 doi:10.20944/preprints202108.0574.v1

Article

Variational Anisotropic Gradient-Domain Image Processing

Ivar Farup [1]

[1] Department of Computer Science, Norwegian University of Science and Technology (NTNU), Gjøvik, Norway; ivar.farup@ntnu.no

Abstract: Gradient-domain image processing is a technique where, instead of operating directly on the image pixel values, the gradient of the image is computed and processed. The resulting image is obtained by reintegrating the processed gradient. This is normally done by solving the Poisson equation, most oftenly by means of a finite difference implementation of the gradient descent method. However, this technique in some cases lead to severe haloing artefacts in the resulting image. To deal with this, local or anisotropic diffusion has been added as an ad-hoc modification of the Poisson equation. In this paper, we show that a version of anisotropic gradient-domain image processing can result from a more general variational formulation through the minimisation of an action potential formulated in terms of the eigenvalues of the structure tensor of the differences between the processed gradient and the gradient of the original image. An example application of local contrast enhancement illustrates the behaviour of the method.

Keywords: variational methods; anisotropic diffusion; gradient-domain image processing; local contrast enhancement

1. Introduction and Background

In 2002, Fattal et al. [1] introduced the method of gradient-domain high-dynamic-range compression. The technique consisted in first computing the gradient field $u^\rho_{0,i}$ of a high-dynamic-range image $u^\rho_0 : \Omega \to \mathcal{C}$, where $\Omega \subset \mathbb{R}^2$ is the image domain and $\mathcal{C} \subset \mathbb{R}^3$ is the colour space.[1] The gradient was then rescaled non-linearly as $v^\rho_i = f(u^\rho_{0,i})$. The resulting vector field v^ρ_i, which is no longer necessarily a gradient field, was then reintegrated by solving the Poisson equations

$$u^\rho_{,ii} = v^\rho_{i,i} \tag{1}$$

to obtain the compressed image. This is often solved by gradient descent,

$$\frac{\partial u^\rho}{\partial t} = u^\rho_{,ii} - v^\rho_{i,i} \tag{2}$$

The Poisson equations can be obtained through a variational approach by minimising the action functionals

$$E(u^\rho) = \frac{1}{2}\int_\Omega (u^\rho_{,i} - v^\rho_i)(u^\rho_{,i} - v^\rho_i)\, d\Omega = \int_\Omega \mathcal{L}(u^\rho, u^\rho_{,i})\, d\Omega \tag{3}$$

where $\Omega \subset \mathbb{R}^2$ is the image domain. This is done by solving the corresponding Euler–Lagrange equations,

$$\partial_i \left(\frac{\partial \mathcal{L}}{\partial u^\rho_{,i}} \right) - \frac{\partial \mathcal{L}}{\partial u^\rho} = 0 \tag{4}$$

[1] We will use a component notation throughout. Greek superscripts will denote coordinates in the colour space, and Latin subscripts will denote spatial coordinates in the image domain. A comma before an index denotes partial differentiation with respect to that component. Einstein's summation convention, i.e., summation over repeated indices, will be applied unless otherwise specified.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 31 August 2021 doi:10.20944/preprints202108.0574.v1

leading to Equation (1).

Perez et al. [2] soon generalised this into the technique called *Poisson Image Editing*, and showed that it allowed for a broad range of image processing applications such as image inpainting, seamless cloning, texture transfer, feature exchange, insertion of transparent objects through gradient mixing, texture flattening, local illumination correction, and seamless tiling. All this is obtained by various processing of the gradient or gradients of the original images to obtain the vector field v_i^ρ used in Equation (1).

One significant challenge of Poisson Image Editing is that it tends to produce visual haloing or blurring artefacts in the reconstructed images. The case of $v_i^\rho = 0$ in Equation (1) is often used for image denoising. For greyscale images, in order to stop the diffusion at edges, and thus reduce the blurring, Perona and Malik [3] introduced an image-dependent, local, non-linear diffusion method[2] described by the equation

$$\frac{\partial u}{\partial t} = \nabla \cdot (D(s)\nabla u) = \partial_i(D(s)u_{,i}) \tag{5}$$

where $s = |\nabla u|^2$ describing the image 'structure' has been introduced. They proposed two different diffusion coefficients with different properties,

$$D(s) = \exp\left(-\frac{s}{K^2}\right) \tag{6}$$

$$D(s) = \frac{1}{1 + s/K^2} \tag{7}$$

Instead of designing PDEs directly, Rudin et al. [4] used a variational approach like Equation (3) and introduced the concept of total variation. They showed that the minimisation of the action functional

$$E = \int_\Omega |\nabla u|\, d\Omega \tag{8}$$

leads to the PDE

$$\frac{\partial u}{\partial t} = \nabla \cdot \left(\frac{\nabla u}{|\nabla u|}\right) \tag{9}$$

Comparing with the Perona–Malik diffusion, Equation (5), we see that it can be written in the same form choosing

$$D(s) = \frac{1}{\sqrt{s}} \tag{10}$$

For total variation and Perona–Malik diffusion, the extension to colour images is not that straight forward. Blomgren and Chan [5] introduced the concept of colour total variation by using an action functional as an ℓ_2 norm of the action functionals in Equation (8) for each of the colour channels resulting in

$$E = \sqrt{\sum_i E_i^2} \tag{11}$$

where E_i is the total variation for each colour channel according to Equation (8).

Later approaches have been based on the structure tensor by Di Zenzo [6] and Bigun and Granlund [7], with components

$$s_{ij} = u^\mu_{,i} u^\mu_{,j} \tag{12}$$

[2] Despite the fact that the method is isotropic (but non-linear and local), the authors termed it 'anisotropic diffusion' in the original publication [3]. This has caused considerable confusion in the terminology in the following literature.

 doi:10.20944/preprints202108.0574.v1

Saprio and Ringach [8] proposed to use the eigenvalues

$$\lambda_{\pm} = \frac{1}{2}\left(s_{11} + s_{22} \pm \sqrt{(s_{11} - s_{22})^2 + 4s_{12}^2}\right) \tag{13}$$

and the corresponding eigenvectors $\theta_{\pm}$ of the structure tensor as a basis for constructing the diffusion equations. In terms of these eigenvalues, an alternative to the colour total variation by Blomgren and Chan [5] can be obtained by the action functional

$$E = \int_{\Omega} \sqrt{\lambda_+ + \lambda_-}\, d\Omega = \int_{\Omega} \sqrt{s}\, d\Omega \tag{14}$$

For greyscale images, where $\lambda_+ = |\nabla u|^2$ and $\lambda_- = 0$, this reduces to total variation. For colour images, the corresponding Euler–Lagrange equations become

$$\frac{\partial u^{\mu}}{\partial t} = \partial_i \left(\frac{u^{\mu}_{,i}}{\sqrt{u^{\nu}_{,j} u^{\nu}_{,j}}} \right) \tag{15}$$

which again is on the form of Equation (5) with $D(s) = 1/\sqrt{s} = 1/\sqrt{\lambda_+ + \lambda_-} = 1/\sqrt{s_{11} + s_{22}} = 1/\sqrt{u^{\nu}_{,j} u^{\nu}_{,j}}$. Notice the coupling between the colour channels introduced by the sum in the denominator of Equation (15).

Tschumperlé and Deriche [9] extended this approach to a *anisotropic* diffusion by introducing the general Lagrangian density $\psi(\lambda_+, \lambda_-)$ in the action functional

$$E = \int_{\Omega} \psi(\lambda_+, \lambda_-)\, d\Omega \tag{16}$$

The corresponding Euler–Lagrange equations are

$$\frac{\partial u^{\mu}}{\partial t} = \partial_k (D_{kl} u^{\mu}_{,l}) \tag{17}$$

where D_{kl} are the components of the diffusion tensor

$$D_{kl} = 2 \frac{\partial \psi}{\partial \lambda_p} \theta_{pk} \theta_{pl} \tag{18}$$

where the index p is introduced for summing over the two eigenvalues of the difference structure tensor (despite the fact that they, technically speaking, are not coordinates), and $\theta_{\pm}$ are the eigenvectors of the structure sensor s_{ij}. It should be noted that this actually encompasses all previously presented diffusion methods as follows:

$$\psi(\lambda_+, \lambda_-) = \sqrt{\lambda_+ + \lambda_-} \tag{19}$$

gives the solution of Sapiro and Ringach [8] for colour images and total variation of Rudin et al. [4] for greyscale images,

$$\psi(\lambda_+, \lambda_-) = -K^2 \exp(-s/K^2) \text{ and} \tag{20}$$

$$\psi(\lambda_+, \lambda_-) = K^2 \ln(1 + s/K^2) \tag{21}$$

give the two equations of Perona and Malik [3], and

$$\psi(\lambda_+, \lambda_-) = s/2 \tag{22}$$

gives the classical linear diffusion, Equation (2).

Choosing

$$\psi(\lambda_+, \lambda_-) = \phi(\lambda_+ + \lambda_-) = \phi(s) \tag{23}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 31 August 2021 doi:10.20944/preprints202108.0574.v1

in general leads to isotropic equations, since then $\partial\psi/\partial\lambda_+ = \partial\psi/\partial\lambda_- = \phi'(s)$ and the diffusion tensor in Equation (18) reduces to the scalar diffusion coefficient $D = 2\phi'(s)$. With

$$\psi(\lambda_+, \lambda_-) = \phi^+(\lambda_+) + \phi^-(\lambda_-) \tag{24}$$

the diffusion in the mutually orthogonal directions of maximal and minimal change can be controlled independently, like used by, e.g., Farup [10].

For other application of Poisson image editing, i.e., applications of Equation (2) where $v_i^\rho \neq 0$, extensions to anisotropic and edge-preserving methods have been obtained by ad-hoc modifications of Equation (2). This has been done for e.g., colour gamut mapping [11], colour image demosaicing [12], colour-to-greyscale conversion [13], and colour image daltonisation [10]. Common to these is that they solve an equation on the form

$$\frac{\partial u^\rho}{\partial t} = \partial_k (D_{kl}(u^\rho_{,l} - v^\rho_l)) \tag{25}$$

Where D_{kl} are the components of a diffusion tensor constructed from the structure tensor like, e.g., Equation (18).

However, a unifying variational formulation of anisotropic gradient-domain image processing has not yet been given. In this paper, we provide this by introducing the *difference structure tensor* based on the difference of the original image gradient $u^\rho_{,i}$ and the vector field v_i^ρ and follow the process of Tschumperlé and Deriche [9]. The resulting PDE is similar, but not identical, to the ones obtained by the ad-hoc modification of Poisson Image editing, Equation (25). For the application to local contrast enhancement, we show that the two approaches gives very similar results.

2. Variational Anisotropic Gradient-Domain Formulation

The variational formulation of anisotropic diffusion by Tschumperlé and Deriche [9] was based on the structure tensor s_{ij}, Equation (18). In analogy, we introduce the *difference structure tensor*,

$$s'_{ij} = (u^\mu_{,i} - v^\mu_i)(u^\mu_{,j} - v^\mu_j) \tag{26}$$

with eigenvalues

$$\lambda'_\pm = \frac{1}{2}\left(s'_{11} + s'_{22} \pm \sqrt{(s'_{11} - s'_{22})^2 + 4{s'_{12}}^2}\right) \tag{27}$$

and corresponding eigenvectors $\theta'_\pm$.

In analogy with Tschumperlé and Deriche [9], we define the action potential in terms of the eigenvalues of this difference structure tensor,

$$E = \int_\Omega \psi(\lambda'_+, \lambda'_-)\, d\Omega \tag{28}$$

The corresponding Euler-Lagrange equations are

$$\partial_i \left(\frac{\partial\psi}{\partial u^\rho_{,i}}\right) - \frac{\partial\psi}{\partial u^\rho} = 0 \tag{29}$$

It is clear that $\partial\psi/\partial u^\rho = 0$. To derive the explicit form of the Euler–Lagrange equations, we will need

$$\frac{\partial\psi}{\partial u^\rho_{,i}} = \frac{\partial\psi}{\partial\lambda'_p}\frac{\partial\lambda'_p}{\partial s'_{kl}}\frac{\partial s'_{kl}}{\partial u^\rho_{,i}} \tag{30}$$

where the index p is again used for summing over the two eigenvalues of the difference structure tensor. The first factor, $\partial\psi/\partial\lambda'_p$, can be computed directly since the Lagrangian ψ is defined explicitly in terms of the eigenvalues of the difference structure tensor, and will thus depend on the design of the Lagrangian density.

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 31 August 2021 doi:10.20944/preprints202108.0574.v1

The second factor can be computed implicitly following the method of Tschumperlé and Deriche [9] as follows

$$\begin{aligned}\delta_i^k \delta_j^l &= \frac{\partial s'_{kl}}{\partial s'_{ij}} \\ &= \frac{\partial \lambda'_p}{\partial s'_{ij}} \theta'_{pk} \theta'_{pl} + \lambda'_p \frac{\partial \theta'_{pk}}{\partial s'_{ij}} \theta'_{pl} + \lambda'_p \theta'_{pk} \frac{\partial \theta'_{pl}}{\partial s'_{ij}}\end{aligned} \tag{31}$$

Multiplying with θ'_{mk} and θ'_{ml}, summing over the k and l indexes, using that $\theta'_{mk}\theta'_{pk} = \delta_{pm}$, and $\theta'_{pk}(\partial\theta'_{pk}/\partial s'_{ij}) = 0$, the latter due to the orthonormality of the θ's gives

$$\frac{\partial \lambda_p}{\partial s'_{kl}} = \theta'_{pk}\theta'_{pl} \tag{32}$$

(no sum).

The last term, $\partial s'_{kl}/\partial u^\rho_{,i}$ deviates from the derivation of Tschumperlé and Deriche due to the definition of the difference structure tensor, Equation (26),

$$\frac{\partial s'_{kl}}{\partial u^\rho_{,i}} = \delta_{ik}(u^\rho_{,l} - v^\rho_l) + \delta_{il}(u^\rho_{,k} - v^\rho_k) \tag{33}$$

Inserted into Equation (30) and exploiting the symmetry of Equation (32) gives

$$\frac{\partial \psi}{\partial u^\rho_{,k}} = 2\frac{\partial \psi}{\partial \lambda'_p}\theta'_{pk}\theta'_{pl}(u^\rho_{,k} - v^\rho_k) = D'_{kl}(u^\rho_{,k} - v^\rho_k) \tag{34}$$

where the diffusion tensor

$$D'_{kl} = 2\frac{\partial \psi}{\partial \lambda'_p}\theta'_{pk}\theta'_{pl} \tag{35}$$

has been defined.

Inserting this into Equation (29) and solving by gradient descent, gives the variational anisotropic gradient-domain image processing PDE:

$$\frac{\partial u^\rho}{\partial t} = \partial_k\left(D'_{kl}(u^\rho_{,k} - v^\rho_k)\right) \tag{36}$$

The form of this equation is similar to the one based on the ad-hoc approach, Equation (25). The only difference is that, in this case, the diffusion tensor is computed from the difference structure tensor, and not the common structure tensor.

3. Example Application

To demonstrate the behaviour of the solution, we will use the method for gradient-domain contrast enhancement, since this is a technique that is particularly prone to haloing problems. We follow a simple approach and define the contrast enhancement as a scalar multiplication of the original image gradient

$$v^\rho_i = a u^\rho_{0,i} \tag{37}$$

where $a > 1$.

We compare this with the standard Poisson method, Equation (2), and the ad-hoc addition with the diffusion tensor derived from the structure tensor instead of the difference structure tensor, Equation (18). We use $a = 4$ in order to obtain a quite extreme contrast enhancement, and use the Perona–Malik motivated functional

$$\psi(\lambda'_+, \lambda'_-) = K^2 \ln(1 + \lambda'_+/K^2) + K^2 \ln(1 + \lambda'_-/K^2) \tag{38}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 31 August 2021 doi:10.20944/preprints202108.0574.v1

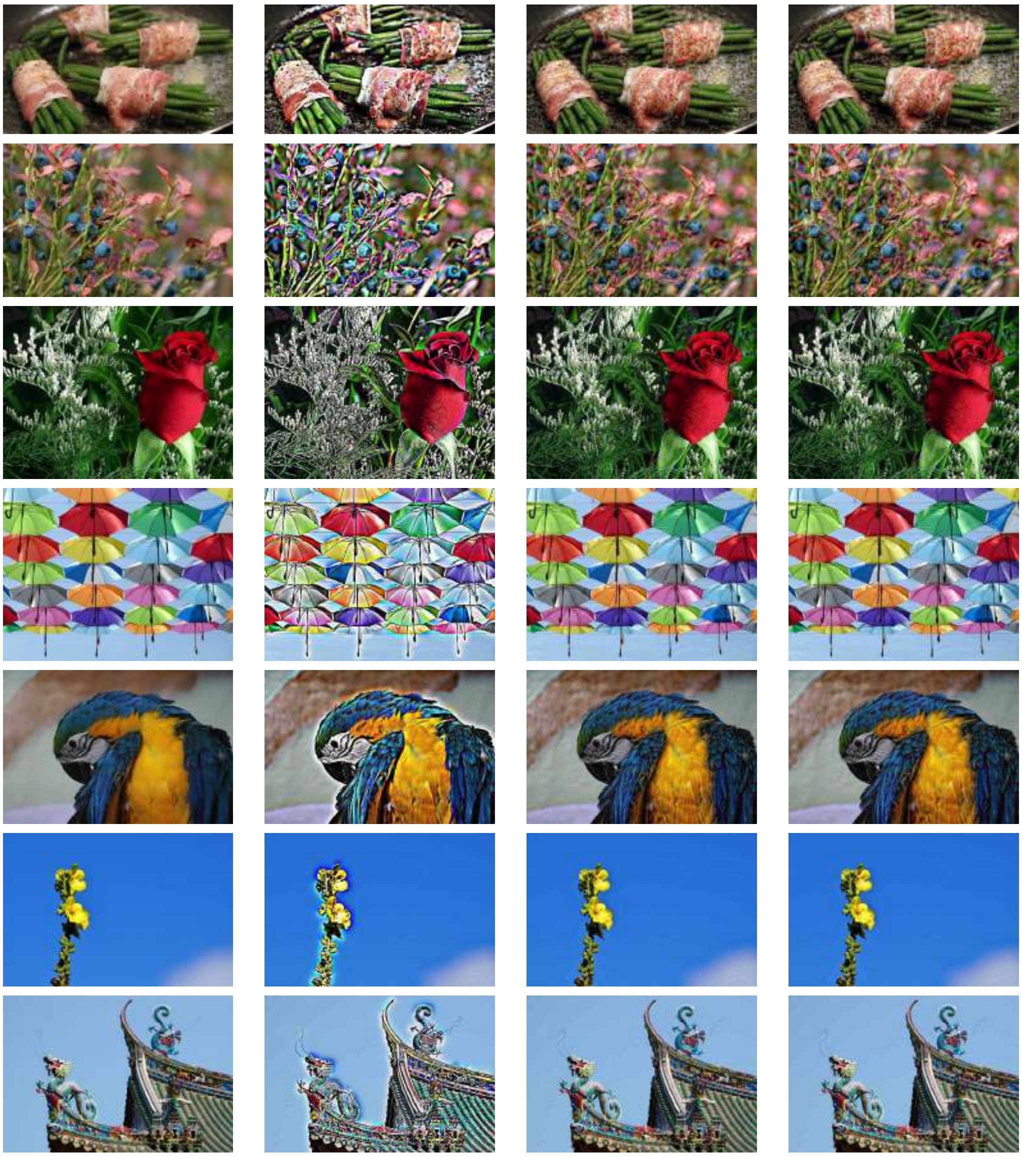

Figure 1. Example results for contrast enhancement with $a = 4$. Original images in the first column, Poisson solution in the second, ad-hoc anisotropic diffusion with $K = 10^{-3}$ in the third, and the proposed variational with $K = 10^{-2}$ in the fourth. All images are available under the CC0 licence.

with $K = 10^{-2}$ for the variational approach based on the difference structure tensor, and $K = 10^{-3}$ for the ad-hoc solution based on the standard structure tensor. Difference constants are needed, since the scaling of the structure tensor will be different in the two cases.

The methods are implemented with a simple finite difference scheme with explicit time integration.[3] Example results are shown in Figures 1 and 2. We can see that with this choice of K, the results of the variational and the ad-hoc approaches are indistinguishable, whereas the Poisson solutions exhibit severe haloing artefacts, as expected.

4. Conclusion

In this paper, a variational formulation of anisotropic gradient-domain image processing has been introduced. It generalises previous formulations of anisotropic processing and also introduces the difference structure tensor. The resulting PDE deviates somewhat from

[3] The code is available at https://github.com/ifarup/variational-anisotropic-gradient-domain.

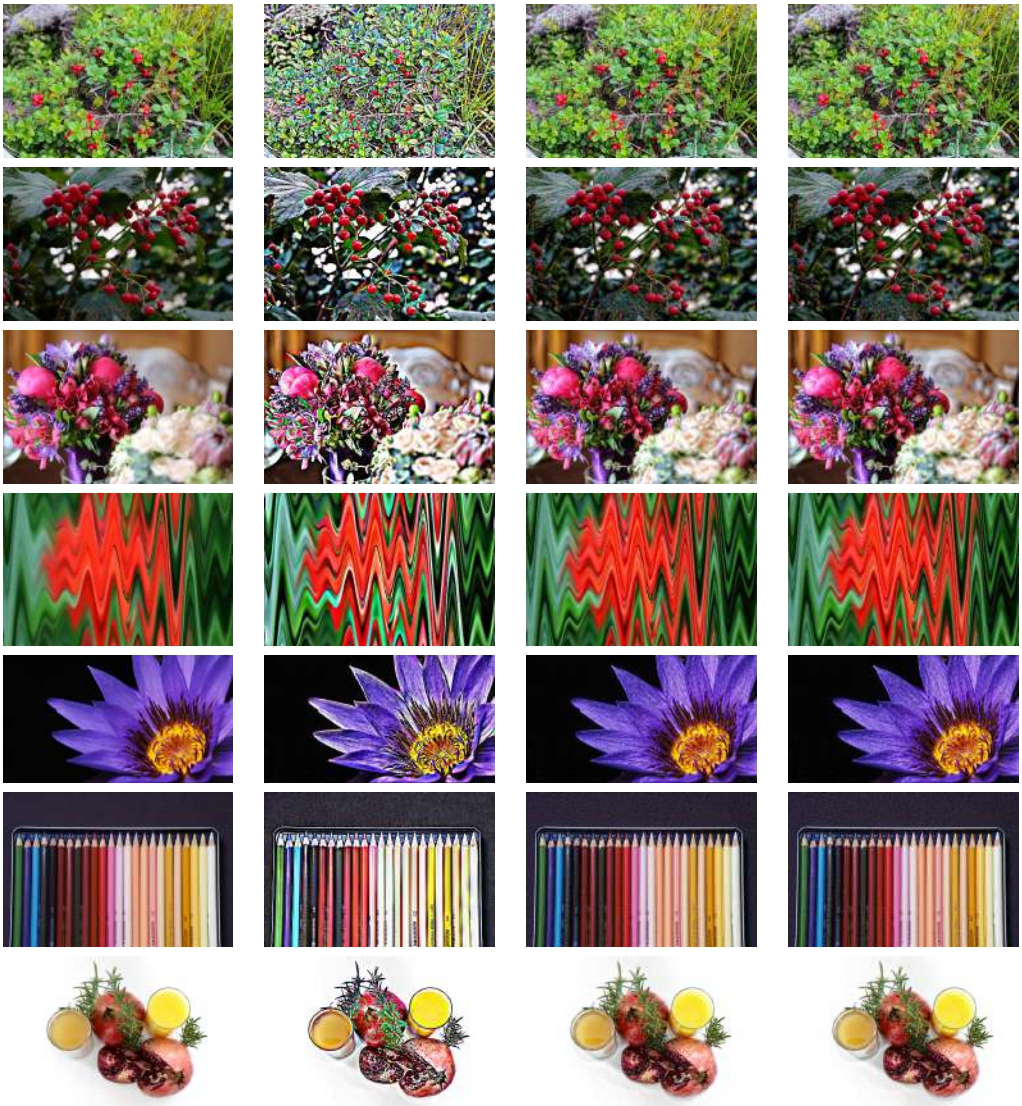

Figure 2. Example results for contrast enhancement with $a = 4$. Original images in the first column, Poisson solution in the second, ad-hoc anisotropic diffusion with $K = 10^{-3}$ in the third, and the proposed variational with $K = 10^{-2}$ in the fourth. All images are available under the CC0 licence.

the one usually found when anisotropic diffusion is added in and ad-hoc manner to do gradient-domain image processing. The difference is that the diffusion tensor is based on the difference structure tensor rather than the conventional structure tensor. The example application of local contrast enhancement illustrates that the behaviour is very similar to what is obtained using the conventional approach. This indicates that the proposed variational formulation is well suited for rigorous derivations of anisotropic gradient-domain image processing for a broad range of applications.

Funding: This research was funded by the Research Council of Norway over the project 'Individualised Color Vision-based Image Optimisation', grant number 287209.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: The code is available at https://github.com/ifarup/variational-anisotropic-gradient-domain.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Fattal, R.; Lischinski, D.; Werman, M. Gradient domain high dynamic range compression. *ACM Transactions on Graphics (TOG)* **2002**, *21*, 249–256.
2. Pérez, P.; Gangnet, M.; Blake, A. Poisson image editing. *ACM Transactions on Graphics* **2003**, *22*, 313–318.
3. Perona, P.; Malik, J. Scale-space and edge detection using anisotropic diffusion. *IEEE Transactions on Pattern Analysis and Machine Intelligence* **1990**, *12*, 629–639.
4. Rudin, L.I.; Osher, S.; Fatemi, E. Nonlinear total variation based noise removal algorithms. *Physica D: nonlinear phenomena* **1992**, *60*, 259–268.
5. Blomgren, P.; Chan, T.F. Color TV: total variation methods for restoration of vector-valued images. *IEEE T. Image Process.* **1998**, *7*, 304–309.
6. Di Zenzo, S. A note on the gradient of a multi-image. *Computer Vision, Graphics, and Image Processing* **1986**, *33*, 116–125. doi:https://doi.org/10.1016/0734-189X(86)90223-9.
7. Bigun, J.; Granlund, G. Optimal orientation detection of linear symmetry. First int. Conf. on Computer Vision, ICCV, (London); IEEE Computer Society Press, , 1987; p. 433–438.
8. Sapiro, G.; Ringach, D.L. Anisotropic Diffusion of Multivalued Images with Applications to Color Filtering. *IEEE T. Image Process.* **1996**, *5*, 1582–1586.
9. Tschumperlé, D.; Deriche, R. Vector-valued image regularization with PDEs: A common framework for different applications. *IEEE Transactions on Pattern Analysis and Machine Intelligence* **2005**, *27*, 506–517.
10. Farup, I. Individualised Halo-Free Gradient-Domain Colour Image Daltonisation. *Journal of Imaging* **2020**, *6*, 116.
11. Alsam, A.; Farup, I. Spatial Colour Gamut Mapping by Means of Anisotropic Diffusion. Computational Colour Imaging Workshop (CCIW); Springer: Berlin, 2011; Vol. 6626, *Lecture Notes in Computer Science*, pp. 113–124.
12. Thomas, J.B.; Farup, I. Demosaicing of Periodic and Random Colour Filter Arrays by Linear Anisotropic Diffusion. *Journal of Imaging Science and Technology* **2018**, *62*, 050401–1–050401–8.
13. Farup, I.; Pedersen, M.; Alsam, A. Colour-to-Greyscale Image Conversion by Linear Anisotropic Diffusion of Perceptual Colour Metrics. Colour and Visual Computing Symposium. IEEE, 2018.