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A Proof of the Riemann Hypothesis Based on MacLaurin Expansion of the Completed Zeta Function

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Abstract The basic idea is to expand the completed zeta function $\xi(s)$ in MacLaurin series (infinite polynomial). Thus, by $\xi(s) = \xi(1-s) = 0$, we have the following infinite polynomial equation

$$\begin{aligned} & \xi(0) + \xi'(0)s + \frac{\xi''(0)}{2!}s^2 + \cdots + \frac{\xi^{(n)}(0)}{n!}s^n + \cdots \\ &= \xi(0) + \xi'(0)(1-s) + \frac{\xi''(0)}{2!}(1-s)^2 + \cdots + \frac{\xi^{(n)}(0)}{n!}(1-s)^n + \cdots \\ &= 0 \end{aligned}$$

which finally leads to $s = 1-s$, $s = \alpha \pm j\beta$, $\beta \neq 0$, then a proof of the Riemann Hypothesis can be achieved.

Keywords Riemann Hypothesis (RH) · Proof · Completed zeta function $\xi(s)$

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1 Introduction and the problem description

It has been 162 years since the Riemann Hypothesis (RH) is proposed in 1859 ^[1]. Many efforts and achievements have been made towards proving the hypothesis, but it is still an open problem ^[2–3].

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The Riemann zeta function is the function of the complex variable s , defined in the half-plane $\Re(s) > 1$ by the absolutely convergent series ^[2]

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad (1)$$

Riemann showed how to extend zeta function to the whole complex plane $\mathbb{C}$ by analytic continuation

$$\zeta(s) = \frac{\pi^{s/2}}{\Gamma(s/2)} \left\{ \frac{1}{s(s-1)} + \int_1^{\infty} (x^{\frac{s}{2}-1} + x^{-\frac{s}{2}-\frac{1}{2}}) \cdot \left(\frac{\theta(x)-1}{2} \right) dx \right\} \quad (2)$$

where $\theta(x) = \sum_{n=-\infty}^{\infty} e^{-n^2 \pi x}$, Γ being the Gamma function in the following equivalent form

$$\frac{1}{\Gamma(s)} = s \cdot e^{\gamma s} \prod_{n=1}^{\infty} \left(1 + \frac{s}{n} \right) e^{-s/n} \quad (3)$$

where γ is Euler's constant.

The connection between the zeta function and prime numbers can be established through the well-known Euler product.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p (1 - p^{-s})^{-1}, \Re(s) > 1 \quad (4)$$

the product being over the prime numbers p .

As shown by Riemann, $\zeta(s)$ extends to $\mathbb{C}$ as a meromorphic function with only a simple pole at $s = 1$, with residue 1, and satisfies the following functional equation

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s) \quad (5)$$

The Riemann zeta function $\zeta(s)$ has zeros at the negative even integers $-2, -4, -6, -8, \dots$ and one refers to them as the **trivial zeros**. The other zeros of $\zeta(s)$ are the complex numbers, i.e., **non-trivial zeros** ^[2].

About the non-trivial zeros of $\zeta(s)$, the following results are well established ^[4].

Lemma 1: Non-trivial zeroes of $\zeta(s)$, noted as $\rho = \alpha + j\beta$ have the following properties

- 1) The number of non-trivial zeroes is infinity;
- 2) $\beta \neq 0$;
- 3) $0 \leq \alpha \leq 1$;
- 4) $\rho, \bar{\rho}, 1 - \bar{\rho}, 1 - \rho$ are all non-trivial zeroes.

For further study, a completed zeta function $\xi(s)$ is defined as

$$\xi(s) = \frac{1}{2}s(1-s)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) \quad (6)$$

It is well-known that $\xi(s)$ is an entire function of order 1.

Replacing s with $1-s$ in Eq.(6), and considering Eq.(5), we have the following functional equation

$$\xi(s) = \xi(1-s) \quad (7)$$

Considering the definition of $\xi(s)$, i.e., Eq. (6), and recalling Eq.(3), the trivial zeros of $\zeta(s)$ are canceled by the poles of $\Gamma(\frac{s}{2})$. The zero of $s-1$ and the pole of $\zeta(s)$ cancel; the zero $s=0$ and the pole of $\Gamma(\frac{s}{2})$ cancel [5–6]. Thus, all the zeros of $\xi(s)$ must be the nontrivial zeros of $\zeta(s)$, and vice versa. That means the following result, i.e., Lemma 2.

Lemma 2: Zeros of $\xi(s)$ coincide with nontrivial zeros of $\zeta(s)$.

It is well-known that $\xi(s)$ can be expressed by the following infinite product, which was first proposed by Riemann. However, it was Hadamard who showed the validity of this infinite product expansion [7]

$$\xi(s) = \xi(0) \prod_{\rho} \left(1 - \frac{s}{\rho}\right) \quad (8)$$

where ρ are precisely the roots of the Riemann zeta function $\zeta(s)$, the factor ρ and $1-\rho$ are paired.

This paper will use another infinite expansion of $\xi(s)$, i.e., MacLaurin series (infinite polynomial) to open the door to the proof of RH. The following two statements for RH are equivalent.

Statement 1 of RH: The non-trivial zeros of $\zeta(s)$ have real part equal to $\frac{1}{2}$.

Statement 2 of RH: All the zeros of $\xi(s)$ have real part equal to $\frac{1}{2}$.

2 A proof of RH

This section is planned to give a proof of the Statement 2 of RH. For this purpose, we need the following result on infinite polynomial equation.

Lemma 3: Given two infinite polynomials

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots \quad (9)$$

and

$$f(y) = a_0 + a_1y + a_2y^2 + \cdots + a_ny^n + \cdots \quad (10)$$

where x and y are variables (either real or complex), $a_0, a_1, a_2, \cdots, a_n, \cdots \in \mathbb{R}$ are all real coefficients, and $n \in \mathbb{N}$ are integers.

Then we have

$$f(x) = f(y) \Leftrightarrow x = y \quad (11)$$

Proof: The proof is delivered by mathematical induction.

First, it is obvious that Lemma 3 is true for $n = 1$, i.e.,

$$a_0 + a_1x = a_0 + a_1y \Leftrightarrow x = y \quad (12)$$

Second, suppose Lemma 3 is true for $n = m$, then we only need to prove Lemma 3 is true for $n = m + 1$.

Thus we begin with the following Eq.(13)

$$a_0 + a_1x + \cdots + a_mx^m = a_0 + a_1y + \cdots + a_my^m \Leftrightarrow x = y \quad (13)$$

Now, let's consider

$$a_0 + a_1x + \cdots + a_{m+1}x^{m+1} = a_0 + a_1y + \cdots + a_{m+1}y^{m+1} \quad (14)$$

i.e.,

$$x(a_1 + a_2x \cdots + a_{m+1}x^m) = y(a_1 + a_2y + \cdots + a_my^{m+1}) \quad (15)$$

It is trivial that $x = y = 0$ makes Lemma 3 to be true, thus, we only consider $x \neq 0, y \neq 0$. Then by Eq.(15), we get

$$\frac{x}{y} = \frac{a_1 + a_2y \cdots + a_{m+1}y^m}{a_1 + a_2x \cdots + a_{m+1}x^m} \quad (16)$$

Without loss of generality, set

$$\frac{x}{y} = \frac{a_1 + a_2y \cdots + a_{m+1}y^m}{a_1 + a_2x \cdots + a_{m+1}x^m} = k \neq 0 \quad (17)$$

where k is a real or complex number.

Then Eq.(17) is equivalent to the following Eq.(18)

$$\begin{cases} ka_1 + ka_2x \cdots + ka_{m+1}x^m = a_1 + a_2y + \cdots + a_{m+1}y^m \\ x = ky \end{cases} \quad (18)$$

According to Eq.(13) and the arbitrariness of its coefficients, we see that Eq.(18) holds if and only if $k = 1(x = y)$, i.e.,

$$a_0 + a_1x + \cdots + a_{m+1}x^{m+1} = a_0 + a_1y + \cdots + a_{m+1}y^{m+1} \Leftrightarrow x = y \quad (19)$$

Then we conclude: Lemma 3 is true for $n = m + 1$.

Finally, by mathematical induction, Lemma 3 is true for any natural number n .

That completes the proof of Lemma 3.

Proof of RH: The details are delivered in three steps as follows.

Step 1: Since $\xi(s)$ is an entire function, it is analytic in the whole complex plane $\mathbb{C}$. Then $\xi(s)$ can be expanded in MacLaurin series (infinite polynomial) at $s = 0$, i.e.

$$\xi(s) = \xi(0) + \xi'(0)s + \frac{\xi''(0)}{2!}s^2 + \cdots + \frac{\xi^{(n)}(0)}{n!}s^n + \cdots, |s| < \infty \quad (20)$$

It is obvious that $\frac{\xi^{(n)}(0)}{n!} = \frac{\xi^{(n)}(s)}{n!} \Big|_{s=0}$, $n = 0, 1, 2, \dots$ are all real numbers.

Thus, all the zeros of $\xi(s)$ are the roots of the following infinite algebraic equation with real coefficients.

$$0 = \xi(0) + \xi'(0)s + \frac{\xi''(0)}{2!}s^2 + \cdots + \frac{\xi^{(n)}(0)}{n!}s^n + \cdots \quad (21)$$

According to the well established theory of algebraic equation with real number coefficients, complex roots always come in pairs (complex conjugate). Further by Lemma 2, all the zeros of $\xi(s)$ are complex pairs, then we denote the roots of Eq.(21) as $s = \alpha \pm j\beta$, $\beta \neq 0$.

Step2: Replacing s with $1 - s$ in Eq.(20) yields

$$\xi(1-s) = \xi(0) + \xi'(0)(1-s) + \frac{\xi''(0)}{2!}(1-s)^2 + \cdots + \frac{\xi^{(n)}(0)}{n!}(1-s)^n + \cdots, |s| < \infty \quad (22)$$

Therefore, all the zeros of $\xi(1-s)$ are the roots of the following infinite algebraic equation with real coefficients.

$$0 = \xi(0) + \xi'(0)(1-s) + \frac{\xi''(0)}{2!}(1-s)^2 + \cdots + \frac{\xi^{(n)}(0)}{n!}(1-s)^n + \cdots \quad (23)$$

where $s = \alpha \pm j\beta$, $\beta \neq 0$.

Step 3: Due to $\xi(s) = \xi(1-s)$, we have

$$\begin{aligned} & \xi(0) + \xi'(0)s + \frac{\xi''(0)}{2!}s^2 + \cdots + \frac{\xi^{(n)}(0)}{n!}s^n + \cdots \\ &= \xi(0) + \xi'(0)(1-s) + \frac{\xi''(0)}{2!}(1-s)^2 + \cdots + \frac{\xi^{(n)}(0)}{n!}(1-s)^n + \cdots \end{aligned} \quad (24)$$

Based on Lemma 3, Eq.(24), ie., $\xi(s) = \xi(1-s)$ indicates

$$s = 1 - s \quad (25)$$

Furthermore, from Step 1 and Step 2, $\xi(s) = \xi(1-s) = 0$ implies

$$s = 1 - s, s = \alpha \pm j\beta, \beta \neq 0 \quad (26)$$

It follows that

$$\alpha \pm j\beta = 1 - \alpha \pm j\beta \Rightarrow \alpha = 1 - \alpha \Rightarrow \alpha = \frac{1}{2} \quad (27)$$

Since all the zeros of $\xi(s)$ are the solution of equation $\xi(s) = \xi(1-s) = 0$, then we conclude that all the zeros of the completed zeta function $\xi(s)$ have real part equal to $\frac{1}{2}$.

That completes the proof of the Statement 2 of RH.

Remark: According to Lemma 2, we know that the Statement 1 of RH is also true, i.e., The non-trivial zeros of the Riemann zeta function $\zeta(s)$ have real part equal to $\frac{1}{2}$.

3 Conclusion

A proof of the Riemann Hypothesis is presented based on a new road map: First, the completed zeta function $\xi(s)$ is expressed as MacLaurin series (infinite polynomial); Second, by Lemma 3, $\xi(s) = \xi(1-s) = 0$ means

$$s = 1 - s, s = \alpha \pm j\beta, \beta \neq 0 \Rightarrow \alpha \pm j\beta = 1 - \alpha \pm j\beta \Rightarrow \alpha = \frac{1}{2}$$

Then we conclude that the celebrated Riemann Hypothesis is true.

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