A high-frequency phase feature for the measurement of magnetic permeability using eddy current sensor

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Abstract – Electromagnetic sensing has been used for diverse applications of non-destructive testing, including the surface inspection, measurement of properties, object characterization. However, the measurement accuracy could be significantly influenced by the lift-off between sensors and samples. To address the issue caused by lift-offs, various strategies have been proposed for the permeability measurement of ferromagnetic steels, which mainly involves different sensor designs and signal features (e.g., the zero-crossing feature). In this paper, a single high-frequency scenario for the permeability retrieval is introduced. By combining the signal of two sensing pairs, the retrieval of magnetic permeability is less affected by the lift-off of sensors. Unlike the previous strategy on reducing the lift-off effect (directly taking the phase term out of the integration) using the Dodd-Deeds analytical method, the proposed method is based on a high-frequency linear feature of the phase term. Therefore, this method has the merit of high accuracy and fast processing for the permeability retrieval (a simplified version of Dodd-Deeds analytical formulas after the integration). Experimental measurement has been carried out on the impedance measurement of designed sensors interrogating ferromagnetic dual-phase steels. For sensor lift-offs of up to 10 mm, the error of the permeability retrieval is controlled within 4 % under the optimal frequency.

Index Terms – electromagnetic sensing, lift-off, eddy current, magnetic permeability, non-destructive testing.

Introduction

As one of the promising methods in the field of non-destructive testing (NDT), the electromagnetic (EM) sensing technique [1-9] has used to exploit the electromagnetic characteristics of steels, including the EM properties, surface inspection, B-H saturation, hysteresis of ferromagnetic materials [10-20]. Both the pulsed eddy current (PEC) [9,10,12,18,20-23] and multi-frequency eddy current techniques [24-26] have been used to retrieve the permeability from the measured impedance. However, the lift-off of eddy current probes could significantly influence the measurement signal and thus the retrieval of magnetic permeabilities.

Novel sensor designs, post-processing techniques, and measurement manners have been proposed for reducing the permeability measurement error caused by the lift-off effect [18,20-24]. Giguère et al. have found a feature - lift-off point of intersection (LOI) from the pulsed eddy current signal, which allows for an accurate evaluation of the ferromagnetic materials [20]. The LOI feature has been widely applied in the estimation of coating thickness, and measurement of ferromagnetic materials [21-23]. Tian et al. have proposed a four-point potential drop method in the form of multi-frequency spectra, which has its merit of simultaneously measuring both the electrical conductivity and magnetic permeability but restrained by its contact-sensing manner [24]. Besides, Tian and Sophian have used two references from air measurement and defect-free sample measurement for reducing the lift-off effect [18]. These methods apply to the evaluations of meta properties using PEC techniques.

Previously, both the iterative method [25], multi-frequency analysis – zero-crossing frequency feature (ZCF) [26], finite-element analysis (FEA) [27], conductivity invariance phenomenon (CIP) [28], and real-time embedded algorithm based on novel sensor design [29,30] have been proposed for the reconstruction of the ferromagnetic plate. From the iterative method [25], multiple parameters (including the conductivity, permeability, thickness, and lift-offs) are retrieved by analysing both the real and imaginary part of the impedance. The ZCF feature is proved sensitive to the test piece [26], where algorithms have been proposed to incorporate the ZCF feature for compensating the lift-off effect on the permeability retrieval. The FEA method [27] is based on a customised MATLAB scripted solver, which can further retrieve the phase fractions of alloys based on the predicted permeability. However, both the zero-crossing frequency feature, finite-element analysis, and iterative method are hampered by their long processing time, and need to recalibrate for different sensor setups. From the CIP feature [28], it is found that on an optimal sensor-sample lift-off, the measured impedance is barely affected by the
electrical conductivities of tested steels. The CIP feature is only valid for the drive-pickup (where the transmitter and receiver are horizontally separated) sensor structure.

To reduce the lift-off effect on the retrieval of magnetic permeabilities, a high-frequency feature has been proposed in this paper. With a simplified Dodd-Deeds analytical method, the proposed scenario has hastened the computation (after the integration), which applies to the real-time retrieval of permeabilities or monitor of permeability inhomogeneities. Compared to the previous method on reducing the lift-off effect of permeability predictions [30], a similar triple-coil sensor setup but different retrieval strategies have been used. In the previous method, a real-time embedded algorithm is used, which is based on the slow change rate of the phase term $\phi$ with respect to $\alpha$ when using the Dodd-Deeds analytical approach. However, the change rate of the phase term is found still considerable for certain values of properties and frequencies. Thus, it needs to recalibrate for different frequencies and samples when using the previous method. Moreover, the retrieval in the previous method [30] is based on the phase of the impedance, which has a relatively lower sensitivities (compared to the impedance) to the variation of steel properties. In this paper, unlike the previous approach of assuming the slow change rate of the phase term $\phi$, a high-frequency feature (linear relationship) has been found between the ratio (imaginary part over real part) of the phase term $\phi$ and the integration variable $\alpha$. Thus, compared to the method reported in [30], the proposed method has a relatively high sensitivity (and accuracy, as the permeability is retrieved from the impedance instead of phase) does not need to recalibrate for different materials and frequencies. The Experimental measurement on ferromagnetic dual-phase alloys have been carried out for the verification of the proposed method.

**Methodology – high-frequency phase feature for the magnetic permeability measurement**

Both the A-V form finite element formulation [14] and the Dodd-Deeds method [31] have been widely utilised for the simulation of the electromagnetic field and sensor response. The finite element method (FEM) is a general approach for field computation in the media with arbitrary geometry and anisotropic properties. However, the solution of using FEM is significantly determined by the mesh resolution. A finely discretised mesh could result in hours of processing for the convergence. The Dodd-Deed method has been proposed for decades but is still the most dominant method for the interaction analysis between a circular coil and a conductive plate or co-axial cylindrical rod.

In Fig. 1, three coils, with a single transmitter coil and two sensing coils (one receiver coil and one reference coil), are equally wound (same number of turns, and types of copper wires with insulating coatings) on slots (of the same radius) of a ceramic rod. The permeability retrieval is based on the impedance measurement from two sensing pairs (transmitter-receiver and transmitter-reference). Considering the sensitivities to the sample, the receiver is designed close to the test piece. The reference coil is symmetrically deployed with respect to the transmitter, to ensure that the permeability retrieval applies to a decent range of lift-offs. For circular coils above the ferromagnetic plate, as shown in Fig. 1, the impedance change (minus the background signal when the sensor is in the free space) test by transmitter-receiver and transmitter-reference sensing coils are shown in (1) and (2).

$$
\Delta Z_1 = j\omega k \int_{0}^{\infty} \frac{1}{\alpha^2} \Lambda_1(\alpha) \phi(\alpha) d\alpha
$$
\[ \Delta Z_2 = j\omega K \int_0^\infty \frac{p^2(\alpha)}{\alpha^5} A_2(\alpha) \phi(\alpha) d\alpha \]  

(2)

In (1) (2), assume all the coils have the same number of turns N, coil height h, and inner and outer radii \( r_1 \) and \( r_2 \). Then,

\[ K = \frac{\pi N^2 \mu_0}{h^2 (r_2 - r_1)^2} \]  

(3)

In (3), \( \mu_0 \) is the vacuum permeability. \( \bar{r} \) is the mean value of the outer and inner radii for all the coils (both transmitter, receiver, and reference coils).

\[ \bar{r} = \frac{r_1 + r_2}{2} \]  

(4)

\[ P(\alpha) = \int_{ar_1}^{ar_2} t_1(\tau) d\tau \]  

(6)

\[ A_1(\alpha) = e^{-\alpha(2h + 3g) + \beta h} \left(e^{-ah} - 1\right)^2 \]  

(7)

\[ A_2(\alpha) = e^{-\alpha(2h + 3g + 3h)} \left(e^{-ah} - 1\right)^2 \]  

(8)

\[ \phi(\alpha) = \frac{\alpha_1 - \mu_1 \alpha}{\alpha_1 + \mu_1 \alpha} \]  

(9)

\[ \alpha_1 = \sqrt{\alpha^2 + j\omega \mu_1 \mu_0} \]  

(10)

Where, \( J_1 \) denotes the first-order Bessel function of the first kind. \( \tau \) and \( \alpha \) are the variables for the integration. \( \omega \) denotes the operation frequency. \( \sigma \) is the electrical conductivity of the ferromagnetic sample. \( \mu_1 \) is the relative permeability of the ferromagnetic sample. \( h \) denotes the sensor lift-off, which is the lower height of the receiver depicted in Fig. 1.

In the previous work, compared to the varying rate of \( \frac{p^2(\alpha)}{\alpha^6} A_1(\alpha) \) or \( \frac{p^2(\alpha)}{\alpha^6} A_2(\alpha), \phi(\alpha) \) was assumed to be almost immune to \( \alpha \). Thus \( \phi(\alpha) \) was regarded as a constant for varying \( \alpha \) and taken out from the integration [26,29,30]. However, in general, the real part of the phase term \( \phi(\alpha) \) is found to be almost linear to \( \alpha \) under the high working frequency.

![Graph](image)

Fig. 2. Real part of \( \phi(\alpha) \) versus \( \alpha \) under high frequencies

In Fig. 2, the real part of \( \phi(\alpha) \) vary slowly with \( \alpha \) especially under high frequencies, which can be approximated as

\[ \text{Re}(\phi(\alpha)) = -e^{-2\alpha t} \]  

(11)

\( G \) in (11) is a small value function and determined by \( \omega, \sigma, \) and \( \mu_1 \).

Therefore, under high working frequencies, the real part and imaginary part of the impedance for transmitter-receiver and transmitter-reference sensing coil - (1) and (2) are shown in the following equations.

\[ \text{Re}(\Delta Z_1) = -\omega \bar{r} K \int_0^\infty \frac{p_1(\alpha)}{\alpha^6} e^{-\alpha(2h + \beta h) + \beta h} \left(e^{-ah} - 1\right)^2 \text{Im}(\phi(\alpha)) d\alpha \]  

(12)

\[ \text{Im}(\Delta Z_1) = -\omega \bar{r} K \int_0^\infty \frac{p_1(\alpha)}{\alpha^6} e^{-\alpha(2h + \beta h) + \beta h} \left(e^{-ah} - 1\right)^2 \text{Re}(\phi(\alpha)) d\alpha \]  

(13)

\[ \text{Re}(\Delta Z_2) = -\omega \bar{r} K \int_0^\infty \frac{p_1(\alpha)}{\alpha^6} e^{-\alpha(2h + 3g + 3h) + \beta h} \left(e^{-ah} - 1\right)^2 \text{Im}(\phi(\alpha)) d\alpha \]  

(14)

\[ \text{Im}(\Delta Z_2) = -\omega \bar{r} K \int_0^\infty \frac{p_1(\alpha)}{\alpha^6} e^{-\alpha(2h + 3g + 3h) + \beta h} \left(e^{-ah} - 1\right)^2 \text{Re}(\phi(\alpha)) d\alpha \]  

(15)
Moreover, as illustrated in Fig. 3, the term \( \frac{\text{Im}(\phi(\alpha))}{\text{Re}(\phi(\alpha))} \) is found to be linear with the variable \( \alpha \) under high frequencies.

\[
\text{Im}(\phi(\alpha)) = F \text{Re}(\phi(\alpha))
\]

(16)

\( F \) in (16) is the proportional function, which is also determined by \( \omega, \sigma, \) and \( \mu_1 \).

\[
F = \lim_{\alpha \to 0} \frac{\text{Im}(\phi(\alpha))}{\text{Re}(\phi(\alpha))} = \sqrt{\frac{2\mu_1}{\omega\sigma}}
\]

(17)

Combine (16) and (17) with equations from (12) to (15), the real part and imaginary part of the impedance from transmitter-receiver and transmitter-reference sensing coils are shown in the following equations.

\[
\text{Re}(\Delta Z_1) = \omega FK \int_0^\infty \frac{p^2(\alpha)}{\alpha^5} e^{-\alpha(2(\alpha+G)+g+b)}(e^{-\alpha h} - 1)^2 \, d\alpha
\]

(18)

\[
\text{Im}(\Delta Z_1) = -\omega FK \int_0^\infty \frac{p^2(\alpha)}{\alpha^5} e^{-\alpha(2(\alpha+G)+g+b)}(e^{-\alpha h} - 1)^2 \, d\alpha
\]

(19)

\[
\text{Re}(\Delta Z_2) = \omega FK \int_0^\infty \frac{p^2(\alpha)}{\alpha^5} e^{-\alpha(2(\alpha+G)+3g+3h)}(e^{-\alpha h} - 1)^2 \, d\alpha
\]

(20)

\[
\text{Im}(\Delta Z_2) = -\omega FK \int_0^\infty \frac{p^2(\alpha)}{\alpha^5} e^{-\alpha(2(\alpha+G)+3g+3h)}(e^{-\alpha h} - 1)^2 \, d\alpha
\]

(21)

Fig. 4 The Bessel series \( \frac{p^2(\alpha)}{\alpha^5} e^{-\alpha(2(\alpha+G)+g+b)}(e^{-\alpha h} - 1)^2 \) can be approximated as \( e^{-2\alpha^2\sin^2\left(\frac{\alpha^2}{2\alpha_0}\right)} \).
As Fig. 4 depicts, the Bessel series in (18) and (20) \( \int_{0}^{\infty} \frac{p^2(\alpha)}{\alpha^5} e^{-\alpha(2l_0+g+h)}(e^{-\alpha h} - 1)^2 \, d\alpha \) can be estimated as the product between the lift-off attenuation factor \( e^{-2\alpha l_0} \) and the squared sinusoidal functions that peaks at \( \alpha_0 - e^{-2\alpha_0 l_0} \sin^2 \left( \frac{\alpha_0}{2} \right) \).

\[
\int_{0}^{\infty} \frac{p^2(\alpha)}{\alpha^5} e^{-\alpha(2l_0+g+h)}(e^{-\alpha h} - 1)^2 \, d\alpha = Z_m \int_{0}^{2\alpha_0} e^{-2\alpha l_0+G} \sin^2 \left( \frac{\alpha}{2\alpha_0} \right) \, d\alpha
\]

\[
\pi^2 \left( 1 - e^{-4\alpha_0(l_0+G)} \right)
\]

\[
= Z_m \frac{4(l_0+G)(4\alpha_0^2(l_0+G)^2+\pi^2)}{4(l_0+G+g+h)(4\alpha_0^2(l_0+G+g+h)^2+\pi^2)}
\]

In (22) and (23), \( \alpha_0^2(l_0+G)^2 \ll \pi^2, \alpha_0^2(l_0+G+g+h)^2 \ll \pi^2. \) Therefore, (22) and (23) can be further approximated as

\[
\int_{0}^{\infty} \frac{p^2(\alpha)}{\alpha^5} e^{-\alpha(2l_0+G+g+h)}(e^{-\alpha h} - 1)^2 \, d\alpha = \Delta Z_m \frac{1 - e^{-4\alpha_0(l_0+G)}}{4(l_0+G)}
\]

\[
\int_{0}^{\infty} \frac{p^2(\alpha)}{\alpha^5} e^{-\alpha(2l_0+G+g+h)}(e^{-\alpha h} - 1)^2 \, d\alpha = \Delta Z_m \frac{1 - e^{-4\alpha_0(l_0+G+g+h)}}{4(l_0+G+g+h)}
\]

In (24) and (25), \( \alpha_0 \) is named as the spatial frequency, which is a constant value for a specific sensor geometry. \( Z_m \) is used to normalise the squared sinusoidal functions and defined as

\[
Z_m = \frac{p^2(\alpha_0)}{\alpha_0^5} e^{-\alpha_0(2l_0+g+h)}(e^{-\alpha_0 h} - 1)^2
\]

Through mathematical manipulations (Appendix at the end), the permeability of the steel is retrieved by using the following equation.

\[
\mu_l = \frac{F_{00} \sigma_{\mu_0}}{2} = \frac{2R_1 R_2 (g+h)^2 \sigma_{\mu_0}}{Z_m \pi^2 R_0^2 (R_1 - R_2)^2}
\]

In (27), \( R_1 = \text{Re}(\Delta Z_1), R_2 = \text{Re}(\Delta Z_2). \) Therefore, in (27), the magnetic permeability of the ferromagnetic plate can be reconstructed from the measured resistance from transmitter-receiver and transmitter-reference sensing coils.

**Sensor and experiment setup**

Experiments on the impedance measurement of the designed sensor deployed above the dual-phase (ferrite-austenite) alloy have been carried out for the test of the proposed method – (27).

**Fig. 5** Air-core sensor above the ferromagnetic sample is connected to the impedance analyser for the impedance measurement

As shown in Fig. 5, the sensor contains three coils – transmitter, receiver, and reference coils spirally wounded on the middle, bottom, and top buckets of the ceramic rod. The coils parameters including the number of turns, inner/outer diameters, height, and lift-offs are exhibited in Table 1. Three coils have the same turns and height. The lift-off is achieved by inserting the lift-off spacers between the sensor and sample, which ranges from 1 mm to 10 mm with 1 mm for each spacer. As listed in Table 2, the dual-phase (ferrite-austenite) alloys are DP 600 and DP 1000 with different phase fractions of the ferrite. The actual values of the relative magnetic permeability are 222 and 122 for DP 600 and DP 1000. The electrical conductivity of the alloy is measured from the four-terminal sensing method. The sensor is connected to the Zurich impedance analyser for the measurement of the multi-frequency impedance spectra. The working frequency is from 1.0 kHz to 2.0 MHz. It was found the signals have a poor Signal-to-Noise Ratio (SNR) for lower frequencies (particularly for frequencies lower than 1.0 kHz). Besides, the working frequency of the instrument ranges from 1 mHz to 5 MHz, where the resonance effect (starts from approximately 7 MHz) barely happens.
### Results and discussions

In Fig. 6, the electrical resistance under the working frequencies of from 1.0 kHz to 2.0 MHz has been illustrated. It can be seen that the resistance of the sensor above DP 600 alloy is slightly larger than that of DP 1000 alloy. In addition, the measured resistance is reduced for an increased sensor lift-off, which can be explained by the lift-off decay factor \( e^{-2\alpha g} \) in equation (18) and (20). Since the reference coil is much away from the sample compared to the receiver coil, less interaction exists between the sample and transmitter-reference sensing coil (TR2). Consequently, the electrical resistance measured from transmitter-reference sensing coil (TR2) is much smaller than that from transmitter-receiver sensing coil (TR1), which can also be explicated by referring to the different term \( e^{-\alpha(g+h)} \) and \( e^{-3\alpha(g+h)} \) between (20) and (18).

![Multi-frequency spectra of the electrical resistance (real part of impedance) under the sensor lift-off of 4 mm and 7 mm](image)

Fig. 6 Multi-frequency spectra of the electrical resistance (real part of impedance) under the sensor lift-off of 4 mm and 7 mm (a) DP 600 alloy (b) DP 1000 alloy

![Multi-frequency spectra of the inductance (imaginary part of impedance over the angular frequency \( \omega \)) under the frequency range from 1.0 kHz to 2.0 MHz. Similar to the trend of the multi-frequency](image)

Fig. 7 demonstrates the inductance (the imaginary part of impedance or reactance over the angular frequency \( \omega \)) under the frequency range from 1.0 kHz to 2.0 MHz. Similar to the trend of the multi-frequency...
resistance spectrum, the inductance is shown to be decreased on an increased sensor lift-off. Moreover, the measured inductance from transmitter-reference sensing coil ($TR_2$) is significantly smaller than that from transmitter-receiver sensing coil ($TR_1$) due to the less coupled effect between the sensing coil and sample. It can be observed that the inductance curve crossed zero at a certain frequency, which is named as the zero-crossing frequency (ZCF) feature. In our previous work, the ZCF feature is linked to the magnetic permeability of the ferromagnetic sample based on the low varying rate of the phase term $\phi(\alpha)$. However, the phase term $\phi$ is proved can vary fast under a certain value of parameters (Fig. 2). Moreover, the ZCF method needs to be achieved under the sweeping frequency manner, which is time-consuming and need to recalibrate for real-time measurement when using different sensor setups.

Fig. 8 Reconstructed relative magnetic permeability under different working frequencies with the sensor lift-off of 4 mm, 7 mm, and 10 mm
(a) DP 600 alloy (b) DP 1000 alloy

Fig. 8 depicts the reconstructed relative magnetic permeability of DP 600 and DP 1000 alloy under different working frequencies. The permeability is derived from the measured electrical resistance and the proposed algorithm (27). It can be seen that an increase in lift-off will result in a slightly reduced permeability value. Since the proposed method is based on the high-frequency phase feature (Fig. 2 and Fig. 3), the reconstructed permeability is significantly lower than the expected/actual permeability. From Fig. 8 (a) and (b), it can be found the optimal frequency for the reconstruction is around 1.0 MHz for both DP 600 and DP 1000 alloy.

Fig. 9 Retrieved relative magnetic permeability using the proposed method under different working frequencies (520 kHz, 1.1 MHz, and 2.0 MHz) with the sensor lift-off from 1 mm to 10 mm
(a) DP 600 alloy (b) DP 1000 alloy

As the reconstructed magnetic permeability is more precious under high frequency – around 1 MHz, reconstruction results from three working frequencies (520 kHz, 1.1 MHz, and 2.0 MHz) have been extracted for the illumination of the lift-off effect. It can be seen that an increased lift-off will lead to a slightly reduced but gradually stable value. Comparing the three curves for both steel alloys, it is found that the optimal frequency for a high-accuracy reconstruction with a small range of lift-off (less than 10 mm) is around 1.1 MHz. Under the optimal frequency – 1.1 MHz, the error of permeability retrieval is controlled within 4 %.
off will result in a distorted value due to the resolution of the first-order Padé approximation in (27) and (28). The optimal frequency for the permeability retrieval is sensor-dependent. For the sensor size listed in Table 1, the optimal frequency is around 1.1 MHz. In practical measurements of different sensor dimensions, it is suggested to use a benchmark sample with nominal property to find the optimal frequency (or optimal frequency range) for the permeability retrieval method.

Conclusions

In this paper, a high-frequency phase feature has been found in the phase term \( \phi \) of the Dodd-Deed analytical formulation. Unlike the previous technique of directly taking the phase term out of the integration, both the real part and imaginary part of the phase term have been approximated by combined linear and exponential functions. The eddy current sensor contains three coils, with the transmitter, receiver, and reference coil wounded on the middle, bottom, and top bucket of the ceramic rod. By combining the measured electrical resistance from both transmitter-receiver and transmitter-reference coils, the relative permeability of the ferromagnetic plate can be reconstructed. From the carried-out experiments on the dual-phase alloy, the deviation on the permeability retrieval using the proposed method is within 4 % for lift-offs of up to 10 mm. Further increased sensor lift-off will lead to distortion due to the Padé approximation utilised in the proposed algorithm. The designed air-core sensor can only be used for the initial permeability measurement of the ferromagnetic specimen due to its weak induced magnetic field. An air-core sensor with a thicker wire could be used for the B-H saturation measurement of the ferromagnetic sample. As the skin depth of the eddy current is much smaller than the thickness of sample plates under the optimal frequency (1.1 MHz), further increased thickness will not affect the accuracy of the permeability retrieval. Since the proposed method is under the relatively small magnetic field generated from the excitation current, the magnetization or the hysteresis (saturation) effect is not encountered. For the measurement of some ferromagnetic samples with higher permeability (e.g., electrical steel, which could have relative permeabilities of over 1000), the sample may be easily magnetised under a small magnetic field. Thus, in the future, more experiments will be carried out for the analysis of hysteresis effects on the permeability retrieval.

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Appendix

For a small sensor lift-off of less than 10 mm, according to the Padé approximation, \( e^{-4a_0(l_0 + G)} \) and \( e^{-4a_0(l_0 + G + g + h)} \) in (24) and (25) can be estimated as

\[
e^{-4a_0(l_0 + G)} = \frac{4a_0(l_0 + G) - 2}{4a_0(l_0 + G) + 2}
\]  
\[
e^{-4a_0(l_0 + G + g + h)} = \frac{4a_0(l_0 + G + g + h) - 2}{4a_0(l_0 + G + g + h) + 2}
\]

Substitute (28) and (29) into (24) and (25),

\[
\int_0^1 \frac{p^2(\alpha)}{\alpha^5} e^{-a(2(l_0 + G) + g + h)}(e^{-ah} - 1)^2 \, d\alpha = Z_m \frac{a_0}{2a_0(l_0 + G + g + h) + 1}
\]
\[
\int_0^1 \frac{p^2(\alpha)}{\alpha^5} e^{-a(2(l_0 + G) + 3g + 3h)}(e^{-ah} - 1)^2 \, d\alpha = Z_m \frac{a_0}{2a_0(l_0 + G + g + h) + 1}
\]

Combine (30) and (31) with (18) and (20),

\[
\text{Re}(\Delta Z_1) = Z_m \frac{a_0}{2a_0(l_0 + G) + 1}
\]
\[
\text{Re}(\Delta Z_2) = Z_m \frac{a_0}{2a_0(l_0 + G + g + h) + 1}
\]

Let \( R_1 = \text{Re}(\Delta Z_1), R_2 = \text{Re}(\Delta Z_2). \) The solution of \( (l_0 + G) \) and \( F \) can be derived.
Finally, the relative permeability of the ferromagnetic plate can be obtained by substituting (35) into (17).

\[ \mu_1 = \frac{2R_1^2R_2^2(g + h)^2\sigma_0}{Z_m^2\pi^2\mu_0(R_1 - R_2)^2} \]  

(36)

References


