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A Proof of the Riemann Hypothesis Based on MacLaurin Expansion of the Completed Zeta Function

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Abstract The basic idea is to expand the completed zeta function $\xi(s)$ in MacLaurin series (infinite polynomial). Thus, by $\xi(s) = \xi(1-s) = 0$, we have the following infinite polynomial equation

$$\begin{aligned}&\xi(0) + \xi^{'}(0)s + \frac{\xi^{''}(0)}{2!}s^2 + \cdots + \frac{\xi^{(n)}(0)}{n!}s^n + \cdots \\ =&\xi(0) + \xi^{'}(0)(1-s) + \frac{\xi^{''}(0)}{2!}(1-s)^2 + \cdots + \frac{\xi^{(n)}(0)}{n!}(1-s)^n + \cdots \\ =&0\end{aligned}$$

which finally leads to $s = 1 - s, s = \alpha \pm j\beta; 1 - \alpha \pm j\beta, \beta \neq 0$, then a proof of the Riemann Hypothesis can be achieved.

Keywords Riemann Hypothesis (RH) · Proof · Completed zeta function $\xi(s)$

Mathematics Subject Classification (2020) 11M26

1 Introduction and the problem description

It has been 162 years since the Riemann Hypothesis (RH) is proposed in 1859 [1]. Many efforts and achievements have been made towards proving the hypothesis, but it is still an open problem [2−3].

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The Riemann zeta function is the function of the complex variable s, defined in the half-plane $\Re(s) > 1$ by the absolutely convergent series [2]

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

Riemann showed how to extend zeta function to the whole complex plane $\mathbb{C}$ by analytic continuation

$$\zeta(s) = \frac{\pi^{s/2}}{\Gamma(s/2)}\{\frac{1}{s(s-1)} + \int_1^{\infty}(x^{\frac{s}{2}-1} + x^{-\frac{s}{2}-\frac{1}{2}}) \cdot (\frac{\theta(x)-1}{2})dx\} \tag{2}$$

where $\theta(x) = \sum_{-\infty}^{\infty} e^{-n^2\pi x}$, Γ being the Gamma function in the following equivalent form

$$\frac{1}{\Gamma(s)} = s \cdot e^{\gamma s} \prod_{n=1}^{\infty}(1 + \frac{s}{n})e^{-s/n} \tag{3}$$

where γ is Euler's constant.

The connection between the zeta function and prime numbers can be established through the well-known Euler product.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p (1 - p^{-s})^{-1}, \Re(s) > 1 \tag{4}$$

the product being over the prime numbers p.

As shown by Riemann, $\zeta(s)$ extends to $\mathbb{C}$ as a meromorphic function with only a simple pole at $s = 1$, with residue 1, and satisfies the following functional equation

$$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})\zeta(1-s) \tag{5}$$

The Riemann zeta function $\zeta(s)$ has zeros at the negative even integers $-2, -4, -6, -8, \cdots$ and one refers to them as the **trivial zeros**. The other zeros of $\zeta(s)$ are the complex numbers, i.e., **non-trivial zeros** [2].

About the non-trivial zeros of $\zeta(s)$, the following results are well established [4].

Lemma 1: Non-trivial zeroes of $\zeta(s)$, noted as $\rho = \alpha + j\beta$ have the following properties
1) The number of non-trivial zeroes is infinity;
2) $\beta \neq 0$;
3) $0 \leqslant \alpha \leqslant 1$;
4) $\rho, \bar{\rho}, 1 - \bar{\rho}, 1 - \rho$ are all non-trivial zeroes.

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For further study, a completed zeta function $\xi(s)$ is defined as

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) \tag{6}$$

It is well-known that $\xi(s)$ is an entire function of order 1.

Replacing s with $1-s$ in Eq.(6), and considering Eq.(5), we have the following functional equation

$$\xi(s) = \xi(1-s) \tag{7}$$

Considering the definition of $\xi(s)$, i.e., Eq. (6), and recalling Eq.(3), the trivial zeros of $\zeta(s)$ are canceled by the poles of $\Gamma(\frac{s}{2})$. The zero of $s-1$ and the pole of $\zeta(s)$ cancel; the zero $s=0$ and the pole of $\Gamma(\frac{s}{2})$ cancel [5−6]. Thus, all the zeros of $\xi(s)$ must be the nontrivial zeros of $\xi(s)$, and vice versa. That means the following result, i.e., Lemma 2.

Lemma 2: Zeros of $\xi(s)$ coincide with nontrivial zeros of $\zeta(s)$.

According to Lemma 2, the following two statements for RH are equivalent.

Statement 1 of RH: The non-trivial zeros of $\zeta(s)$ have real part equal to $\frac{1}{2}$.

Statement 2 of RH: All the zeros of $\xi(s)$ have real part equal to $\frac{1}{2}$.

It is well-known that $\xi(s)$ can be expressed by the following infinite product, which was first proposed by Riemann. However, it was Hadamard who showed the validity of this infinite product expansion [7]

$$\xi(s) = \xi(0)\prod_{\rho}(1-\frac{s}{\rho}) \tag{8}$$

where ρ are precisely the roots of the Riemann zeta function $\zeta(s)$, the factor ρ and $1-\rho$ are paired.

This paper will use another infinite expansion of $\xi(s)$, i.e., MacLaurin series (infinite polynomial) to open the door to the proof of RH.

2 A proof of RH

This section is planned to give a proof of the Statement 2 of RH. For this purpose, we need the following result on infinite polynomial equation.
Lemma 3: Given two infinite polynomials

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots \tag{9}$$

and

$$f(y) = a_0 + a_1 y + a_2 y^2 + \cdots + a_n y^n + \cdots \tag{10}$$

where x and y are variables(either real or complex), $a_0, a_1, a_2, \cdots, a_n, \cdots \in \mathbb{R}$ are all real coefficients, and $n \in \mathbb{N}$ are integers.

Then we have

$$f(x) = f(y) \Leftrightarrow x = y \tag{11}$$

Proof: The proof is delivered by mathematical induction.
First, it is obvious that Lemma 3 is true for $n = 1$, i.e.,

$$a_0 + a_1 x = a_0 + a_1 y \Leftrightarrow x = y \tag{12}$$

Second, suppose Lemma 3 is true for $n = m$, then we only need to prove Lemma 3 is true for $n = m + 1$.
Thus we begin with the following Eq.(13)

$$a_0 + a_1 x + \cdots + a_m x^m = a_0 + a_1 y + \cdots + a_m y^m \Leftrightarrow x = y \tag{13}$$

Now, let's consider

$$a_0 + a_1 x + \cdots + a_{m+1} x^{m+1} = a_0 + a_1 y + \cdots + a_{m+1} y^{m+1} \tag{14}$$

i.e.,

$$x(a_1 + a_2 x \cdots + a_{m+1} x^m) = y(a_1 + a_2 y + \cdots + a_m y^{m+1}) \tag{15}$$

It is trivial that $x = y = 0$ makes Lemma 3 to be true, thus, we only consider $x \neq 0, y \neq 0$. Then by Eq.(15), we get

$$\frac{x}{y} = \frac{a_1 + a_2 y \cdots + a_{m+1} y^m}{a_1 + a_2 x \cdots + a_{m+1} x^m} \tag{16}$$

Without loss of generality, set

$$\frac{x}{y} = \frac{a_1 + a_2 y \cdots + a_{m+1} y^m}{a_1 + a_2 x \cdots + a_{m+1} x^m} = k \neq 0 \tag{17}$$

where k is a real or complex number to be determined.
Then Eq.(17) is equivalent to the following Eq.(18)

$$\begin{cases} ka_1 + ka_2 x \cdots + ka_{m+1} x^m = a_1 + a_2 y + \cdots + a_{m+1} y^m \\ x = ky \end{cases} \tag{18}$$

According to Eq.(13) and the arbitrariness of its coefficients, we see that Eq.(18) holds if and only if $k = 1 (x = y)$, i.e.,

$$a_0 + a_1 x + \cdots + a_{m+1} x^{m+1} = a_0 + a_1 y + \cdots + a_{m+1} y^{m+1} \Leftrightarrow x = y \tag{19}$$

Then we conclude: Lemma 3 is true for $n = m + 1$.
Finally, by mathematical induction, Lemma 3 is true for any natural number n.

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That completes the proof of Lemma 3.

Proof of RH: The details are delivered in three steps as follows.

Step 1: Since $\xi(s)$ is an entire function, it is analytic in the whole complex plane $\mathbb{C}$. Then $\xi(s)$ can be expanded in MacLaurin series (infinite polynomial) at $s = 0$, i.e.

$$\xi(s) = \xi(0) + \xi^{'}(0)s + \frac{\xi^{''}(0)}{2!}s^2 + \cdots + \frac{\xi^{(n)}(0)}{n!}s^n + \cdots, |s| < \infty \tag{20}$$

It is obvious that $\frac{\xi^{(n)}(0)}{n!} = \left.\frac{\xi^{(n)}(s)}{n!}\right|_{s=0}, n = 0, 1, 2, \cdots$ are all real numbers. Thus, all the zeros of $\xi(s)$ are the roots of the following infinite algebraic equation with real coefficients.

$$0 = \xi(0) + \xi^{'}(0)s + \frac{\xi^{''}(0)}{2!}s^2 + \cdots + \frac{\xi^{(n)}(0)}{n!}s^n + \cdots \tag{21}$$

According to the well established theory of algebraic equation with real number coefficients, complex roots always come in pairs (complex conjugate). Further by Lemma 2, all the zeros of $\xi(s)$ are complex pairs, then we denote the roots of Eq.(21) as $s = \alpha \pm j\beta, \beta \neq 0$.

Step2: Replacing s with $1 - s$ in Eq.(20) yields

$$\xi(1-s) = \xi(0)+\xi^{'}(0)(1-s)+\frac{\xi^{''}(0)}{2!}(1-s)^2+\cdots+\frac{\xi^{(n)}(0)}{n!}(1-s)^n+\cdots, |s| < \infty \tag{22}$$

Therefore, all the zeros of $\xi(1-s)$ are the roots of the following infinite algebraic equation with real coefficients.

$$0 = \xi(0) + \xi^{'}(0)(1 - s) + \frac{\xi^{''}(0)}{2!}(1 - s)^2 + \cdots + \frac{\xi^{(n)}(0)}{n!}(1 - s)^n + \cdots \tag{23}$$

where $s = 1 - \alpha \pm j\beta, \beta \neq 0$.

Step 3: Then we have by $\xi(s) = \xi(1 - s)$ that

$$\begin{aligned} &\xi(0) + \xi^{'}(0)s + \frac{\xi^{''}(0)}{2!}s^2 + \cdots + \frac{\xi^{(n)}(0)}{n!}s^n + \cdots \\ =&\xi(0) + \xi^{'}(0)(1 - s) + \frac{\xi^{''}(0)}{2!}(1 - s)^2 + \cdots + \frac{\xi^{(n)}(0)}{n!}(1 - s)^n + \cdots \end{aligned} \tag{24}$$

By Lemma 3, Eq.(24) gives

$$s = 1 - s \tag{25}$$

Of course, the roots of $\xi(s) = 0$, i.e., $s = \alpha \pm j\beta$, and the roots of $\xi(1 - s) = 0$, i.e., $s = 1 - \alpha \pm j\beta$, should obey Eq.(25), then we obtain

$$s = 1 - s, s = \alpha \pm j\beta \tag{26}$$

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and

$$s = 1 - s, s = 1 - \alpha \pm j\beta \tag{27}$$

It follows from both Eq.(26) and Eq.(27) that

$$\alpha \pm j\beta = 1 - \alpha \pm j\beta \quad \Rightarrow \quad \alpha = 1 - \alpha \quad \Rightarrow \quad \alpha = \frac{1}{2} \tag{28}$$

Then we conclude that all the zeros of the completed zeta function $\xi(s)$ have real part equal to $\frac{1}{2}$.

That completes the proof of the Statement 2 of RH.

Remark: According to Lemma 2, we know that the Statement 1 of RH is also true, i.e., The non-trivial zeros of the Riemann zeta function $\zeta(s)$ have real part equal to $\frac{1}{2}$.

3 Conclusion

A proof of the Riemann Hypothesis is presented based on a new road map: First, the completed zeta function $\xi(s)$ is expressed as MacLaurin series (infinite polynomial); Second, by Lemma 3, $\xi(s) = \xi(1 - s) = 0$ means

$$s = 1 - s, s = \alpha \pm j\beta; 1 - \alpha \pm j\beta, \beta \neq 0 \quad \Rightarrow \quad \alpha = \frac{1}{2}$$

Then we conclude that the celebrated Riemann Hypothesis is true.

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