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ON GRADED WEAKLY S-PRIME IDEALS

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ABSTRACT. Let R be a commutative graded ring with unity, S be a multiplicative subset of homogeneous elements of R and P be a graded ideal of R such that $P\bigcap S=\emptyset$. In this article, we introduce several results concerning graded S-prime ideals. Then we introduce the concept of graded weakly S-prime ideals which is a generalization of graded weakly prime ideals. We say that P is a graded weakly S-prime ideal of R if there exists $s\in S$ such that for all $x,y\in h(R)$, if $0\neq xy\in P$, then $sx\in P$ or $sy\in P$. We show that graded weakly S-prime ideals have many acquaintance properties to these of graded weakly prime ideals.

1. Introduction

Throughout this article, G will be a group with identity e and R be a commutative ring with nonzero unity 1. Then R is called G-graded if $R=\bigoplus_{g\in G} R_g$ with $R_gR_h\subseteq R_{gh}$ for all $g,h\in G$ where R_g is an additive subgroup of R for all $g\in G$. The elements of R_g are

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called homogeneous of degree g. If $a \in R$, then a can be written uniquely as $\sum_{g\in G} a_g$, where a_g is the component of a in R_g. The component R_e is a subring of R and $1 \in R_e$. The set of all homogeneous elements of R is $h(R) = \bigcup_{g\in G} R_g$. Let P be an ideal of a graded ring R. Then P is called a graded ideal if $P = \bigoplus_{g\in G}(P \cap R_g)$, i.e., for $a \in P$, $a = \sum_{g\in G} a_g$ where $a_g \in P$ for all $g \in G$. It is not necessary that every ideal of a graded ring is a graded ideal. For more details and terminology, see [6, 7].

Let P be a proper graded ideal of R. Then the graded radical of P is denoted by $Grad(P)$ and it is defined as follows:

$$Grad(P) = \left\{ x = \sum_{g\in G} x_g \in R : \text{ for all } g \in G, \text{ there exists } n_g \in \mathbb{N} \text{ such that } x_g^{n_g} \in P \right\}.$$

Note that $Grad(P)$ is always a graded ideal of R (see [9]).

A proper graded ideal P of R is said to be graded prime if for $x, y \in h(R)$, $xy \in P$ implies $x \in P$ or $y \in P$ [9]. Graded prime ideals play a very important role in the commutative graded rings theory. There are several ways to generalize the concept of a graded prime ideal, for example, Atani in [2] defined the concept of graded weakly prime ideals; a proper graded ideal P of R is said to be a graded weakly prime ideal if whenever $0 \neq ab \in P$ for some $a, b \in h(R)$, then either $a \in P$ or $b \in P$.

Let $S \subseteq R$ be a multiplicative set and P be an ideal of R such that $P \bigcap S = \emptyset$. In [5], P is said to be an S-prime ideal of R if there exists $s \in S$ such that for all $x, y \in R$, if $xy \in P$, then $sx \in P$ or $sy \in P$. Then in [1], P is said to be a weakly S-prime ideal of R if there exists $s \in S$ such that for all $x, y \in R$, if $0 \neq xy \in P$, then $sx \in P$ or $sy \in P$.

Let R be a graded ring, $S \subseteq h(R)$ be a multiplicative set and P be a graded ideal of R such that $P \bigcap S = \emptyset$. In this article, in Section Two, motivated by [5, 10], we study the concept of graded S-prime ideals. We say that P is a graded S-prime ideal of R if there exists $s \in S$ such that for all $x, y \in h(R)$, if $xy \in P$, then $sx \in P$ or $sy \in P$. Clearly, every S-prime ideal is graded S-prime, we prove that the converse is not necessarily true (Example 2.2). Also, it is obvious that every graded prime ideal that is disjoint with S is graded S-prime, we prove that the converse is not necessarily true (Example 2.3). Note that if S consists of units of $h(R)$, then the notions of graded S-prime and graded prime ideal coincide. We show that graded S-prime ideals have many analog properties to these of graded prime ideals. In Section Three, motivated by [1], we introduce the concept of graded weakly S-prime ideals. We say that P is a graded weakly S-prime ideal of R if there exists $s \in S$ such that for all $x, y \in h(R)$, if $0 \neq xy \in P$, then $sx \in P$ or $sy \in P$. Clearly, every weakly S-prime ideal is graded weakly S-prime, we prove that the converse is not necessarily true (Example 3.2). Also, it is obvious that every graded weakly prime ideal that is disjoint with S is graded weakly S-prime, we prove that the converse is not necessarily true (Example 3.4). Note that if S consists of units of $h(R)$, then the notions of graded weakly S-prime and graded weakly prime ideal coincide. Also, it is evident that every graded S-prime ideal is graded weakly S-prime, we prove that the converse is not necessarily true (Example 3.3). We show that graded weakly S-prime ideals have many acquaintance properties to these of graded weakly prime ideals.

2. Graded S-Prime Ideals

In this section, motivated by [5, 10], we study the concept of graded S-prime ideals. We prove that graded S-prime ideals have many analog properties to these of graded prime ideals.

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Definition 2.1. *([10]) Let R be a graded ring, $S \subseteq h(R)$ be a multiplicative set and P be a graded ideal of R such that $P \bigcap S = \emptyset$. We say that P is a graded S-prime ideal of R if there exists $s \in S$ such that for all $x, y \in h(R)$, if $xy \in P$, then $sx \in P$ or $sy \in P$.*

Clearly, every S-prime ideal is graded S-prime, but the converse is not necessarily true, see the following example:

Example 2.2. *Consider $R = \mathbb{Z}[i]$ and $G = \mathbb{Z}_2$. Then R is G-graded by $R_0 = \mathbb{Z}$ and $R_1 = i\mathbb{Z}$. Consider the graded ideal $I = 5R$ of R. We show that I is a graded prime ideal of R. Let $xy \in I$ for some $x, y \in h(R)$.*

Case (1): $x, y \in R_0$. In this case, $x, y \in \mathbb{Z}$ such that 5 divides xy, and then either 5 divides x or 5 divides y as 5 is a prime, which implies that either $x \in I$ or $y \in I$.

Case (2): $x, y \in R_1$. In this case, $x = ia$ and $y = ib$ for some $a, b \in \mathbb{Z}$ such that 5 divides $xy = -ab$, and then 5 divides ab in $\mathbb{Z}$, and again either 5 divides a or 5 divides b, which implies that either 5 divides $x = ia$ or 5 divides $y = ib$, and hence either $x \in I$ or $y \in I$.

Case (3): $x \in R_0$ and $y \in R_1$. In this case, $x \in \mathbb{Z}$ and $y = ib$ for some $b \in \mathbb{Z}$ such that 5 divides $xy = ixb$ in R, that is $ixb = 5(\alpha + i\beta)$ for some $\alpha, \beta \in \mathbb{Z}$, which gives that $xb = 5\beta$, that is 5 divides xb in $\mathbb{Z}$, and again either 5 divides x or 5 divides b, and then either 5 divides x or 5 divides $y = ib$ in R, and hence either $x \in I$ or $y \in I$.

So, I is a graded prime ideal of R. Consider the graded ideal $P = 10R$ of R and the multiplicative subset $S = \{2^n : n$ is a non-negative integer$\}$ of $h(R)$. We show that P is a graded S-prime ideal of R. Note that $P \bigcap S = \emptyset$. Let $xy \in P$ for some $x, y \in h(R)$. Then 10 divides xy in R. Then $xy \in I$, and then $x \in I$ or $y \in I$ as I is graded prime, which implies that $2x \in P$ or $2y \in P$. Therefore, P is a graded S-prime ideal of R. On the other hand, P is not an S-prime ideal of R since $3 - i, 3 + i \in R$ with $(3 - i)(3 + i) \in P$, $s(3 - i) \notin P$ and $s(3 + i) \notin P$ for each $s \in S$.

It is obvious that every graded prime ideal that is disjoint with S is graded S-prime. However, in Example 2.2, we proved that P is a graded S-prime ideal of R. On the other hand, P is not a graded prime ideal of R since $2, 5 \in h(R)$ with $2.5 \in P$, $2 \notin P$ and $5 \notin P$. Moreover, in the next example, we introduce another example on graded S-prime ideal which is not graded prime. In fact, if S consists of units of $h(R)$, then the notions of graded prime and graded S-prime ideals coincide.

Example 2.3. *Consider $R = \mathbb{Z}[X]$ and $G = \mathbb{Z}$. Then R is G-graded by $R_j = \mathbb{Z}X^j$ for $j \geq 0$ and $R_j = \{0\}$ otherwise. Consider the graded ideal $P = 9XR$ of R and the multiplicative subset $S = \{3^n : n$ is a non-negative integer$\}$ of $h(R)$. We show that P is a graded S-prime ideal of R. Note that $P \bigcap S = \emptyset$. Let $f(X)g(X) \in P$ for some $f(X), g(X) \in h(R)$. Then X divides $f(X)g(X)$, and then X divides $f(X)$ or X divides $g(X)$, which implies that $9f(X) \in P$ or $9g(X) \in P$. Therefore, P is a graded S-prime ideal of R. On the other hand, P is not a graded prime ideal of R since $3, 3X \in h(R)$ with $3.3X \in P$, $3 \notin P$ and $3X \notin P$.*

By ([10], Lemma 4.1), if I is a graded ideal of R, then $(I : a) = \{r \in R : ra \in I\}$ is a graded ideal of R for each $a \in h(R)$.

Proposition 2.4. *Let R be a graded ring, $S \subseteq h(R)$ be a multiplicative set and I be a graded ideal of R such that $I \bigcap S = \emptyset$. Then I is a graded S-prime ideal of R if and only if $(I : s)$ is a graded prime ideal of R for some $s \in S$.*

Proof. Suppose that I is a graded S-prime ideal of R. Then there exists $s \in S$ such that for any $x, y \in h(R)$, if $xy \in I$, then $sx \in I$ or $sy \in I$. We show that $(I : s)$ is a graded prime ideal of R. Let $xy \in (I : s)$ for some $x, y \in h(R)$. Then $sxy = (sx)y \in I$, and then $s^2x \in I$ or $sy \in I$. If $sy \in I$, then $y \in (I : s)$. Assume that $s^2x \in I$. Then $s^3 \in I$ or

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$sx \in I$. If $s^3 \in I$, then $s^3 \in I \bigcap S$, which is a contradiction. So, $sx \in I$, which yields that $x \in (I : s)$. Therefore, $(I : s)$ is a graded prime ideal of R. Conversely, let $xy \in I$ for some $x, y \in h(R)$. Then $sxy \in I$, and then $xy \in (I : s)$, which implies that $x \in (I : s)$ or $y \in (I : s)$, and hence $sx \in I$ or $sy \in I$. Therefore, I is a graded S-prime ideal of R. □

Let R be a G-graded ring. Then $T = R[X]$ is G-graded by $T_g = R_g[X]$ for all $g \in G$. Note that $f(x) \in T$ is a homogeneous element if all coefficients of $f(x)$ are homogeneous of the same degree in R. Clearly, an ideal I of R is graded if and only if $I[X]$ is a graded ideal of $R[X]$.

Theorem 2.5. *Let R be a graded ring, $S \subseteq h(R)$ be a multiplicative set and I be an ideal of R such that $I \bigcap S = \emptyset$. Then I is a graded S-prime ideal of R if and only if $I[X]$ is a graded S-prime ideal of $R[X]$.*

Proof. Suppose that $I[X]$ is a graded S-prime ideal of $R[X]$. Then there exists $s \in S$ such that whenever $f(X), g(X) \in h(R[X])$ with $f(X)g(X) \in I[X]$, then $sf(X) \in I[X]$ or $sg(X) \in I[X]$. Let $a, b \in h(R)$ such that $ab \in I$. We have $I \subseteq I[X]$, thus $sa \in I[X]$ or $sb \in I[X]$. So $sa \in I$ or $sb \in I$. Conversely, $(I : s)$ is a graded prime ideal of R for some $s \in S$ by Proposition 2.4, and then $(I : s)[X]$ is a graded prime ideal of $R[X]$. Let $f(X), g(X) \in h(R[X])$ such that $f(X)g(X) \in I[X]$. We have $f(X)g(X) \in I[X] \subseteq (I : s)[X]$. Therefore $f(X) \in (I : s)[X]$ or $g(X) \in (I : s)[X]$. This implies that $sf(X) \in I[X]$ or $sg(X) \in I[X]$. □

Let R be a G-graded ring and I be a graded ideal of R. Then R/I is a G-graded ring by $(R/I)_g = (R_g + I)/I$ for all $g \in G$. Moreover, if P is an ideal of R containing I, then P is a graded ideal of R if and only if P/I is a graded ideal of R/I, see ([10], Lemma 3.2).

Theorem 2.6. *Let R be a graded ring, $S \subseteq h(R)$ be a multiplicative set and I be a graded ideal of R such that $I \bigcap S = \emptyset$. Suppose that $Z(R/I) \bigcap \overline{S} = \emptyset$, where $\overline{S} = \{s + I : s \in S\}$. Then I is a graded S-prime ideal of R if and only if I is a graded prime ideal of R.*

Proof. Suppose that I is a graded S-prime ideal of R. We show that $I = (I : s)$ for each $s \in S$. Let $s \in S$. Clearly, $I \subseteq (I : s)$. Let $x \in (I : s)$. Then $x_g \in (I : s)$ for all $g \in G$ as $(I : s)$ is a graded ideal by ([10], Lemma 4.1), and then $sx_g \in I$ for all $g \in G$, which implies that $(s + I)(x_g + I) = sx_g + I = 0 + I$ for all $g \in G$. Since $Z(R/I) \bigcap \overline{S} = \emptyset$ and $s \notin I$, $x_g \in I$ for all $g \in G$, and so $x \in I$. Hence, $I = (I : s)$ for each $s \in S$, and then I is a graded prime ideal of R by Proposition 2.4. The converse is clear. □

Recall that if R is a G-graded ring and $S \subseteq h(R)$ is a multiplicative set, then $S^{-1}R$ is a G-graded ring with $(S^{-1}R)_g = \left\{\frac{a}{s}, a \in R_h, s \in S \cap R_{hg^{-1}}\right\}$ for all $g \in G$. Moreover, if I is a graded ideal of R, then $S^{-1}I$ is a graded ideal of $S^{-1}R$ [7].

Theorem 2.7. *Let R be a graded ring, $S \subseteq h(R)$ be a multiplicative set and I be a graded ideal of R such that $I \bigcap S = \emptyset$. Suppose that $Z(R) \bigcap S = \emptyset$. Then I is a graded S-prime ideal of R if and only if $S^{-1}I$ is a graded prime ideal of $S^{-1}R$ and $S^{-1}I \bigcap h(R) = (I :_{h(R)} s)$ for some $s \in S$.*

Proof. Since I is graded S-prime, there exists $s \in S$ such that whenever $a, b \in h(R)$ with $ab \in I$, then $sa \in I$ or $sb \in I$. Let $\frac{x}{s_1}\frac{y}{s_2} \in S^{-1}I$ for some $\frac{x}{s_1}, \frac{y}{s_2} \in h(S^{-1}R)$. Then $\frac{x}{s_1}\frac{y}{s_2} = \frac{z}{s_3}$ for some $z \in I$, $s_3 \in S$, and then $xys_3 = s_1s_2z \in I$. So, $sx \in I$ or $ss_3y \in I$, which yields that $\frac{x}{s_1} = \frac{sx}{ss_1} \in S^{-1}I$ or $\frac{y}{s_2} = \frac{ss_3y}{ss_3s_2} \in S^{-1}I$. Hence, $S^{-1}I$ is a graded prime ideal of $S^{-1}R$. Let $a \in (I :_{h(R)} s)$. Then $a \in h(R)$ with $sa \in I$, and then $a = \frac{sa}{s} \in S^{-1}I$. So, $a \in S^{-1}I \bigcap h(R)$, and hence $(I :_{h(R)} s) \subseteq S^{-1}I \bigcap h(R)$. Let $b \in S^{-1}I \bigcap h(R)$. Then $b \in h(R)$ and $b = \frac{w}{t}$ for some $w \in I$, $t \in S$, and then $bt = w \in I$. So, $sb \in I$ or $st \in I$.

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Since $I\bigcap S=\emptyset$, $st\notin I$, and then $sb\in I$, which gives that $b\in(I:_{h(R)}s)$. Therefore, $S^{-1}I\bigcap h(R)=(I:_{h(R)}s)$. Conversely, let $xy\in I$ for some $x,y\in h(R)$. Then $\frac{x}{1}\frac{y}{1}\in S^{-1}I$, and then $\frac{x}{1}\in S^{-1}I$ or $\frac{y}{1}\in S^{-1}I$. If $\frac{x}{1}\in S^{-1}I$, then $\frac{x}{1}=\frac{z}{t}$ for some $z\in I$, $t\in S$, and then $xt=z\in I$, which implies that $x=\frac{xt}{t}\in S^{-1}I\bigcap h(R)$. By assumption, $x\in(I:_{h(R)}s)$ for some $s\in S$, which gives that $sx\in I$. Similarly, if $\frac{y}{1}\in S^{-1}I$, then $sy\in I$. Therefore, I is a graded S-prime ideal of R. □

Example 2.8. *In Example 2.3, we proved that P is a graded S-prime ideal of R. So, by Theorem 2.7, $S^{-1}P$ is a graded prime ideal of $S^{-1}R$, while P is not a graded prime idea of R.*

Proposition 2.9. *Let R be a graded ring, $S\subseteq h(R)$ be a multiplicative set and I be a graded ideal of R such that $I\bigcap S=\emptyset$. Suppose that J is a graded ideal of R such that $J\bigcap S\neq\emptyset$. If I is a graded S-prime ideal of R, then JI is a graded S-prime ideal of R.*

Proof. By ([4], Lemma 2.1), JI is a graded ideal of R. Clearly, $JI\bigcap S=\emptyset$. Since $J\bigcap S\neq\emptyset$, there exists $t\in J\bigcap S$. Since I is a graded S-prime ideal of R, there exists $s\in S$ such that whenever $x,y\in h(R)$ with $xy\in I$, then $sx\in I$ or $sy\in I$. Let $a,b\in h(R)$ such that $ab\in JI$. Then $ab\in I$, and then $sa\in I$ or $sb\in I$, which implies that $ts\in S$ with $tsa\in JI$ or $tsb\in JI$. Hence, JI is a graded S-prime ideal of R. □

Proposition 2.10. *Let R, T be two G-graded rings with $R\subseteq T$ and $S\subseteq h(R)$ be a multiplicative set. If I is a graded S-prime ideal of T, then $I\bigcap R$ is a graded S-prime ideal of R.*

Proof. By ([4], Lemma 2.1), $I\bigcap R$ is a graded ideal of R. Clearly, $(I\bigcap R)\bigcap S=\emptyset$. Since I is a graded S-prime ideal of T, there exists $s\in S$ such that whenever $x,y\in h(T)$ with $xy\in I$, then $sx\in I$ or $sy\in I$. Let $x,y\in h(R)\subseteq h(T)$ with $xy\in I\bigcap R$. Then $xy\in I$, and then $sx\in I$ or $sy\in I$, which implies that $sx\in I\bigcap R$ or $sy\in I\bigcap R$. Hence, $I\bigcap R$ is a graded S-prime ideal of R. □

Theorem 2.11. *Let R be a graded ring, $S\subseteq h(R)$ be a multiplicative set and I be a graded ideal of R such that $I\bigcap S=\emptyset$. Then I is a graded S-prime ideal of R if and only if there exists $s\in S$ such that for all graded ideals K,J of R, if $KJ\subseteq I$, then $sK\subseteq I$ or $sJ\subseteq I$.*

Proof. Suppose that I is a graded S-prime ideal of R. Then there exists $s\in S$ such that whenever $x,y\in h(R)$ with $xy\in I$, then $sx\in I$ or $sy\in I$. Let K,J be graded ideals with $KJ\subseteq I$. Assume that $sK\nsubseteq I$. Then $sx\notin I$ for some $x\in K$, and then $sx_g\notin I$ for some $g\in G$. Note that $x_g\in K$ as K is a graded ideal. Let $y\in J$. Then $y_h\in J$ for all $h\in G$ as J is a graded ideal. For $h\in G$, $x_gy_h\in KJ\subseteq I$, and then $sy_h\in I$ for all $h\in G$, which implies that $sy\in I$, and hence $sJ\subseteq I$. Conversely, let $a,b\in h(R)$ such that $ab\in I$. Then $K=Ra$ and $J=Rb$ are graded ideals of R with $KJ\subseteq I$, and then by assumption, $sK\subseteq I$ or $sJ\subseteq I$, and hence $sa\in I$ or $sb\in I$. Thus, I is a graded S-prime ideal of R. □

Using induction, one can deduce the following from Theorem 2.11:

Corollary 2.12. *Let R be a graded ring, $S\subseteq h(R)$ be a multiplicative set and I be a graded ideal of R such that $I\bigcap S=\emptyset$. Then I is a graded S-prime ideal of R if and only if there exists $s\in S$ such that for all graded ideals $K_1,K_2,...,K_n$ of R, if $K_1.K_2...K_n\subseteq I$, then $sK_j\subseteq I$ for some $1\leq j\leq n$.*

Also, using Corollary 2.12, one can prove the following:

Proposition 2.13. *Let R be a graded ring, $S\subseteq h(R)$ be a multiplicative set and I be a graded ideal of R disjoint with S. Then I is a graded S-prime ideal of R if and only if*

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there exists $s \in S$ such that for all $x_1, x_2, ..., x_n \in h(R)$, if $x_1.x_2...x_n \in I$, then $sx_i \in I$ for some $1 \leq i \leq n$.

Applying Corollary 2.12 with $S = \{1\}$, we have the following well-known fact:

Corollary 2.14. *Let R be a graded ring and I be a proper graded ideal of R. Then I is a graded prime ideal of R if and only if for all graded ideals $I_1, I_2, ..., I_n$ of R, if $I_1.I_2...I_n \subseteq I$, then $I_j \subseteq I$ for some $1 \leq j \leq n$.*

Using Corollary 2.12, we prove the following:

Proposition 2.15. *Let R be a graded ring, $S \subseteq h(R)$ be a multiplicative set and I be a graded ideal of R such that $I \bigcap S = \emptyset$. Suppose that K is a graded ideal of R with $K \subseteq I$. If I is a graded S-prime ideal of R, then there exists $s \in S$ such that $s.Grad(K) \subseteq I$.*

Proof. Since I is a graded S-prime ideal of R, there exists $s \in S$ such that for all graded ideals $J_1, J_2, ..., J_n$ of R, if $J_1.J_2...J_n \subseteq I$, then $sJ_j \subseteq I$ for some $1 \leq j \leq n$ by Corollary 2.12. Let $a \in Grad(K)$. Then $a_g \in Grad(K)$ for all $g \in G$ as $Grad(K)$ is a graded ideal, which gives that $a_g^{n_g} \in K$ for some positive integer n_g, and then for $g \in G$, $J = Ra_g$ is a graded ideal of R with $\underbrace{J.J...J}_{n_g-\text{ times}} \subseteq I$, which implies that $sJ \subseteq I$, and then $sa_g \in I$ for all $g \in G$. So, $sa \in I$, and hence $s.Grad(K) \subseteq I$. □

Proposition 2.16. *Let $K_1, K_2, ..., K_n$ be graded S-prime ideals of R. Then there exists $s \in S$ such that $s.Grad\left(K_1 \bigcap K_2 \bigcap ... \bigcap K_n\right) \subseteq K_1 \bigcap K_2 \bigcap ... \bigcap K_n$.*

Proof. By Proposition 2.15, for each $1 \leq i \leq n$, there exists $s_i \in S$ such that $s_i.Grad(K_i) \subseteq K_i$, and then $s = s_1.s_2...s_n \in S$ with $s.Grad\left(K_1 \bigcap K_2 \bigcap ... \bigcap K_n\right) = s.\left(Grad(K_1) \bigcap Grad(K_2) \bigcap ... \bigcap Grad(K_n)\right) \subseteq K_1 \bigcap K_2 \bigcap ... \bigcap K_n$. □

Recall that if R and T are G-graded rings, then a ring homomorphism $f : R \rightarrow T$ is said to be a graded ring homomorphism if $f(R_g) \subseteq T_g$ for all $g \in G$ [7].

Proposition 2.17. *Let $f : R \rightarrow T$ be a graded ring homomorphism and $S \subseteq h(R)$ be a multiplicative set with $0_T \in f(S)$. If P is a graded $f(S)$-prime ideal of T, then $f^{-1}(P)$ is a graded S-prime ideal of R.*

Proof. By ([8], Lemma 3.11 (1)), $f^{-1}(P)$ is a graded ideal of R. Since P is a graded $f(S)$-prime ideal of T, there exists $s \in S$ such that whenever $a, b \in h(T)$ with $ab \in P$, then $f(s)a \in P$ or $f(s)b \in P$. Clearly, $f^{-1}(P) \bigcap S = \emptyset$. Let $x, y \in h(R)$ such that $xy \in f^{-1}(P)$. Then $f(x), f(y) \in h(T)$ with $f(x)f(y) = f(xy) \in P$, and then $f(s)f(x) = f(sx) \in P$ or $f(s)f(y) = f(sy) \in P$, which yields that $sx \in f^{-1}(P)$ or $sy \in f^{-1}(P)$. Hence, $f^{-1}(P)$ is a graded S-prime ideal of R. □

Proposition 2.18. *Let R be a graded ring, $S \subseteq h(R)$ be a multiplicative set and K be a graded ideal of R disjoint with S. Let I be a proper ideal of R containing K such that $(I/K) \bigcap \overline{S} = \emptyset$, where $\overline{S} = \{s + K : s \in S\}$. Then I is a graded S-prime ideal of R if and only if I/K is a graded $\overline{S}$-prime ideal of R/K.*

Proof. Suppose that I is a graded S-prime ideal of R. Then there exists $s \in S$ such that whenever $x, y \in h(R)$ with $xy \in I$, then $sx \in I$ or $sy \in I$. By ([10], Lemma 3.2), I/K is a graded ideal of R/K. Let $x + K, y + K \in h(R/K)$ such that $(x + K)(y + K) = xy + K \in I/K$. Then $x, y \in h(R)$ such that $xy \in I$, and then $sx \in I$ or $sy \in I$, which gives that $s + K \in \overline{S}$ with $(s + K)(x + K) = sx + K \in I/K$ or $(s + K)(y + K) = sy + K \in I/K$. Therefore, I/K is a graded $\overline{S}$-prime ideal of R/K. Conversely, there exists $s + K \in \overline{S}$ such that whenever $x + K, y + K \in h(R/K)$ with $(x + K)(y + K) \in I/K$, then $(s + K)(x + K) \in I/K$ or $(s + K)(y + K) \in I/K$. By ([10], Lemma 3.2), I is a graded ideal of R. Clearly,

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$I \bigcap S = \emptyset$. Let $x, y \in h(R)$ such that $xy \in I$. Then $x + K, y + K \in h(R/K)$ with $(x + K)(y + K) = xy + K \in I/K$, and then $(s + K)(x + K) = sx + K \in I/K$ or $(s + K)(y + K) = sy + K \in I/K$, which yields that $s \in S$ with $sx \in I$ or $sy \in I$. Thus, I is a graded S-prime ideal of R. □

Theorem 2.19. *Let R be a graded ring and $S \subseteq h(R)$ be a multiplicative set. Suppose that I is a graded ideal of R and $I_1, I_2, ..., I_n$ are graded S-prime ideals of R. If $I \subseteq \bigcup_{k=1}^{n} I_k$, then there exists $s \in S$ and such that $sI \subseteq I_k$ for some $1 \leq k \leq n$.*

Proof. By Proposition 2.4, for all $1 \leq k \leq n$, there exists $s_k \in S$ such that $(I_k : s_k)$ is a graded prime ideal of R. We have $I \subseteq \bigcup_{k=1}^{n} I_k \subseteq \bigcup_{k=1}^{n} (I_k : s_k)$. By the graded prime avoidance theorem ([3], Theorem 2.7), there exists $1 \leq k \leq n$ such that $I \subseteq (I_k : s_k)$, this implies that $s_k I \subseteq I_k$. □

3. Graded Weakly S-Prime Ideals

In this section, motivated by [1], we introduce the concept of graded weakly S-prime ideals. We show that graded weakly S-prime ideals have many acquaintance properties to these of graded weakly prime ideals.

Definition 3.1. *Let R be a graded ring, $S \subseteq h(R)$ be a multiplicative set and P be a graded ideal of R such that $P \bigcap S = \emptyset$. We say that P is a graded weakly S-prime ideal of R if there exists $s \in S$ such that for all $x, y \in h(R)$, if $0 \neq xy \in P$, then $sx \in P$ or $sy \in P$.*

It is evident that every weakly S-prime ideal is graded weakly S-prime, but the converse is not necessarily true, see the following example:

Example 3.2. *In Example 2.2, we proved that P is a graded S-prime ideal of R, so P is a graded weakly S-prime ideal of R. On the other hand, P is not a weakly S-prime ideal of R since $3 - i, 3 + i \in R$ with $0 \neq (3 - i)(3 + i) \in P$, $s(3 - i) \notin P$ and $s(3 + i) \notin P$ for each $s \in S$.*

Clearly, every graded S-prime ideal is graded weakly S-prime, but the converse is not necessarily true, see the following example:

Example 3.3. *Consider $R = \mathbb{Z}_{12}[i]$ and $G = \mathbb{Z}_2$. Then R is G-graded by $R_0 = \mathbb{Z}_{12}$ and $R_1 = i\mathbb{Z}_{12}$. Consider the graded ideal $P = \{0\}$ of R and the multiplicative subset $S = \{1, 3, 9\}$ of $h(R)$. Note that $P \bigcap S = \emptyset$. Clearly, P is a graded weakly S-prime ideal of R. On the other hand, P is not a graded S-prime ideal of R since $2, 6 \in h(R)$ with $2.6 \in P$, $2s \notin P$ and $6s \notin P$ for each $s \in S$.*

Also, it is obvious that every graded weakly prime ideal that is disjoint with S is graded weakly S-prime, but the converse is not necessarily true, see the following example. In fact, if S consists of units of $h(R)$, then the notions of graded weakly prime and graded weakly S-prime ideals coincide.

Example 3.4. *In Example 2.3, we proved that P is a graded S-prime ideal of R, so P is a graded weakly S-prime ideal of R. On the other hand, P is not a graded weakly prime ideal of R since $9, X \in h(R)$ with $0 \neq 9X \in P$, $9 \notin P$ and $X \notin P$.*

Proposition 3.5. *Let R be a graded ring and S be a multiplicative subset of $h(R)$. If I is a graded weakly S-prime ideal of R, then $S^{-1}I$ is a graded weakly prime ideal of $S^{-1}R$.*

Proof. Since $S\bigcap I=\emptyset$, we have that $S^{-1}I\neq S^{-1}R$. Let $0\neq \frac{x}{s_1}\frac{y}{s_2}\in S^{-1}I$ for some $x,y\in h(R)$ and $s_1,s_2\in S$. Then $\frac{x}{s_1}\frac{y}{s_2}=\frac{z}{s_3}$ for some $z\in I$ and $s_3\in S$. So, there is $t\in S$ such that $0\neq ts_3xy=ts_1s_2z\in I$. Since I is graded weakly S-prime, there exists $s\in S$ such that $sts_3\in I$ or $0\neq sxy\in I$. Thus $0\neq sxy\in I$ as $sts_3\notin I$ because $S\bigcap I=\emptyset$. Hence, $0\neq s^2x\in I$ or $sy\in I$, and so $sx\in I$ or $sy\in I$. This implies that $\frac{x}{s_1}=\frac{sx}{ss_1}\in S^{-1}I$ or $\frac{y}{s_2}=\frac{sy}{ss_2}\in S^{-1}I$. Hence, $S^{-1}I$ is a graded weakly prime ideal of $S^{-1}R$. □

Proposition 3.6. *Let R be a graded ring, $S\subseteq h(R)$ be a multiplicative set consisting of regular elements and I be a graded ideal of R such that $I\bigcap S=\emptyset$. Then I is a graded weakly S-prime ideal of R if and only if $(I:s)$ is a graded weakly prime ideal of R for some $s\in S$.*

Proof. Suppose that I is a graded weakly S-prime ideal of R. Then there exists $s\in S$ such that for any $x,y\in h(R)$, if $0\neq xy\in I$, then $sx\in I$ or $sy\in I$. We show that $(I:s)$ is a graded weakly prime ideal of R. Let $0\neq xy\in(I:s)$ for some $x,y\in h(R)$. Then $0\neq sxy=(sx)y\in I$, and then $0\neq s^2x\in I$ or $sy\in I$. If $sy\in I$, then $y\in(I:s)$. Assume that $0\neq s^2x\in I$. Then $s^3\in I$ or $sx\in I$. If $s^3\in I$, then $s^3\in I\bigcap S$, which is a contradiction. So, $sx\in I$, which yields that $x\in(I:s)$. Therefore, $(I:s)$ is a graded weakly prime ideal of R. Conversely, let $0\neq xy\in I$ for some $x,y\in h(R)$. Then $sxy\in I$, and then $0\neq xy\in(I:s)$, which implies that $x\in(I:s)$ or $y\in(I:s)$, and hence $sx\in I$ or $sy\in I$. Therefore, I is a graded weakly S-prime ideal of R. □

As one can see from the proof of Proposition 3.6, it is always true that if $(I:s)$ is a graded weakly prime ideal of R for some $s\in S$ and I is a graded ideal of R with $I\bigcap S=\emptyset$, then I is a graded weakly S-prime ideal of R. The condition that "S consisting of regular elements" was needed for the converse. The next example shows that this condition is a sufficient condition which is not necessary:

Example 3.7. *In Example 3.3, $I=\{0\}$ is a graded weakly S-prime ideal of R. One can see that $s=1\in S$ with $(I:s)=I$ is a graded weakly prime ideal of R. On the other hand, $3\in S$ is a zero divisor since $4\in R$ with $3.4=0$.*

Proposition 3.8. *Let R be a graded ring, S be a multiplicative subset of $h(R)$ consisting of regular elements, and I be a graded ideal of R disjoint with S. Then I is a graded weakly S-prime ideal of R if and only if $S^{-1}I$ is a graded weakly prime ideal of $S^{-1}R$ and there is $s\in S$ such that $(I:_{h(R)}t)\subseteq(I:_{h(R)}s)$ for all $t\in S$.*

Proof. Suppose that I is a graded weakly S-prime ideal of R. Then by Proposition 3.5, $S^{-1}I$ is a graded weakly prime ideal of $S^{-1}R$. Since I is a graded weakly S-prime ideal of R, there exists $s\in S$ such that for any $x,y\in h(R)$, if $0\neq xy\in I$, then $sx\in I$ or $sy\in I$. Let $t\in S$ and $0\neq x\in(I:_{h(R)}t)$. Then $0\neq tx\in I$. Hence, $st\in I$ or $sx\in I$. Since $I\bigcap S=\emptyset$, $st\notin I$ which implies that $sx\in I$. Consequently, $x\in(I:_{h(R)}s)$. Conversely, let $0\neq xy\in I$ for some $x,y\in h(R)$. Then $0\neq\frac{x}{1}\frac{y}{1}\in S^{-1}I$, and then $\frac{x}{1}\in S^{-1}I$ or $\frac{y}{1}\in S^{-1}I$. If $\frac{x}{1}\in S^{-1}I$, then $\frac{x}{1}=\frac{p}{t}$ for some $p\in I$ and $t\in S$, and then $xt=p\in I$, which implies that $x\in(I:_{h(R)}t)\subseteq(I:_{h(R)}s)$ for some $s\in S$ by assumption. Hence, $sx\in I$. Similarly, if $\frac{y}{1}\in S^{-1}I$, then $sy\in I$. Therefore, I is a graded weakly S-prime ideal of R. □

Proposition 3.9. *Let R be a graded ring, S be a multiplicative subset of $h(R)$ consisting of regular elements, and I be a graded ideal of R disjoint with S. Then I is a graded weakly S-prime ideal of R if and only if $S^{-1}I$ is a graded weakly prime ideal of $S^{-1}R$ and $S^{-1}I\bigcap h(R)=(I:_{h(R)}s)$ for some $s\in S$.*

Proof. Suppose that I is a graded weakly S-prime ideal of R. Then by Proposition 3.5, $S^{-1}I$ is a graded weakly prime ideal of $S^{-1}R$. Since I is a graded weakly S-prime ideal of R, there exists $s\in S$ such that for any $x,y\in h(R)$, if $0\neq xy\in I$, then $sx\in I$ or $sy\in I$.

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Let $0 \neq x \in (I :_{h(R)} s)$. Then $sx \in I$ and $x = \frac{xs}{s} \in S^{-1}I$. Hence, $x \in S^{-1}I \bigcap h(R)$. Now, let $0 \neq x \in S^{-1}I \bigcap h(R)$. Then $x \in h(R)$ and $x = \frac{p}{t}$ for some $p \in I$ and $t \in S$. So, $0 \neq tx = p \in I$. Hence, $st \in I$ or $sx \in I$. Thus, $sx \in I$ since $S \bigcap I = \emptyset$. Consequently, $x \in (I :_{h(R)} s)$. Therefore, $S^{-1}I \bigcap h(R) = (I :_{h(R)} s)$. Conversely, Let $x, y \in h(R)$ such that $0 \neq xy \in I$. Since $0 \neq \frac{x}{1}\frac{y}{1} \in S^{-1}I$, we have $\frac{x}{1} \in S^{-1}I$ or $\frac{y}{1} \in S^{-1}I$. If $\frac{x}{1} \in S^{-1}I$, then $\frac{x}{1} = \frac{p}{t}$ for some $p \in I$ and $t \in S$. Hence, $tx = p \in I$ and so $x = \frac{tx}{t} \in S^{-1}I \bigcap h(R)$, and then $x \in (I :_{h(R)} s)$ for some $s \in S$ by assumption, that is $sx \in I$. Similarly, if $\frac{y}{1} \in S^{-1}I$, then $sy \in I$. Therefore, I is a graded weakly S-prime ideal of R. □

Lemma 3.10. *Let R be a graded ring, S be a multiplicative subset of $h(R)$ and I be a graded ideal of R such that $I \bigcap S = \emptyset$. Then there is a graded ideal P of R which is maximal with respect to the properties that $I \subseteq P$ and $P \bigcap S = \emptyset$. Moreover, P is a graded prime ideal of R.*

Proof. Let Γ be the set of all graded ideals J of R such that $I \subseteq J$ and $J \bigcap S = \emptyset$. Then $\Gamma \neq \emptyset$ since $I \in \Gamma$. By Zorn's lemma, Γ has a maximal element P. Suppose that P is not a graded prime ideal of R. Then there exist $x, y \in h(R)$ such that $xy \in P$, $x \notin P$ and $y \notin P$. Then by ([4], Lemma 2.1), $P + Rx$ and $P + Ry$ are graded ideals of R with $P \subset P + Rx$ and $P \subset P + Ry$, and so there exist $s, t \in S$ such that $s \in P + Rx$ and $t \in P + Ry$. Hence, $s = p + rx$ and $t = q + \alpha y$ for some $p, q \in P$ and $r, \alpha \in R$. So, $st = pq + p\alpha y + rxq + r\alpha xy \in P \bigcap S$, which is a contradiction. Therefore, P is a graded prime ideal of R. □

A graded ring R is said to be a graded field if every nonzero homogeneous element of R is unit [10]. Clearly, every field is a graded field, but the converse is not necessarily true, see ([10], Example 3.6). In the same context, a graded ring R is said to be a graded domain if R has no homogeneous zero divisor. Obviously, every domain is a graded domain, but the converse is not necessarily true, see ([10], Example 3.6).

Proposition 3.11. *Let R be a graded ring and S be a multiplicative subset of $h(R)$. Then the following statements are equivalent:*

(1) $\{0\}$ is the only graded weakly S-prime ideal of R.
(2) $\{0\}$ is the only graded S-prime ideal of R.
(3) R is a graded domain and $S^{-1}R$ is a graded field.

Proof. (1) ⇒ (2): Let I be a graded S-prime ideal of R. Then I is a graded weakly S-prime ideal of R, and then $I = \{0\}$. So, $\{0\}$ is the only graded S-prime ideal of R.

(2) ⇒ (3): By Lemma 3.10, there is a graded prime ideal P of R with $P \bigcap S = \emptyset$. Hence, P is a graded S-prime ideal of R. Then $P = \{0\}$, and so R is a graded domain. Let $0 \neq x \in h(R)$ and $s \in S$. We show that $\frac{x}{s}$ is unit in $S^{-1}R$. If $x \in S$, then we have the desired result. Assume that $x \notin S$. If $Rx \bigcap S = \emptyset$, then by Lemma 3.10, there is a graded prime ideal P of R such that $Rx \subseteq P = \{0\}$, a contradiction. So, $Rx \bigcap S \neq \emptyset$. Let $t \in Rx \bigcap S$. Then $t \in S$ and $t = rx$ for some $r \in R$. We have, $\frac{sr}{t} \in S^{-1}R$ and $\frac{x}{s}\frac{sr}{t} = \frac{xsr}{st} = \frac{st}{st} = \frac{1}{1}$. Then $\frac{x}{s}$ is unit in $S^{-1}R$, and hence $S^{-1}R$ is a graded field.

(3) ⇒ (1): Let I be a nonzero graded weakly S-prime ideal of R. Let $0 \neq p \in I$. Then $0 \neq p_g \in I$ for some $g \in G$ since I is a graded ideal. Since $S^{-1}R$ is a graded field, there exists $0 \neq x \in R$ and $s \in S$ such that $\frac{p_g}{1}\frac{x}{s} = \frac{1}{1}$. Since R is a graded domain, we deduce that $p_g x = s \in I \bigcap S = \emptyset$, a contradiction. Consequently, $\{0\}$ is the only graded weakly S-prime ideal of R. □

Proposition 3.12. *Let R be a graded ring and S be a multiplicative subset of $h(R)$. Then every graded weakly S-prime ideal of R is graded prime if and only if R is a graded domain and every graded S-prime ideal of R is graded prime.*

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Proof. Suppose that every graded weakly S-prime ideal of R is graded prime. Since $\{0\}$ is a graded weakly S-prime ideal of R, $\{0\}$ is a graded prime ideal of R, and then R is a graded domain. Also, every graded S-prime ideal of R is graded weakly S-prime, and hence is graded prime by assumption. Conversely, since R is a graded domain, every graded weakly S-prime ideal of R is graded S-prime, and hence is graded prime by assumption. □

Proposition 3.13. *Let R be a graded ring, $S \subseteq h(R)$ be a multiplicative set and I be a graded weakly S-prime ideal of R. If J is a graded ideal of R such that $J \bigcap S \neq \emptyset$, then $I \bigcap J$ is a graded weakly S-prime ideal of R.*

Proof. By ([4], Lemma 2.1), $I \bigcap J$ is a graded ideal of R. Clearly, $(I \bigcap J) \bigcap S = \emptyset$. Let $t \in J \bigcap S$. Assume that $0 \neq xy \in I \bigcap J$ for some $x, y \in h(R)$. Then $0 \neq xy \in I$, and then $sx \in I$ or $sy \in I$ for some $s \in S$, which implies that $st \in S$ with $stx \in I \bigcap J$ or $sty \in I \bigcap J$. Hence, $I \bigcap J$ is a graded weakly S-prime ideal of R. □

Remark 3.14. *Let $S_1 \subseteq S_2$ be multiplicative subsets of $h(R)$ and I be a graded ideal of R disjoint with S_2. Clearly, if I is a graded weakly S_1-prime ideal of R, then I is graded weakly S_2-prime. However, the converse is not necessarily true. To see this, we proved in Example 2.3 that $P = 9XR$ is a graded S_2-prime ideal of R, where $S_2 = \{3^n : n$ is a non-negative integer$\}$, so P is a graded weakly S_2-prime ideal of R. On the other hand, $S_1 = \{1\} \subseteq S_2$ with P is not a graded weakly S_1-prime ideal of R since $9, X \in h(R)$ with $0 \neq 9X \in P$, $9 \notin P$ and $X \notin P$.*

Proposition 3.15. *Let $S_1 \subseteq S_2$ be multiplicative subsets of $h(R)$ such that for any $s \in S_2$, there is an element $t \in S_2$ satisfying $st \in S_1$. If I is a graded weakly S_2-prime ideal of R, then I is a graded weakly S_1-prime ideal of R.*

Proof. Since I is a graded weakly S_2-prime ideal of R, there exists $s \in S_2$ such that if $x, y \in h(R)$ with $0 \neq xy \in I$, then $sx \in I$ or $sy \in I$. Let $x, y \in h(R)$ such that $0 \neq xy \in I$. Then $sx \in I$ or $sy \in I$. By the assumption, $r = st \in S_1$ for some $t \in S_2$, and then $rx \in I$ or $ry \in I$. Consequently, I is a graded weakly S_1-prime ideal of R. □

Let S be a multiplicative subset of $h(R)$, $S^* = \left\{r \in h(R) : \frac{r}{1} \text{ is unit in } S^{-1}R\right\}$ denotes the saturation of S. Note that, S^* is a multiplicative subset of $h(R)$ containing S.

Proposition 3.16. *Let R be a graded ring, S be a multiplicative subset of $h(R)$ and I be a graded ideal of R disjoint with S. Then I is a graded weakly S-prime ideal of R if and only if I is a graded weakly S^*-prime ideal of R.*

Proof. Clearly, $S^* \bigcap I = \emptyset$. We show that for any $a \in S^*$, there is $b \in S^*$ such that $ab \in S$. Let $a \in S^*$. Then $\frac{a}{1}\frac{c}{s} = 1$ for some $s \in S$ and $c \in h(R)$. This implies that $tca = ts \in S$, for some $t \in S$. Now, take $b = tc$. Then, we have $b \in S^*$ with $ab \in S$, and so the desired condition is satisfied. Therefore, by putting $S = S_1$ and $S_2 = S^*$, we conclude immediately the result from Proposition 3.15. □

Proposition 3.17. *Let $f : R \rightarrow T$ be a graded ring homomorphism and S be a multiplicative subset of $h(R)$.*

(1) If f is a graded epimorphism and I is a graded weakly S-prime ideal of R containing $Ker(f)$, then $f(I)$ is a graded weakly $f(S)$-prime ideal of T.
(2) If f is a graded monomorphism and J is a graded weakly $f(S)$-prime ideal of T, then $f^{-1}(J)$ is a graded weakly S-prime ideal of R.

Proof. (1) By ([8], Lemma 3.11 (2)), $f(I)$ is a graded ideal of T. Let $r \in f(S) \bigcap f(I)$. Then $r = f(p) = f(s)$ for some $p \in I$ and $s \in S$. So, $s - p \in Ker(f) \subseteq I$, which implies that $s \in I$, a contradiction. Hence $f(S) \bigcap f(I) = \emptyset$. Now, let

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$0 \neq xy \in f(P)$ for some $x, y \in h(T)$. Then there is $a, b \in h(R)$ such that $f(a) = x, f(b) = y$ and $0 \neq f(ab) = xy \in f(P)$. Since $Ker(f) \subseteq I$, we get $0 \neq ab \in I$, and so $sa \in I$ or $sb \in I$ for some $s \in S$. It means that $f(s)x \in f(P)$ or $f(s)y \in f(I)$. Thus, $f(I)$ is a graded weakly $f(S)$-prime ideal of T.

(2) By ([8], Lemma 3.11 (1)), $f^{-1}(J)$ is a graded ideal of R. Clearly, $f^{-1}(J) \bigcap S = \emptyset$. Let $x, y \in h(R)$ such that $0 \neq xy \in f^{-1}(J)$. Since $Ker(f) = \{0\}$, we get $0 \neq f(xy) = f(x)f(y) \in J$. Then $f(s)f(x) = f(sx) \in J$ or $f(s)f(y) = f(sy) \in J$ for some $s \in S$. Thus, $sx \in f^{-1}(J)$ or $sy \in f^{-1}(J)$, and so we conclude that $f^{-1}(J)$ is a graded weakly S-prime ideal of R.

□

Corollary 3.18. *Let R be a graded ring and S be a multiplicative subset of $h(R)$.*

(1) If $I \subseteq P$ are two graded ideals of R and P is a graded weakly S-prime ideal of R, then P/I is a graded weakly $\overline{S}$-prime ideal of R/I, where $\overline{S} = \{s + I : s \in S\}$.

(2) If P is a graded weakly S-prime ideal of R, then $P \bigcap R_e$ is a weakly S-prime ideal of R_e.

Proof. (1) By ([10], Lemma 3.2), P/I is a graded ideal of R/I. Define $f : R \to R/I$ by $f(r) = r + I$. Then f is a graded epimorphism, and then the result follows by Proposition 3.17 (1).

(2) Define $f : R_e \to R$ by $f(r) = r$. Then f is a graded monomorphism, and then the result follows by Proposition 3.17 (2).

□

Proposition 3.19. *Let R be a graded ring and $S \subseteq h(R)$ a multiplicative set. If every proper graded ideal of R is graded weakly S-prime, then $S \subseteq U(R)$, and hence the concepts of graded weakly S-prime ideals and graded weakly prime ideals coincide.*

Proof. Let $s \in S$. Since every graded maximal ideal of R is graded weakly S-prime, there is no graded maximal ideal of R contains s. This implies that s is unit, and so $S \subseteq U(R)$. □

Definition 3.20. *Let R be a G-graded ring, $g \in G$, $S \subseteq R_e$ be a multiplicative set and P be a graded ideal of R such that $P \bigcap S = \emptyset$. We say that P is a g-S-prime ideal of R if there exists $s \in S$ such that for all $x, y \in R_g$, if $xy \in P$, then $sx \in P$ or $sy \in P$.*

Definition 3.21. *Let R be a G-graded ring, $g \in G$, $S \subseteq R_e$ be a multiplicative set and P be a graded ideal of R such that $P \bigcap S = \emptyset$. We say that P is a g-weakly S-prime ideal of R if there exists $s \in S$ such that for all $x, y \in R_g$, if $0 \neq xy \in P$, then $sx \in P$ or $sy \in P$.*

Remark 3.22. *Clearly, every g-S-prime ideal is g-weakly S-prime. However, in Example 3.3, P is a graded weakly S-prime ideal of R, so P is a g-weakly S-prime ideal of R for all $g \in G$. On the other hand, P is not a 0-S-prime ideal of R since $2, 6 \in R_0$ with $2.6 \in P$, $2s \notin P$ and $6s \notin P$ for each $s \in S$.*

Proposition 3.23. *Let R be a G-graded ring, $g \in G$, S be a multiplicative subset of R_e and I be a g-weakly S-prime ideal of R which is not g-S-prime. Then $I_g^2 = \{0\}$.*

Proof. Since I is a g-weakly S-prime ideal of R, there exists $s \in S$ such that whenever $x, y \in R_g$, $0 \neq xy \in I$ implies $sx \in I$ or $sy \in I$. Suppose that $I_g^2 \neq \{0\}$. We show that I is a g-S-prime ideal of R. Let $x, y \in R_g$ such that $xy \in I$. If $xy \neq 0$, then $sx \in I$ or $sy \in I$. Assume that $xy = 0$. If $xI_g \neq \{0\}$, there is $p \in I_g$ such that $0 \neq xp$, and so $0 \neq xp = x(p + y) \in I$. Hence, $sx \in I$ or $s(p + y) \in I$, and hence $sx \in I$ or $sy \in I$. Similarly, if $yI_g \neq \{0\}$, we obtain the same result. Finally, assume that $xI_g = \{0\}$ and $yP_g = \{0\}$. Since $I_g^2 \neq \{0\}$, there exists $p, q \in I_g$ such that $pq \neq 0$. Thus, $0 \neq pq = (x + p)(y + q) \in I$. Then $s(x + p) \in I$ or $s(y + q) \in I$. Therefore, $sx \in I$ or $sy \in I$. Consequently, we conclude that I is a g-S-prime ideal of R. So, $I_g^2 = \{0\}$. □

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Compare the following corollary with ([2], Proposition 2.2):

Corollary 3.24. *Let R be a G-graded ring, $g \in G$ and I be a g-weakly prime ideal of R which is not g-prime. Then $I_g^2 = \{0\}$.*

Proof. Apply Proposition 3.23 with $S = \{1\}$. □

Recall that a ring R is said to be reduced if $N(R) = \{0\}$.

Corollary 3.25. *Let R be a G-graded ring, $g \in G$, S be a multiplicative subset of R_e and P be a g-weakly S-prime ideal of R which is not g-S-prime. Then $I_g \subseteq Grad(\{0\})$. In particular, if R is reduced, then $I_g = \{0\}$.*

Compare the following theorem with ([2], Proposition 2.3, Theorem 2.10, Theorem 2.12):

Theorem 3.26. *Let R be a G-graded ring, $g \in G$, $S \subseteq R_e$ be a multiplicative subset and I be a graded ideal of R disjoint with S. Then the following statements are equivalent:*

(1) I is a g-weakly S-prime ideal of R.
(2) There exists $s \in S$ such that for each $a \notin (I :_{R_g} s)$ we have either $(I :_{R_g} a) \subseteq (I :_{R_g} s)$ or $(I :_{R_g} a) = (0 :_{R_g} a)$.
(3) There exists $s \in S$ such that for all graded ideals K and J of R, if $0 \neq K_gJ_g \subseteq I$, then $sK_g \subseteq I$ or $sJ_g \subseteq I$.

Proof. (1) ⇒ (2): Since I is a g-weakly S-prime ideal of R, there exists $s \in S$ such that whenever $a, b \in R_g$, $0 \neq ab \in I$ implies $sa \in I$ or $sb \in I$. Let $a \in R_g - (I :_{R_g} s)$. Then $sa \notin I$. Suppose that $(I :_{R_g} a) \neq (0 :_{R_g} a)$. Since $(0 :_{R_g} a) \subseteq (I :_{R_g} a)$, there exists $x \in (I :_{R_g} a)$ with $xa \neq 0$. Thus, $0 \neq xa \in I$. Hence, since $sa \notin I$, we have $sx \in I$. Assume that $b \in (I :_{R_g} a)$. Then $ab \in I$. If $ab \neq 0$, then $sb \in I$, and then $b \in (I :_{R_g} s)$. Suppose that $ab = 0$. Then $0 \neq ax = a(x + b) \in I$, and then $s(x + b) \in I$. Therefore, $sb \in I$. So, $b \in (I :_{R_g} s)$. Consequently, $(I :_{R_g} a) \subseteq (I :_{R_g} s)$.

(2) ⇒ (3): Let K and J be graded ideals of R such that $0 \neq K_gJ_g \subseteq I$. Suppose that $sK_g \nsubseteq I$ and $sJ_g \nsubseteq I$. We show that $K_gJ_g = \{0\}$. Let $a \in K_g - (I :_{R_g} s)$. Then $aJ_g \subseteq I$, and then $J_g \subseteq (I :_{R_g} a)$. Since $J_g \nsubseteq (I :_{R_g} s)$, we get $J_g \subseteq (I :_{R_g} a) = (0 :_{R_g} a)$. Hence, $aJ_g = \{0\}$. Suppose that $a \in K_g \bigcap (I :_{R_g} s)$. Let $b \in J_g$. If $b \notin (I :_{R_g} s)$, then as heretofore, we obtain $bK_g = \{0\}$, and so $ba = 0$. If $b \in (I :_{R_g} s)$, take $r \in J_g$ such that $sr \notin I$. Hence, $r \notin (I :_{R_g} s)$ and $b + r \notin (I :_{R_g} s)$. Hence, $ar = 0$ and $a(b + r) = 0$. Then $ab = 0$. So, $aJ_g = \{0\}$. Consequently, $I_gJ_g = \{0\}$.

(3) ⇒ (1): Let $x, y \in R_g$ with $0 \neq xy \in I$. Then $K = Rx$ and $J = Ry$ are graded ideals of R with $0 \neq K_gJ_g \subseteq I$. Hence, by assumption we have $sK_g \subseteq I$ or $sJ_g \subseteq I$. Thus, $sx \in I$ or $sy \in I$, and so we conclude that I is a g-weakly S-prime ideal of R. □

Proposition 3.27. *Let R be a graded ring, S be a multiplicative subset of R_e and I be an e-weakly S-prime ideal of R that is not e-S-prime. Then $sI_e(Grad(\{0\}))_e = \{0\}$ for some $s \in S$.*

Proof. By Theorem 3.26, there exists $s \in S$ such that for each $a \notin (I :_{R_e} s)$ we have either $(I :_{R_e} a) \subseteq (I :_{R_e} s)$ or $(I :_{R_e} a) = (0 :_{R_e} a)$. Let $x \in (Grad(\{0\}))_e$. If $x \in (I :_{R_e} s)$, then $sx \in I_e$. Thus, by Proposition 3.23, $sxI_e = \{0\}$. Suppose that $x \notin (I :_{R_e} s)$. Then $(I :_{R_e} x) \subseteq (I :_{R_e} s)$ or $(I :_{R_e} x) = (0 :_{R_e} x)$. Since $I \subseteq (I :_{R_e} x)$, the case $(I :_{R_e} x) = (0 :_{R_e} x)$ implies that $xI_e = \{0\}$, and then $sxI_e = \{0\}$. Assume that $(I :_{R_e} x) \subseteq (I :_{R_e} s)$. Let $m > 1$ be a minimal integer such that $x^m = 0$. Then $x^{m-1} \in (I :_{R_e} x) \subseteq (I :_{R_e} s)$. Therefore, $sx^{m-1} \in I_e$. Since $I \bigcap S = \emptyset$, $m - 1 > 1$. If $sx^{m-1} \neq 0$, then $sx \in I_e$, a contradiction. So, $sx^{m-1} = 0$. Let k be minimal such that $sx^k = 0$. Since $sx \neq 0$, we get that $k > 1$. Suppose that there exists $p \in I_e$ such that $sxp \neq 0$. We have $0 \neq sx(x^{k-1} + p) = sxp \in I$. Then, $s(x^{k-1} + p) \in I$. Hence,

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$0 \neq sx^{k-1} \in I$. Hence, $sx \in I$, a contradiction. Thus, $sxI_e = \{0\}$. Consequently, $sI_eGrad(\{0\}) = \{0\}$. □

Corollary 3.28. *Let R be a graded ring, S be a multiplicative subset of R_e and I be an e-weakly S-prime ideal of R that is not e-S-prime. Then $I_e \subseteq Grad(\{0\})$ and $I_e(Grad(\{0\}))_e = \{0\}$.*

Proof. Apply Corollary 3.25 and Proposition 3.27 with $S = \{1\}$. □

Corollary 3.29. *Let R be a graded ring and S be a multiplicative subset of R_e. If I and J are e-weakly S-prime ideals that are not e-S-prime, then $sI_eJ_e = \{0\}$ for some $s \in S$.*

Proof. By Corollary 3.25, $J_e \subseteq Grad(\{0\})$, and then $J_e \subseteq R_e \bigcap Grad(\{0\}) = (Grad(\{0\}))_e$. So, by Proposition 3.27, $sI_eJ_e \subseteq sI_e(Grad(\{0\}))_e = \{0\}$ for some $s \in S$. □

Let R_1 and R_2 be two G-graded rings. Then $R = R_1 \times R_2$ is G-graded by $(R_1 \times R_2)_g = (R_1)_g \times (R_2)_g$ for all $g \in G$. Moreover, $I = I_1 \times I_2$ is a graded ideal of R if and only if I_1 is a graded ideal of R_1 and I_2 is a graded ideal of R_2 ([10], Lemma 3.10).

Proposition 3.30. *Let R_1, R_2 be two G-graded rings, S_1, S_2 be multiplicative subsets of $(R_1)_e$, $(R_2)_e$ respectively, and I_1, I_2 be nonzero graded ideals of R_1, R_2, respectively. Suppose that $I = I_1 \times I_2$ and $S = S_1 \times S_2$. Then the following statements are equivalent:*

(1) I is an e-weakly S-prime ideal of R.
(2) I_1 is an e-S_1-prime ideal of R_1 and $S_2 \bigcap I_2 \neq \emptyset$ or I_2 is an e-S_2-prime ideal of R_2 and $S_1 \bigcap I_1 \neq \emptyset$.
(3) I is an e-S-prime ideal of R.

Proof. (1) $\Rightarrow$ (2): Let $(x, y) \in R_e$ such that $0 \neq (x, y) \in I$. Then $0 \neq (x, y) = (x, 1)(1, y) \in I$. Since I is an e-weakly S-prime ideal of R, then there is $s = (s_1, s_2) \in S$ such that $s(x, 1) = (s_1x, s_2) \in I$ or $s(1, y) = (s_1, s_2y) \in I$. Thus, $S_1 \bigcap I_1 \neq \emptyset$ or $S_2 \bigcap I_2 \neq \emptyset$. Assume that $S_2 \bigcap I_2 \neq \emptyset$. As $I \bigcap S = \emptyset$, we have $I_1 \bigcap S_1 = \emptyset$. Now, we show that I_1 is an e-S_1-prime ideal of R_1. Let $xy \in I_1$ for some $x, y \in (R_1)_e$. Since $S_2 \bigcap I_2 \neq \emptyset$, then there is $0 \neq t \in S_2 \bigcap I_2$, and so we have $0 \neq (x, t)(y, 1) \in I$. Hence, $s(x, t) = (s_1x, s_2t) \in I$ or $s(y, 1) = (s_1y, s_2) \in I$. So, we get $s_1x \in I_1$ or $s_1y \in I_1$, as desired.

(2) $\Rightarrow$ (3): Follows from ([10], Lemma 3.11).

(3) $\Rightarrow$ (1): Obvious. □

Proposition 3.31. *Let R_1, R_2 be two G-graded rings, S_1, S_2 be multiplicative subsets of $(R_1)_e$, $(R_2)_e$ respectively, and $S = S_1 \times S_2$. If every proper graded ideal of R is e-weakly S-prime, then $(R_1)_e$ and $(R_2)_e$ are fields.*

Proof. Let P_1 be a proper graded ideal of R_1. Then $P_1 \times R_2$ is an e-weakly S-prime ideal of R, and so P_1 is an e-S_1-prime ideal of R_1 by Proposition 3.30. Thus, every proper graded ideal of R_1 is e-S_1-prime. So, by ([10], Corollary 2.14), $(R_1)_e$ is a field. Similarly, $(R_2)_e$ is a field. □

Proposition 3.32. *Let R_1, R_2 be two G-graded rings, S_1, S_2 be multiplicative subsets of $(R_1)_e$, $(R_2)_e$ respectively, and $S = S_1 \times S_2$. If R_1 and R_2 are graded fields, then every proper graded ideal of R is e-weakly S-prime.*

Proof. Since R_1 and R_2 are graded fields, we have exactly three proper graded ideals of R, that are $\{0\} \times \{0\}$, $\{0\} \times R_2$ and $R_1 \times \{0\}$ which are e-weakly-S-prime by Proposition 3.30. □

Assume that M is an R-module. Then M is said to be G-graded if $M = \bigoplus_{g \in G} M_g$ with $R_gM_h \subseteq M_{gh}$ for all $g, h \in G$ where M_g is an additive subgroup of M for all $g \in G$. The elements of M_g are called homogeneous of degree g. It is clear that M_g is an R_e-submodule of M for all $g \in G$. We assume that $h(M) = \bigcup_{g\in G} M_g$. Let N be an R-submodule of a graded R-module M. Then N is said to be graded R-submodule if $N = \bigoplus_{g\in G}(N \cap M_g)$, i.e., for $x \in N$, $x = \sum_{g\in G} x_g$ where $x_g \in N$ for all $g \in G$. It is known that an R-submodule of a graded R-module need not be graded. For more terminology see [6, 7].

Let M be an R-module. The idealization $R(+)M = \{(r, m) : r \in R \text{ and } m \in M\}$ of M is a commutative ring with componentwise addition and multiplication; $(x, m_1)+(y, m_2) = (x+y, m_1+m_2)$ and $(x, m_1)(y, m_2) = (xy, xm_2+ym_1)$ for each $x, y \in R$ and $m_1, m_2 \in M$. Let G be an abelian group and M be a G-graded R-module. Then $X = R(+)M$ is G-graded by $X_g = R_g(+)M_g$ for all $g \in G$. Note that, X_g is an additive subgroup of X for all $g \in G$. Also, for $g, h \in G$, $X_gX_h = (R_g(+)M_g)(R_h(+)M_h) = (R_gR_h, R_gM_h + R_hM_g) \subseteq (R_{gh}, M_{gh} + M_{hg}) \subseteq (R_{gh}, M_{gh}) = X_{gh}$ as G is abelian [11]. Moreover, if P is an ideal of R and N is an R-submodule of M such that $PM \subseteq N$, then $P(+)N$ is a graded ideal of $R(+)M$ if and only if P is a graded ideal of R and N is a graded R-submodule of M ([11], Proposition 3.3). Clearly, if S is a multiplicative subset of $h(R)$, then $S(+)\{0\}$ and $S(+)h(M)$ are multiplicative subset of $h(R(+)M)$.

Proposition 3.33. *Let G be an abelian group, R be a G-graded ring, M be a G-graded R-module, S be a multiplicative subset of $h(R)$ and I be a graded ideal of R disjoint with S. Then the following statements are equivalent:*

(1) $I(+)M$ is a graded weakly $(S(+)\{0\})$-prime ideal of $R(+)M$.
(2) $I(+)M$ is a graded weakly $(S(+)h(M))$-prime ideal of $R(+)M$.
(3) I is a graded weakly S-prime ideal of R associated to $s \in S$ and whenever $x, y \in h(R)$ with $xy = 0$, but $sx \notin I$ and $sy \notin I$, then $x \in Ann_R(M)$ and $y \in Ann_R(M)$.

Proof. (1) ⇒ (2): Follows by Remark 3.14 since $S(+)\{0\} \subseteq S(+)h(M)$.

(2) ⇒ (3): Let $x, y \in h(R)$ such that $0 \neq xy \in I$. Then $(0, 0) \neq (x, 0)(y, 0) \in I(+)M$. As $I(+)M$ is a graded weakly $(S(+)h(M))$-prime ideal of $R(+)M$, there is $(s, n) \in S(+)h(M)$ such that $(s, n)(x, 0) = (sx, xn) \in I(+)M$ or $(s, n)(y, 0) = (sy, yn) \in I(+)M$. Thus, $sx \in I$ or $sy \in I$, and so I is a graded weakly S-prime ideal of R. Suppose that $xy = 0$ with $sx \notin I$ and $sy \notin I$. Assume that $x \notin Ann_R(M)$. Then there is $m \in M$ such that $xm \neq 0$, which gives that $xm_g \neq 0$ for some $g \in G$, and so we have $(0, 0) \neq (x, 0)(y, m_g) \in I(+)M$. Hence, $(s, n)(x, 0) = (sx, xn) \in I(+)M$ or $(s, n)(y, m_g) = (sy, sm_g + yn) \in I(+)M$, a contradiction. Therefore, $x \in Ann_R(M)$ and $y \in Ann_R(M)$.

(3) ⇒ (1): Let $(0, 0) \neq (x, m)(y, n) \in I(+)M$, where $(x, m), (y, n) \in h(R(+)M)$. If $xy \neq 0$, then $sx \in I$ or $sy \in I$, and hence $(s, 0)(x, m) \in I(+)M$ or $(s, 0)(y, n) \in I(+)M$. Assume that $xy = 0$ with $sx \notin I$ and $sy \notin I$. Then $x, y \in Ann_R(M)$. Consequently, we get $(x, m)(y, n) = (0, 0)$, a contradiction. Therefore, $I(+)M$ is a graded weakly $(S(+)\{0\})$-prime ideal of $R(+)M$. □

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