n+1}, \alpha_{m+1}, s)) = 1. \tag{5}$$

The equation 4 gives rise

$$\begin{aligned} a_{n+1}(s) = \inf_{m>n} \mathcal{M}(\alpha_{n+1}, \alpha_{m+1}, s) &\geq \inf_{m>n} \xi(\mathcal{M}(S\alpha_n, S\alpha_m, s), \mathcal{M}(\alpha_n, \alpha_m, s)) \\ &\geq \inf_{m>n} \mathcal{M}(\alpha_n, \alpha_m, s) = a_n(s). \end{aligned}$$

Taking $n \to \infty$ to the above relation and using equation 5, we get

$$\lim_{n\to\infty} \inf_{m>n} \mathcal{M}(\alpha_n, \alpha_m, s) = a(s) = 1,$$

which is a contradiction to the assumption ($a(s) > 1$ for some $s > 0$). This contradiction concludes that, for each $t > 0$, $\lim_{n,m\to\infty} \mathcal{M}(\alpha_n, \alpha_m, t) = 1$, that is, $\{\alpha_n\}$ is a Cauchy sequence. Due to the $\mathcal{M}$-completeness of the fuzzy metric space $(\mathcal{K}, \mathcal{M}, *)$, there is $\gamma \in \mathcal{K}$ such that

$$\lim_{n\to\infty} \mathcal{M}(\alpha_n, \gamma, t) = 1,$$

for all $t > 0$. The continuity of the mapping S implies that $\lim_{n\to\infty} \mathcal{M}(S\alpha_n, S\gamma, t) = 1$, for each $t > 0$, and hence

$$\lim_{n\to\infty} \mathcal{M}(\alpha_{n+1}, S\gamma, t) = \lim_{n\to\infty} \mathcal{M}(S\alpha_n, S\gamma, t) = 1,$$

for all $t > 0$. Therefore, $S\gamma = \gamma$, due to the uniqueness of the limit. □

The continuity assumption of S in the Theorem 1 can be replaced by another suitable condition. Precisely, we state and prove the following theorem:

Theorem 2. *Let $(\mathcal{K}, \mathcal{M}, *)$ be an $\mathcal{M}$-complete FMS and $\sigma : \mathcal{K} \times \mathcal{K} \times (0, \infty) \to [0, \infty)$. Assume that $S : \mathcal{K} \to \mathcal{K}$ is fuzzy $(\sigma, \mathcal{Z})$-contractive mapping satisfy the following assumptions:*

(a) *S is σ-admissible,*
(b) *the quadruple $(\mathcal{K}, \mathcal{M}, S, \xi)$ owns the property (S),*
(c) *there exists $\alpha_0 \in \mathcal{K}$ with $\sigma(\alpha_0, S\alpha_0, t) \leq 1$, for each $t > 0$,*
(d) *for each sequence $\{\alpha_n\}$ of $\mathcal{K}$ with the property that $\sigma(\alpha_n, \alpha_{n+1}, t) \leq 1$, for each $t > 0$, there exists $k_0 \in \mathbb{N}$ such that $\sigma(\alpha_n, \alpha_m, t) \leq 1$, for each $m, n \in \mathbb{N}$ with $m > n \geq k_0$, $t > 0$.*
(e') *if $\{\alpha_n\}$ be a sequence in $\mathcal{K}$ such that $\lim_{n\to\infty} \alpha_n = \alpha \in \mathcal{K}$ and $\sigma(\alpha_n, \alpha_{n+1}, t) \leq 1$, for each $n \in \mathbb{N}$ and $t > 0$, then $\sigma(\alpha_n, \alpha, t) \leq 1$.*

Then T possesses a fixed point.

Proof. The frame of the proof is same as that in the previous theorem (Theorem 1). So, for a Cauchy sequence $\{\alpha_n\}$ in a complete FMS $(\mathcal{K}, \mathcal{M}, *)$, there exists $\gamma \in \mathcal{K}$ such that

$$\lim_{n\to\infty} (\alpha_n, \gamma, t) = 1, \ \forall \, t > 0. \tag{6}$$

Also, we have $\sigma(\alpha_n, \alpha_{n+1}, t) \leq 1$, for each $n \in \mathbb{N}$ and $t > 0$, and hence as a consequences of the condition (e'), we get

$$\sigma(\alpha_n, \gamma, t) \leq 1, \tag{7}$$

for each $n \in \mathbb{N}$ and $t > 0$. Now, we have to show that S admits a fixed point (say γ). On contrary, assume that $S\gamma \neq \gamma$. For all $n \in \mathbb{N}$. Without loss of generality one can assume that $\alpha_n \neq \gamma$ and $\alpha_n \neq S\gamma$. Then, there is $s > 0$ such that

$$\mathcal{M}(\alpha_n, \gamma, s) < 1, \ \ \mathcal{M}(\alpha_n, S\gamma, s) < 1 \ \text{and} \ \mathcal{M}(S\alpha_n, S\gamma, s) < 1, \ \forall \, n \in \mathbb{N}. \tag{8}$$

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Using (7), (8) and (2), we obtain

$$\begin{aligned}\mathcal{M}(\alpha_{n+1}, S\gamma, s) &\geq \sigma(\alpha_n, \gamma, s)\mathcal{M}(S\alpha_n, S\gamma, s)\\ &\geq \xi(\mathcal{M}(S\alpha_n, S\gamma, s), \mathcal{M}(\alpha_n, \gamma, s))\\ &> \mathcal{M}(\alpha_n, \gamma, s). \end{aligned} \tag{9}$$

Taking $n \to \infty$ and making use of (6), we obtain $\mathcal{M}(\gamma, S\gamma, s) \geq 1$, a contradiction. Therefore, for all $t > 0$, $\mathcal{M}(\gamma, S\gamma, t) = 1$, that is, γ is the fixed point of S. □

In order to support the obtained results, we provide an example. Precisely, we show that our results (Theorems 1 and 2) can be used to cover this example while [25, Theorem 3.13] is not applicable.

Example 4. *Consider* $\mathcal{K} = \{A_1 = (0,0), A_2 = (1,0), A_3 = (1,2), A_4 = (0,1), A_5 = (1,3)\} \subseteq \mathbb{R}^2$. *Define the fuzzy metric* $\mathcal{M}$ *as*

$$\mathcal{M}(\alpha, \beta, t) = e^{\frac{-2d(\alpha,\beta)}{t}}, \ \forall\, \alpha, \beta \in \mathcal{K}, t > 0,$$

where $d(\alpha, \beta)$ *be the Euclidean metric on* $\mathbb{R}^2$. *It is obvious that* $(\mathcal{K}, \mathcal{M}, *)$ *is an* $\mathcal{M}$*-complete FMS w.r.t to the product t-norm. Let* $T : \mathcal{K} \to \mathcal{K}$ *be defined by*

$$T(\alpha) = \begin{cases} A_1, & \text{if } \alpha \in \{A_1, A_3, A_4, A_5\},\\ A_5, & \text{if } \alpha = A_2. \end{cases}$$

Also define $\sigma : \mathcal{K} \times \mathcal{K} \times (0, \infty) \to (0, \infty)$ *by*

$$\sigma(\alpha, \beta, t) = \begin{cases} e^{\frac{2}{t}(2\sqrt{10}-3)}, & \text{if } \alpha = A_2 \,\&\, \beta \in \{A_1, A_3, A_4, A_5\}\\ & \text{or } \alpha = \beta = A_2\\ 1 & \text{otherwise,} \end{cases}$$

and $\xi : (0, 1] \times (0, 1] \to \mathbb{R}$ *by*

$$\xi(l, s) = \frac{s}{l}, \text{ for all } l, s \in (0, 1].$$

For all $\alpha, \beta \in \mathcal{K}$, *we have*

$$\mathcal{M}(T\alpha, T\beta, t) = \begin{cases} e^{\frac{-2}{t}(\sqrt{10})}, & \text{if } \alpha = A_2 \,\&\, \beta \in \{A_1, A_3, A_4, A_5\}\\ & \text{or } \alpha \in \{A_1, A_3, A_4, A_5\} \,\&\, \beta = A_2\\ 1 & \text{otherwise.} \end{cases}$$

Let $\alpha, \beta \in \mathcal{K}$ *such that* $\sigma(\alpha, \beta, t) \leq 1$. *Then* $\alpha, \beta \in \{A_1, A_3, A_4, A_5\}$, *and by the definition of* T, *we conclude that* $T\alpha = T\beta \in \{A_1, A_3, A_4, A_5\}$, *and hence* $\sigma(T\alpha, T\beta) = 1$. *Therefore, the mapping* T *is* σ*-admissible. Also,* $A_4 \in \mathcal{K}$ *and* $\sigma(A_4, TA_4, t) = \alpha(A_4, A_1, t) = 1$, *for each* $t > 0$.

Further, let $\{\alpha_n\} \subseteq \mathcal{K}$ *such that* $\lim_{n\to\infty} \alpha_n = \alpha$ *with* $k_0 = 1$ *and* $\sigma(\alpha_n, \alpha_{n+1}, t) \leq 1$, *for each* $n \in \mathbb{N}$. *It follows that* $\alpha_n \in \{A_1, A_3, A_4, A_5\}$, *for each* $n \in \mathbb{N}$. *if we assume that* $\alpha = A_2$, *then we get*

$$\mathcal{M}(\alpha_n, \alpha, t) = e^{\frac{-2d(\alpha_n,\alpha)}{t}} < 1, \ \text{ for all } t > 0,$$

which is a contradiction to the assumption that $\lim_{n\to\infty} \alpha_n = \alpha$. *Thus, we have* $\alpha \in \{A_1, A_3, A_4, A_5\}$. *Therefore,* $\sigma(\alpha_n, \alpha, t) \leq 1$ *and* $\sigma(\alpha_n, \alpha_m, t) \leq 1$ *for all* $m, n \in \mathbb{N}$ *and* $t > 0$. *Also, it is obvious that the quadruple* $(\mathcal{K}, \mathcal{M}, T, \xi)$ *has the property* (S).

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Finally, to show that T is a fuzzy $(\sigma, \mathcal{Z})$-contractive, we only need to consider the case $\alpha = A_2$ and $\beta \in \{A_1, A_3, A_4, A_5\}$. In this case $\sigma(\alpha, \beta, t) = e^{\frac{2}{t}(2\sqrt{10}-1)}$, and hence

$$\begin{aligned} e^{\frac{2}{t}(\sqrt{10}-3)} = e^{\frac{2}{t}(2\sqrt{10}-3)} \cdot e^{\frac{-2}{t}(\sqrt{10})} &= \sigma(\alpha,\beta,t)\mathcal{M}(T\alpha,T\beta,t) \\ &\geq \xi(\mathcal{M}(T\alpha,T\beta,t),\mathcal{M}(\alpha,\beta,t)) \\ &= \frac{\mathcal{M}(\alpha,\beta,t)}{\mathcal{M}(T\alpha,T\beta,t)} = \frac{e^{\frac{-6}{t}}}{e^{\frac{-2}{t}(\sqrt{10})}} = e^{\frac{2}{t}(\sqrt{10}-3)}, \end{aligned}$$

which shows that T is a fuzzy $(\sigma, \mathcal{Z})$-contractive. Therefore, all the hypothesis of Theorem 1 are satisfied. This insure that the mapping T admits a fixed point (namely $x = A_1$).

However, T is not a fuzzy $\mathcal{Z}$-contractive mapping. On contrary, we assume T is fuzzy $\mathcal{Z}$-contractive w.r.t to some $\xi \in \mathcal{Z}$. Take $\alpha = A_2$ and $\beta = A_4$. As $\mathcal{M}(\alpha, \beta, t) = e^{\frac{-2\sqrt{2}}{t}} \in (0,1)$ and $\mathcal{M}(T\alpha, T\beta, t) = e^{\frac{-2\sqrt{10}}{t}} \in (0,1)$. From the contractive condition and the definition of ξ, we have

$$\begin{aligned} e^{\frac{-2\sqrt{10}}{t}} = \mathcal{M}(T\alpha,T\beta,t) &\geq \xi(\mathcal{M}(T\alpha,T\beta,t),\mathcal{M}(\alpha,\beta,t)) \\ &> \mathcal{M}(\alpha,\beta,t) = e^{\frac{-2\sqrt{2}}{t}}, \end{aligned}$$

for all $t > 0$, which is a contradiction. Hence, T is not a fuzzy $\mathcal{Z}$-contractive mapping.

In Theorems 1 and 2, in order to insure the uniqueness of the fixed point, we add one more sufficient condition to the hypothesis of the theorems. precisely, we take into account the following condition:

(h) for each $\alpha, \beta \in Fix(\mathcal{K})$, then $\sigma(\alpha, \beta, t) \leq 1$, for all $t > 0$.

Theorem 3. *In addition to the hypothesis of Theorems 1 and 2, assume that the condition (h) is held. Then the fixed point of T is unique.*

Proof. Theorems 1 and 2 insure the existence of a fixed point of T. Assume that γ_1 and γ_2 are two such that $T\gamma_1 = \gamma_1 \neq \gamma_2 = T\gamma_2$. Then, there exists $s > 0$ such that $\mathcal{M}(\gamma_1, \gamma_2, s) < 1$.
As T is a fuzzy $(\sigma, \mathcal{Z})$-contractive mapping, in view of the definition of ξ and condition (h), we have

$$\mathcal{M}(\gamma_1,\gamma_2,t) \geq \sigma(\gamma_1,\gamma_2,t)\mathcal{M}(\gamma_1,\gamma_2,t) \geq \xi(\mathcal{M}(\gamma_1,\gamma_2,t),\mathcal{M}(\gamma_1,\gamma_2,t)) > \mathcal{M}(\gamma_1,\gamma_2,t),$$

a contradiction. Therefore, $\mathcal{M}(\gamma_1, \gamma_2, t) = 1$, for all $t > 0$, that is, $\gamma_1 = \gamma_2$. □

Remark 4. *Observe that the mapping defined in Examples 4 satisfies the condition (h), and hence according to Theorem 3, T admits a unique fixed point (namely $x = A_1$).*

Corollary 1. *[25] Let $(\mathcal{K}, \mathcal{M}, *)$ be an $\mathcal{M}$-complete FMS and $S : \mathcal{K} \to \mathcal{K}$. Assume that S be a fuzzy $\mathcal{Z}$-contractive and the quadruple $(\mathcal{K}, \mathcal{M}, S, \xi)$ owns the property (S). Then S possesses a unique fixed point.*

Proof. The result follows from Remark 2 and Theorem 3. □

4. Conclusions

We enlarge the class of fuzzy $\mathcal{Z}$-contractive mappings introduced in [25], by introducing the family of fuzzy $(\sigma, \mathcal{Z})$-contractive mappings. The initiated class of mapping cover all concepts introduced in [7–9,25,26]. Our newly introduced notion has been utilized to prove some results in $\mathcal{M}$-complete fuzzy metric spaces. Finally, an example was adopted to demonstrate that our newly presented results are a proper extension of Shukla et al. [25] results.

 doi:10.20944/preprints202108.0577.v1

Author Contributions: All the authors contributed equally and significantly in writing this article. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the Department of the second author. .

Institutional Review Board Statement: : Not applicable.

Informed Consent Statement: : Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The first author is thankful to Taiz University, Government of Yemen for awarding Ph.D Scholarships.

Conflicts of Interest: The authors declare no conflicts of interest.

Abbreviations

The following abbreviations are used in this manuscript:

FMS   Fuzzy Metric Space

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