A Proof of the Riemann Hypothesis Based on MacLaurin Expansion of the Completed Zeta Function

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Abstract The basic idea is to expand the completed zeta function $\xi(s)$ in MacLaurin series (infinite polynomial), which can be further expressed as infinite product by conjugate complex roots. Then, according to Lemma 3, Lemma 4, and Lemma 5, the functional equation $\xi(s) = \xi(1-s)$ leads to $(s-\alpha_i)^2 = (1-s-\alpha_i)^2$ with solution $\alpha_i = \frac{1}{2}$, where α_i are the real parts of the zeros of $\xi(s)$, i.e., $s_i = \alpha_i \pm j\beta_i, i \in \mathbb{N}$. Thus a proof of the Riemann Hypothesis is achieved.

Keywords Riemann Hypothesis (RH) · Proof · Completed zeta function $\xi(s)$

Mathematics Subject Classification (2020) 11M26

1 Introduction and the problem description

It has been 162 years since the Riemann Hypothesis (RH) was proposed in 1859 [1]. Many efforts and achievements have been made towards proving the hypothesis, but it is still an open problem [2−3].

The Riemann zeta function is the function of the complex variable s, defined in the half-plane $\Re(s) > 1$ by the absolutely convergent series [2]

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

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Riemann showed how to extend zeta function to the whole complex plane $\mathbb{C}$ by analytic continuation

$$\zeta(s)=\frac{\pi^{s/2}}{\Gamma(s/2)}\{\frac{1}{s(s-1)}+\int_{1}^{\infty}(x^{\frac{s}{2}-1}+x^{-\frac{s}{2}-\frac{1}{2}})\cdot(\frac{\theta(x)-1}{2})dx\} \tag{2}$$

where $\theta(x)=\sum_{-\infty}^{\infty}e^{-n^2\pi x}$, Γ being the Gamma function in the following equivalent form

$$\frac{1}{\Gamma(s)}=s\cdot e^{\gamma s}\prod_{n=1}^{\infty}(1+\frac{s}{n})e^{-s/n} \tag{3}$$

where γ is Euler's constant.

The connection between the zeta function and prime numbers can be established through the well-known Euler product.

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\prod_{p}(1-p^{-s})^{-1},\Re(s)>1 \tag{4}$$

the product being over the prime numbers p.

As shown by Riemann, $\zeta(s)$ extends to $\mathbb{C}$ as a meromorphic function with only a simple pole at $s=1$, with residue 1, and satisfies the following functional equation

$$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)=\pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})\zeta(1-s) \tag{5}$$

The Riemann zeta function $\zeta(s)$ has zeros at the negative even integers $-2,-4,-6,-8,\cdots$ and one refers to them as the **trivial zeros**. The other zeros of $\zeta(s)$ are the complex numbers, i.e., **non-trivial zeros** [2].

About the non-trivial zeros of $\zeta(s)$, the following results are well established [4].

Lemma 1: Non-trivial zeroes of $\zeta(s)$, noted as $\rho=\alpha+j\beta$ have the following properties
1) The number of non-trivial zeroes is infinity;
2) $\beta\neq 0$;
3) $0\leqslant\alpha\leqslant 1$;
4) $\rho,\bar{\rho},1-\bar{\rho},1-\rho$ are all non-trivial zeroes.

For further study, a completed zeta function $\xi(s)$ is defined as

$$\xi(s)=\frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) \tag{6}$$

It is well-known that $\xi(s)$ is an entire function of order 1.

Replacing s with $1-s$ in Eq.(6), and considering Eq.(5), we have the following functional equation

$$\xi(s) = \xi(1-s) \tag{7}$$

Considering the definition of $\xi(s)$, i.e., Eq. (6), and recalling Eq.(3), the trivial zeros of $\zeta(s)$ are canceled by the poles of $\Gamma(\frac{s}{2})$. The zero of $s-1$ and the pole of $\zeta(s)$ cancel; the zero $s=0$ and the pole of $\Gamma(\frac{s}{2})$ cancel [5−6]. Thus, all the zeros of $\xi(s)$ must be the nontrivial zeros of $\xi(s)$, and vice versa. Then we have the following Lemma 2.

Lemma 2: Zeros of $\xi(s)$ coincide with non-trivial zeros of $\zeta(s)$.

According to Lemma 2, the following two statements for RH are equivalent.

Statement 1 of RH: All the non-trivial zeros of $\zeta(s)$ have real part equal to $\frac{1}{2}$.

Statement 2 of RH: All the zeros of $\xi(s)$ have real part equal to $\frac{1}{2}$.

To prove the RH, the natural thinking is to estimate the number of zeros of $\zeta(s)$ in some certain closed areas according to the Argument Principle. Along this train of thought, there are some famous research works. Let $N(T)$ denote the number of zeros of $\zeta(s)$ inside the rectangle: $0 \leq \alpha \leq 1, 0 \leq \beta \leq T$, and let $N_0(T)$ denote the number of zeros of $\zeta(s)$ on the line $\alpha = \frac{1}{2}, 0 \leq \beta \leq T$. Selberg proved that there exist positive constants c and T_0, such that $N_0(T) > cN(T), (T > T_0)$[7], later on, Levinson proved that $c \geq \frac{1}{3}$[8], and Conrey proved that $c \geq \frac{2}{5}$[9].

Two types of infinite expansions of $\xi(s)$, i.e., MacLaurin series (infinite polynomial) and infinite product expansion by conjugate roots, will be adopted in this paper to open another door to the proof of RH.

The idea is motivated by Euler's work on

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{6} \tag{8}$$

This interesting and famous result is deduced mainly based on the above-mentioned two types of infinite expansions

$$\begin{aligned} \frac{\sin x}{x} &= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots \\ &= (1 - \frac{x^2}{\pi^2})(1 - \frac{x^2}{4\pi^2})(1 - \frac{x^2}{9\pi^2}) \cdots \end{aligned} \tag{9}$$

This paper is also motivated by the fact that $\xi(s)$ can be expressed by the following infinite product, which was first proposed by Riemann. However, it was Hadamard [10] who showed the validity of this infinite product expansion.

$$\xi(s) = \xi(0) \prod_{\rho} (1 - \frac{s}{\rho}) \tag{10}$$

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where ρ are precisely the non-trivial roots of the Riemann zeta function $\zeta(s)$, the factor ρ and $1-\rho$ are paired.

2 A proof of RH

This section is planned to give a proof of the Statement 2 of RH. For this purpose, we need the following results (Lemma 3, Lemma 4, and Lemma 5) on polynomial equations.

Lemma 3: Given two infinite polynomials

$$f(s) = a_0 + a_2(s-\alpha)^2 + a_4(s-\alpha)^4 + a_6(s-\alpha)^6 + \cdots + a_{2n}(s-\alpha)^{2n} + \cdots \quad (11)$$

and

$$f(1-s) = a_0 + a_2(1-s-\alpha)^2 + a_4(1-s-\alpha)^4 + a_6(1-s-\alpha)^6 + \cdots + a_{2n}(1-s-\alpha)^{2n} + \cdots \quad (12)$$

where s is a complex variable, $\alpha, a_0, a_2, a_4, a_6 + \cdots, a_{2n} \cdots \in \mathbb{R}$ are all real numbers, and $n \in \mathbb{N}$ are integers.

Then we have

$$f(s) = f(1-s) \Leftrightarrow (s-\alpha)^2 = (1-s-\alpha)^2 \quad (13)$$

where " $\Leftrightarrow$ " is the equivalent sign.

Proof: The proof is delivered by mathematical induction.
First, it is obvious that Lemma 3 is true for $n = 2$, i.e.,

$$a_0 + a_2(s-\alpha)^2 = a_0 + a_2(1-s-\alpha)^2 \Leftrightarrow (s-\alpha)^2 = (1-s-\alpha)^2 \quad (14)$$

Second, suppose Lemma 3 is true for $n = 2m$, then we only need to prove Lemma 3 is true for $n = 2m+2$.
Thus we begin with the following Eq.(15)

$$a_0+a_2(s-\alpha)^2+\cdots+a_{2m}(s-\alpha)^{2m} = a_0+a_2(1-s-\alpha)^2+\cdots+a_{2m}(1-s-\alpha)^{2m} \Leftrightarrow (s-\alpha)^2 = (1-s-\alpha)^2 \quad (15)$$

Now, let's consider

$$a_0+a_2(s-\alpha)^2+\cdots+a_{2m+2}(s-\alpha)^{2m+2} = a_0+a_2(1-s-\alpha)^2+\cdots+a_{2m+2}(1-s-\alpha)^{2m+2} \quad (16)$$

i.e.,

$$(s-\alpha)^2\{a_2+a_4(s-\alpha)^2+\cdots+a_{2m+2}(s-\alpha)^{2m}\} = (1-s-\alpha)^2\{a_2+a_4(1-s-\alpha)^2+\cdots+a_{2m+2}(1-s-\alpha)^{2m}\} \quad (17)$$

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Since we consider s being complex numbers, then $s = \alpha$ and $s = 1-\alpha$ are trivial cases (real numbers), so we only consider $s \neq \alpha, s \neq 1-\alpha$. Then by Eq.(17), we get

$$\frac{(s-\alpha)^2}{(1-s-\alpha)^2} = \frac{a_2 + a_4(1-s-\alpha)^2 + \cdots + a_{2m+2}(1-s-\alpha)^{2m}}{a_2 + a_4(s-\alpha)^2 + \cdots + a_{2m+2}(s-\alpha)^{2m}} \tag{18}$$

Without loss of generality, set

$$\frac{(s-\alpha)^2}{(1-s-\alpha)^2} = \frac{a_2 + a_4(1-s-\alpha)^2 + \cdots + a_{2m+2}(1-s-\alpha)^{2m}}{a_2 + a_4(s-\alpha)^2 + \cdots + a_{2m+2}(s-\alpha)^{2m}} = k \neq 0 \tag{19}$$

where k is a real or complex number to be determined.
Then Eq.(19) is equivalent to the following Eq.(20)

$$\begin{cases} ka_2 + ka_4(s-\alpha)^2 + \cdots + ka_{2m+2}(s-\alpha)^{2m} = a_2 + a_4(1-s-\alpha)^2 + \cdots + a_{2m+2}(1-s-\alpha)^{2m} \\ (s-\alpha)^2 = k(1-s-\alpha)^2 \end{cases} \tag{20}$$

According to Eq.(15) and the arbitrariness of its coefficients, we see that Eq.(20) holds if and only if $k = 1$, i.e.,

$$\begin{aligned} & a_0 + a_2(s-\alpha)^2 + \cdots + a_{2m+2}(s-\alpha)^{2m+2} = a_0 + a_2(1-s-\alpha)^2 + \cdots + a_{2m+2}(1-s-\alpha)^{2m+2} \\ & \Leftrightarrow \\ & (s-\alpha)^2 = (1-s-\alpha)^2 \end{aligned} \tag{21}$$

Then we conclude that Lemma 3 is true for $n = 2m+2$.
Finally, by mathematical induction, Lemma 3 is true for any natural number $2n$.
That completes the proof of Lemma 3.

Remark 1: The equality condition of two polynomials like Eq.(11) and Eq.(12) only depends on their lowest no-constant terms, i.e., the quadratic terms.

Lemma 4: Given two polynomials

$$f(s) = a_0 + a_1(s-b_1)^2 + a_2(s-b_2)^2 + a_3(s-b_3)^2 + \cdots + a_n(s-b_n)^2 + \cdots \tag{22}$$

and

$$f(1-s) = a_0 + a_1(1-s-b_1)^2 + a_2(1-s-b_2)^2 + a_3(1-s-b_3)^2 + \cdots + a_n(1-s-b_n)^2 + \cdots \tag{23}$$

where s is a complex variable, $a_0, a_1, b_1, a_2, b_2 \cdots, a_n, b_n, \cdots \in \mathbb{R}$ are all real numbers.

Then we have

$$f(s) = f(1-s) \Leftrightarrow (s-b_i)^2 = (1-s-b_i)^2, i \in \mathbb{N} \tag{24}$$

Proof: The proof is delivered by mathematical induction.

First, Lemma 4 is true for $n = 1$, i.e.,

$$a_0 + a_1(s - b_1)^2 = a_0 + a_1(1 - s - b_1)^2 \Leftrightarrow (s - b_1)^2 = (1 - s - b_1)^2 \qquad (25)$$

Second, suppose Lemma 4 is true for $n = m$, then we only need to prove Lemma 4 is true for $n = m + 1$.

Thus we begin with the following Eq.(26)

$$\begin{aligned} & a_0 + a_1(s - b_1)^2 + \cdots + a_m(s - b_m)^2 = a_0 + a_1(1 - s - b_1)^2 + \cdots + a_m(1 - s - b_m)^2 \\ & \Leftrightarrow \\ & (s - b_i)^2 = (1 - s - b_i)^2, i = 1, 2, 3, \cdots, m \end{aligned} \qquad (26)$$

Besides, it is obvious that

$$\begin{aligned} & a_{m+1}(s - b_{m+1})^2 = a_{m+1}(1 - s - b_{m+1})^2 \\ & \Leftrightarrow \\ & (s - b_{m+1})^2 = (1 - s - b_{m+1})^2 \end{aligned} \qquad (27)$$

Putting Eq.(26) and Eq.(27) together, we obtain

$$\begin{aligned} & a_0 + a_1(s - b_1)^2 + \cdots + a_m(s - b_m)^2 + a_{m+1}(s - b_{m+1})^2 \\ & = a_0 + a_1(1 - s - b_1)^2 + \cdots + a_m(1 - s - b_m)^2 + a_{m+1}(1 - s - b_{m+1})^2 \\ & \Leftrightarrow \\ & (s - b_i)^2 = (1 - s - b_i)^2, i = 1, 2, 3, \cdots, m, m + 1 \end{aligned} \qquad (28)$$

Then we conclude that Lemma 4 is true for $n = m + 1$.

Finally, by mathematical induction, Lemma 4 is true for any natural number n.

That completes the proof of Lemma 4.

Lemma 5: Given two polynomials

$$\begin{aligned} f(s) = a_0 &+ a_{21}(s - b_1)^2 + a_{22}(s - b_2)^2 + a_{23}(s - b_3)^2 + \cdots + a_{2n}(s - b_n)^2 + \cdots \\ &+ a_{41}(s - b_1)^4 + a_{42}(s - b_2)^4 + a_{43}(s - b_3)^4 + \cdots + a_{4n}(s - b_n)^4 + \cdots \\ &+ a_{61}(s - b_1)^6 + a_{62}(s - b_2)^6 + a_{63}(s - b_3)^6 + \cdots + a_{6n}(s - b_n)^6 + \cdots \\ &\cdots \end{aligned} \qquad (29)$$

and

$$\begin{aligned} f(1-s) = a_0 &+ a_{21}(1-s-b_1)^2 + a_{22}(1-s-b_2)^2 + a_{23}(1-s-b_3)^2 + \cdots + a_{2n}(1-s-b_n)^2 + \cdots \\ &+ a_{41}(1-s-b_1)^4 + a_{42}(1-s-b_2)^4 + a_{43}(1-s-b_3)^4 + \cdots + a_{4n}(1-s-b_n)^4 + \cdots \\ &+ a_{61}(1-s-b_1)^6 + a_{62}(1-s-b_2)^6 + a_{63}(1-s-b_3)^6 + \cdots + a_{6n}(1-s-b_n)^6 + \cdots \\ &\cdots \end{aligned} \tag{30}$$

where s is a complex variable, $a_0, a_{ki}, b_i \in \mathbb{R}$ are all real numbers, $k, i \in \mathbb{N}$ are positive integers.

Then we have

$$f(s) = f(1-s) \Leftrightarrow (s-b_i)^2 = (1-s-b_i)^2, i \in \mathbb{N} \tag{31}$$

Proof: Although polynomials Eq.(29) and Eq.(30) seem to be complicated, they are of the same type with Eq.(11) and Eq.(12) in Lemma 3. By Lemma 3 and Remark 1, the equality condition of $f(s)$ and $f(1-s)$ are determined by their lowest non-constant terms, i.e., the quadratic terms. Then by Lemma 4, we conclude that Lemma 5 is true.

To be more specific, comparing the like terms of Eq.(29) and Eq.(30), i.e., comparing Eq.(29) and Eq.(30) line by line, it is obvious by Lemma 4 that

$$f(s) = f(1-s) \Rightarrow (s-b_i)^2 = (1-s-b_i)^2, i \in \mathbb{N}$$

It is also obvious that

$$(s-b_i)^2 = (1-s-b_i)^2, i \in \mathbb{N} \Rightarrow f(s) = f(1-s)$$

That completes the proof of Lemma 5.

Remark 2: In Lemma 5, if the 4th power terms and higher power terms are composed of the product of quadratic factors, i.e., $(s-b_i)^2$ and $(1-s-b_i)^2$, then the conclusion remains unchanged.

Next, we present the proof of RH.

Proof of RH: The details are delivered in three steps as follows.

Step 1: Since $\xi(s)$ is an entire function, it is analytic in the whole complex plane $\mathbb{C}$. Then $\xi(s)$ can be expanded in MacLaurin series (infinite polynomial) at $s = 0$, i.e.

$$\xi(s) = \xi(0) + \xi^{'}(0)s + \frac{\xi^{''}(0)}{2!}s^2 + \cdots + \frac{\xi^{(n)}(0)}{n!}s^n + \cdots, |s| < \infty \tag{32}$$

 doi:10.20944/preprints202108.0146.v8

It is obvious that $\frac{\xi^{(n)}(0)}{n!} = \frac{\xi^{(n)}(s)}{n!}\Big|_{s=0}, n = 0, 1, 2, \cdots$ are all real numbers. Thus, all the zeros of $\xi(s)$ are the roots of the following infinite algebraic equation with real coefficients.

$$0 = \xi(0) + \xi^{'}(0)s + \frac{\xi^{''}(0)}{2!}s^2 + \cdots + \frac{\xi^{(n)}(0)}{n!}s^n + \cdots \tag{33}$$

According to the well established theory of algebraic equation with real number coefficients, complex roots always come in pairs (complex conjugates). Further by Lemma 2, all the zeros of $\xi(s)$ are complex conjugate pairs, then we denote the roots of Eq.(33) as $s_i = \alpha_i \pm j\beta_i, \beta_i \neq 0, i \in \mathbb{N}$.

Therefore, based on Eq.(10), Eq.(32) can be rewritten as infinite product by conjugate complex roots $\alpha_i \pm j\beta_i, i \in \mathbb{N}$

$$\begin{aligned}
\xi(s) &= \xi(0)\prod_{i=1}^{\infty}(1 - \frac{s}{\alpha_i + j\beta_i})(1 - \frac{s}{\alpha_i - j\beta_i}) \\
&= \xi(0)\prod_{i=1}^{\infty}(1 - \frac{2\alpha_i s}{\alpha_i^2 + \beta_i^2} + \frac{s^2}{\alpha_i^2 + \beta_i^2}) \\
&= \xi(0)\prod_{i=1}^{\infty}\frac{\beta_i^2}{\alpha_i^2 + \beta_i^2}\Big(1 + \frac{(s - \alpha_i)^2}{\beta_i^2}\Big) \\
&= \xi(0)\prod_{i=1}^{\infty}\frac{\beta_i^2}{\alpha_i^2 + \beta_i^2}\prod_{i=1}^{\infty}\Big(1 + \frac{(s - \alpha_i)^2}{\beta_i^2}\Big) \\
&= \xi(0)\prod_{i=1}^{\infty}\frac{\beta_i^2}{\alpha_i^2 + \beta_i^2} + \xi(0)\prod_{i=1}^{\infty}\frac{\beta_i^2}{\alpha_i^2 + \beta_i^2}\Big(\frac{(s - \alpha_1)^2}{\beta_1^2} + \frac{(s - \alpha_2)^2}{\beta_2^2} + \cdots\Big) + \cdots
\end{aligned} \tag{34}$$

According to Lemma 1, $\beta_i \neq 0; 0 \leqslant \alpha_i \leqslant 1$, then it is obvious that

$$0 < \prod_{i=1}^{\infty}\frac{\beta_i^2}{\alpha_i^2 + \beta_i^2} \leqslant 1 \tag{35}$$

Thus Eq.(34) is an infinite polynomial with real coefficients.

Step2: Replacing s with $1 - s$ in Eq.(34) yields another infinite polynomial with real coefficients.

$$\begin{aligned}
\xi(1 - s) &= \xi(0)\prod_{i=1}^{\infty}\frac{\beta_i^2}{\alpha_i^2 + \beta_i^2}\Big(1 + \frac{(1 - s - \alpha_i)^2}{\beta_i^2}\Big) \\
&= \xi(0)\prod_{i=1}^{\infty}\frac{\beta_i^2}{\alpha_i^2 + \beta_i^2} + \xi(0)\prod_{i=1}^{\infty}\frac{\beta_i^2}{\alpha_i^2 + \beta_i^2}\Big(\frac{(1 - s - \alpha_1)^2}{\beta_1^2} + \frac{(1 - s - \alpha_2)^2}{\beta_2^2} + \cdots\Big) + \cdots
\end{aligned} \tag{36}$$

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Step 3: We have by $\xi(s) = \xi(1-s)$ that

$$\xi(0)\prod_{i=1}^{\infty}\frac{\beta_i^2}{\alpha_i^2+\beta_i^2} + \xi(0)\prod_{i=1}^{\infty}\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}\Big(\frac{(s-\alpha_1)^2}{\beta_1^2} + \frac{(s-\alpha_2)^2}{\beta_2^2} + \cdots\Big) + \cdots$$
$$=\xi(0)\prod_{i=1}^{\infty}\frac{\beta_i^2}{\alpha_i^2+\beta_i^2} + \xi(0)\prod_{i=1}^{\infty}\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}\Big(\frac{(1-s-\alpha_1)^2}{\beta_1^2} + \frac{(1-s-\alpha_2)^2}{\beta_2^2} + \cdots\Big) + \cdots \tag{37}$$

By Lemma 5 with Remark 2, Eq.(37) means

$$(s-\alpha_i)^2 = (1-s-\alpha_i)^2, i \in \mathbb{N} \tag{38}$$

Solving Eq.(38), we get

Solution 1:

$$(s-\alpha_i) = -(1-s-\alpha_i) \Rightarrow \alpha_i = \frac{1}{2}, i \in \mathbb{N} \tag{39}$$

Solution 2:

$$(s-\alpha_i) = 1-s-\alpha_i \Rightarrow \ s = \frac{1}{2} \tag{40}$$

Noticing that Solution 2 leads to a contradiction that if $s = \frac{1}{2}$, then there exist no complex roots for $\xi(s) = \xi(1-s) = 0$.

Thus, Solution 1, i.e., $\alpha_i = \frac{1}{2}, i \in \mathbb{N}$ is the only solution of Eq.(38), further of Eq.(37), even further of $\xi(s) = \xi(1-s)$. That means all the zeros of the completed zeta function $\xi(s)$ have real part equal to $\frac{1}{2}$.

That completes the proof of the Statement 2 of RH.

Remark 3: According to Lemma 2, we know that the Statement 1 of RH is also true, i.e., All the non-trivial zeros of the Riemann zeta function $\zeta(s)$ have real part equal to $\frac{1}{2}$.

Remark 4: By Lemma 1, 2 pairs of complex zeros of $\zeta(s)$ exist simultaneously, i.e., $\rho = \alpha + j\beta, \bar{\rho} = \alpha - j\beta, 1-\rho = 1-\alpha-j\beta, 1-\bar{\rho} = 1-\alpha+j\beta$. With the proof of RH, i.e., $\alpha = \frac{1}{2}$, these 2 pairs of zeros actually is only one pair, because $\rho = 1-\bar{\rho} = \frac{1}{2}+j\beta, \bar{\rho} = 1-\rho = \frac{1}{2}-j\beta$. Thus Lemma 1 could be modified more precisely as follows.

Lemma 1*: Non-trivial zeroes of $\zeta(s)$, noted as $\rho = \alpha + j\beta$ have the following properties
1) The number of non-trivial zeroes is infinity;
2) $\beta \neq 0$;
3) $0 \leqslant \alpha \leqslant 1$;
4) $\rho = 1-\bar{\rho}, \bar{\rho} = 1-\rho$ are all non-trivial zeroes.

3 Conclusion

A proof of the Riemann Hypothesis is presented based on a new road map: First, the completed zeta function $\xi(s)$ is expressed as MacLaurin series (infinite polynomial), and further expressed as infinite product by conjugate complex roots; Second, based on Lemma 3, Lemma 4, and Lemma 5, the functional equation $\xi(s) = \xi(1-s)$ leads to $(s-\alpha_i)^2 = (1-s-\alpha_i)^2$ with solution $\alpha_i = \frac{1}{2}$.

Then we conclude that the celebrated Riemann Hypothesis is true.

References

1. Riemann, B. (1859), Über die Anzahl der Primzahlen unter einer gegebenen Grösse. Monatsberichte der Deutschen Akademie der Wissenschaften zu Berlin, 2, 671-680.
2. E. Bombieri (2000), Problems of the millennium: The Riemann Hypothesis, CLAY
3. Peter Sarnak (2004), Problems of the Millennium: The Riemann Hypothesis, CLAY
4. Tom M. Apostol (1998), Introduction to Analytic Number Theory, New York: Springer
5. Chengdong Pan, Chengbiao Pan (2016), Basic Analytic Number Theory (in Chinese), 2nd Edition, Harbin Institute of Technology Press
6. Reyes, Ernesto Oscar (2004), The Riemann zeta function, Master Thesis of California State University, San Bernardino, Theses Digitization Project. 2648. https://scholarworks.lib.csusb.edu/etd-project/2648
7. A. Selberg, On the zeros of the zeta-function of Riemann, Der Kong. Norske Vidensk. Selsk. Forhand. 15 (1942), 59-62; also, Collected Papers, Springer- Verlag, Berlin - Heidelberg - New York 1989, Vol. I, 156-159.
8. N. Levinson, More than one-third of the zeros of the Riemann zetafunction are on $\sigma = 1/2$, Adv. Math. 13 (1974), 383-436.
9. J.B. Conrey, More than two fifths of the zeros of the Riemann zeta function are on the critical line, J. reine angew. Math. 399 (1989), 1-26.
10. Hadamard, J. (1893), Étude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann. Journal de mathématiques pures et appliquées, 9, 171-216.