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A Proof of the Riemann Hypothesis Based on a New Expression of the Completed Zeta Function

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Article

A Proof of the Riemann Hypothesis Based on a New Expression of the Completed Zeta Function

A Proof of the Riemann Hypothesis

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Abstract: The Riemann Hypothesis (RH) is proved based on a new absolutely convergent expression of $\xi(s)$, which was obtained from the Hadamard product, through paring ρ_i and $\bar{\rho}_i$, and taking the possible multiple zeros into consideration with their real (unique and unchangeable) multiplicities, i.e. $\xi(s) = \xi(0)\prod_{\rho}(1-\frac{s}{\rho}) = \xi(0)\prod_{i=1}^{\infty}(1-\frac{s}{\rho_i})(1-\frac{s}{\bar{\rho}_i}) = \xi(0)\prod_{i=1}^{\infty}\left(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}\right)^{m_i}$ where $\xi(0) = \frac{1}{2}$, $\rho_i = \alpha_i + j\beta_i$ and $\bar{\rho}_i = \alpha_i - j\beta_i$ are the complex conjugate zeros of $\xi(s)$, $0 < \alpha_i < 1$ and $\beta_i \neq 0$ are real numbers, $m_i \geq 1$ is the real multiplicity of ρ_i, $0 < |\beta_1| \leq |\beta_2| \leq |\beta_3| \leq \cdots$. Then, according to the functional equation $\xi(s) = \xi(1-s)$, we have $\prod_{i=1}^{\infty}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{m_i} = \prod_{i=1}^{\infty}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{m_i}$ which, owing to the uniqueness and unchangeableness of m_i, is finally equivalent to (for more details, see the proof of Lemma 3.) $\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{m_i} = \left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{m_i} \Leftrightarrow \alpha_i = \frac{1}{2}, 0 < |\beta_1| < |\beta_2| < |\beta_3| < \cdots$ Thus, we conclude that the RH is true.

Keywords: riemann hypothesis; hadamard product; new expression of the completed zeta function

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1. Introduction

The Riemann Hypothesis [1] is one of the most important unsolved problems in mathematics. Although many efforts and achievements have been made towards proving this celebrated hypothesis, it still remains an open problem [2,3]. The Riemann zeta function is originally defined in the half-plane $\Re(s) > 1$ by the absolutely convergent series [2]

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}, \Re(s) > 1 \tag{1}$$

The connection between the above-defined Riemann zeta function and prime numbers was discovered by Euler, i.e., the famous Euler product

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s} = \prod_{p}(1-p^{-s})^{-1}, \Re(s) > 1 \tag{2}$$

where p runs over the prime numbers.

Riemann showed in his paper in 1859 how to extend the zeta function to the whole complex plane $\mathbb{C}$ by analytic continuation, i.e.

$$\zeta(s) = \frac{\Gamma(1-s)}{2\pi i}\int_{\infty}^{\infty}\frac{(-x)^s}{e^x-1}\cdot\frac{dx}{x} \tag{3a}$$

where " $\int_{\infty}^{\infty}$ " is the symbol adopted by Riemann to represent the contour integral from $+\infty$ to $+\infty$ around a domain which includes the value 0 but no other point of discontinuity of the integrand in its interior.

Or equivalently,

$$\zeta(s) = \frac{\pi^{s/2}}{\Gamma(s/2)}\{\frac{1}{s(s-1)} + \int_{1}^{\infty}(x^{\frac{s}{2}-1}+x^{-\frac{s}{2}-\frac{1}{2}})\cdot(\frac{\theta(x)-1}{2})dx\} \tag{3b}$$

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where $\theta(x) = \sum_{-\infty}^{\infty} e^{-n^2\pi x}$ is the Jaccobi theta function, Γ is the Gamma function in the following Weierstrass expression

$$\frac{1}{\Gamma(s)} = s \cdot e^{\gamma s} \prod_{n=1}^{\infty} (1 + \frac{s}{n}) e^{-s/n} \tag{4}$$

where γ is the Euler-Mascheroni constant.

As shown by Riemann, $\zeta(s)$ extends to $\mathbb{C}$ as a meromorphic function with only a simple pole at $s = 1$, with residue 1, and satisfies the following functional equation

$$\pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma(\frac{1-s}{2}) \zeta(1-s) \tag{5}$$

The Riemann zeta function $\zeta(s)$ has zeros at the negative even integers: $-2, -4, -6, -8, \cdots$ and one refers to them as the **trivial zeros**. The other zeros of $\zeta(s)$ are the complex numbers, i.e., **non-trivial zeros** [2].

In 1896, Hadamard [4] and Poussin [5] independently proved that no zeros could lie on the line $\Re(s) = 1$, together with the functional equation $\xi(s) = \xi(1-s)$ and the fact that there are no zeros with real part greater than 1, this showed that all non-trivial zeros must lie in the interior of the **critical strip** $0 < \Re(s) < 1$. Later on, Hardy (1914) [6], Hardy and Littlewood (1921) [7] showed that there are infinitely many zeros on the **critical line** $\Re(s) = \frac{1}{2}$.

To give a summary of the related research on the RH, we have the following results on the properties of the non-trivial zeros of $\zeta(s)$ [4–9].

Lemma 1: Non-trivial zeroes of $\zeta(s)$, noted as $\rho = \alpha + j\beta$, have the following properties
1) The number of non-trivial zeroes is infinity;
2) $\beta \neq 0$;
3) $0 < \alpha < 1$;
4) $\rho, \bar{\rho}, 1-\bar{\rho}, 1-\rho$ are all non-trivial zeroes.

As further study, a completed zeta function $\xi(s)$ is defined as

$$\xi(s) = \frac{1}{2} s(s-1) \pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) \zeta(s) \tag{6}$$

It is well-known that $\xi(s)$ is an entire function of order 1. This implies $\xi(s)$ is analytic, and can be expressed as infinite polynomial, in the whole complex plane $\mathbb{C}$. In addition, replacing s with $1-s$ in Eq.(6), and combining Eq.(5), we obtain the following functional equation

$$\xi(s) = \xi(1-s) \tag{7}$$

Considering the definition of $\xi(s)$, and recalling Eq.(4), the trivial zeros of $\zeta(s)$ are canceled by the poles of $\Gamma(\frac{s}{2})$. The zero of $s-1$ and the pole of $\zeta(s)$ cancel; the zero $s = 0$ and the pole of $\Gamma(\frac{s}{2})$ cancel [9,10]. Thus, all the zeros of $\xi(s)$ are exactly the nontrivial zeros of $\zeta(s)$. Then we have the following Lemma 2.

Lemma 2: The zeros of $\xi(s)$ coincide with the non-trivial zeros of $\zeta(s)$.
Accordingly, the following two statements of the RH are equivalent.

Statement 1: All the non-trivial zeros of $\zeta(s)$ have real part equal to $\frac{1}{2}$.

Statement 2: All the zeros of $\xi(s)$ have real part equal to $\frac{1}{2}$.

To prove the RH, a natural thinking is to estimate the numbers of non-trivial zeros of $\zeta(s)$ inside or outside some certain areas according to Argument Principle. Along this train of thought, there are many research works. Let $N(T)$ denote the number of non-trivial zeros of $\zeta(s)$ inside the rectangle: $0 < \alpha < 1, 0 < \beta \leq T$, and let $N_0(T)$ denote the number of non-trivial zeros of $\zeta(s)$ on the line $\alpha = \frac{1}{2}, 0 < \beta \leq T$. Selberg proved that there exist positive constants c and T_0, such that $N_0(T) > cN(T), (T > T_0)$ [11], later on, Levinson proved that $c \geq \frac{1}{3}$ [12], Lou and Yao proved that

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$c \geq 0.3484$ [13], Conrey proved that $c \geq \frac{2}{5}$ [14], Bui, Conrey and Young proved that $c \geq 0.41$ [15], Feng proved that $c \geq 0.4128$ [16], Wu proved that $c \geq 0.4172$ [17].

On the other hand, many non-trivial zeros have been calculated by hand or by computer programs. Among others, Riemann found the first three non-trivial zeros [18]. Gram found the first 15 zeros based on Euler-Maclaurin summation [19]. Titchmarsh calculated the 138^{th} to 195^{th} zeros using the Riemann-Siegel formula [20,21]. Here are the first three (pairs of) non-trivial zeros: $\frac{1}{2} \pm j14.1347251$; $\frac{1}{2} \pm j21.0220396$; $\frac{1}{2} \pm j25.0108575$.

The idea of this paper is originated from Euler's work on proving the following famous equality

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{6} \tag{8}$$

This interesting result is deduced by comparing the like terms of two types of infinite expressions, i.e., infinite polynomial and infinite product, as shown in the following

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots = (1 - \frac{x^2}{\pi^2})(1 - \frac{x^2}{4\pi^2})(1 - \frac{x^2}{9\pi^2}) \cdots \tag{9}$$

Then the author of this paper conjectured that $\xi(s)$ should be factored into $\left(1 + \frac{(s-\alpha_i)^2}{\beta_i^2}\right)$ or something like that, which was verified by paring ρ_i and $\bar{\rho}_i$ in the Hadamard product of $\xi(s)$, i.e. $(1 - \frac{s}{\rho_i})(1 - \frac{s}{\bar{\rho}_i}) = \frac{\beta_i^2}{\alpha_i^2 + \beta_i^2}\left(1 + \frac{(s-\alpha_i)^2}{\beta_i^2}\right)$

The Hadamard product of $\xi(s)$ as shown in Eq.(10) was first proposed by Riemann, however, it was Hadamard who showed the validity of this infinite product expansion [22].

$$\xi(s) = \xi(0) \prod_{\rho} (1 - \frac{s}{\rho}) \tag{10}$$

where $\xi(0) = \frac{1}{2}$, ρ runs over all zeros of the completed zeta function $\xi(s)$.

Hadamard pointed out that to ensure the absolute convergence of the infinite product expansion, ρ and $1 - \rho$ are paired. Later in Section 3, we will show that ρ and $\bar{\rho}$ can also be paired to ensure the absolute convergence of the infinite product expansion.

2. Lemmas

In this section, we first explain the concept of the real multiplicity of a zero of $\xi(s)$. And then we prove Lemma 3 to support the proof of the RH.

Multiple zeros of $\xi(s)$ and their real multiplicities: As shown in Figure 1, the multiple zeros of $\xi(s)$ are defined in terms of the quadruplet, i.e., $\rho, \bar{\rho}, 1 - \rho, 1 - \bar{\rho}$.

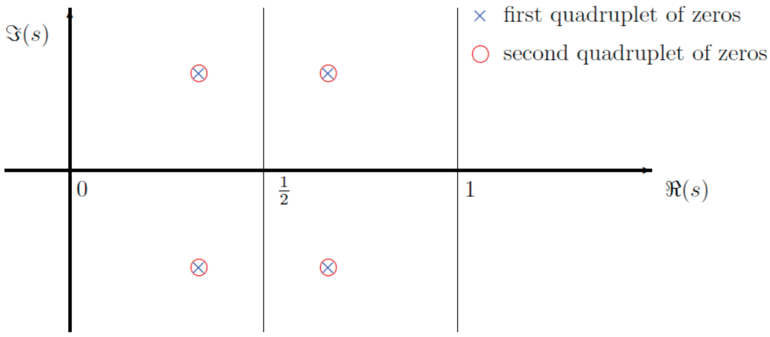

Figure 1. Illustration of the multiple zeros of $\xi(s)$.

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There are two different expressions of factors of $\xi(s)/\xi(1-s)$ for the multiple zeros in Figure 1, respectively, i.e., $\left(1+\frac{(s-\alpha_1)^2}{\beta_1^2}\right)^2/\left(1+\frac{(1-s-\alpha_1)^2}{\beta_1^2}\right)^2$, or $\left(1+\frac{(s-\alpha_1)^2}{\beta_1^2}\right)\left(1+\frac{(s-\alpha_2)^2}{\beta_2^2}\right)/\left(1+\frac{(1-s-\alpha_1)^2}{\beta_1^2}\right)\left(1+\frac{(1-s-\alpha_2)^2}{\beta_2^2}\right)$ with $\alpha_1+\alpha_2=1, \beta_1^2=\beta_2^2$.

To exclude the latter expression, we stipulate that zero ρ_i related factors of $\xi(s)/\xi(1-s)$ take the unique form of $\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{m_i}/\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{m_i}$, where $m_i \geq 1$ is the real multiplicity of ρ_i, here "real" means unique and unchangeable. In Figure 1, the real multiplicity of ρ_1 is 2, i.e., $m_1 = 2$.

Remark: Although the real multiplicity m_i of zero ρ_i is unknown, it is an objective existence, unique, and unchangeable. This is the key point in the proof of Lemma 3.

Lemma 3: Given two absolutely convergent infinite products

$$f(s) = \prod_{i=1}^{\infty}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{m_i} \tag{11}$$

and

$$f(1-s) = \prod_{i=1}^{\infty}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{m_i} \tag{12}$$

where s is a complex variable, $\rho_i = \alpha_i + j\beta_i$ and $\bar{\rho}_i = \alpha_i - j\beta_i$ are the complex conjugate zeros of $\xi(s)$, $0 < \alpha_i < 1$ and $\beta_i \neq 0$ are real numbers, $m_i \geq 1$ is the real multiplicity of ρ_i, $0 < |\beta_1| \leq |\beta_2| \leq |\beta_3| \leq \cdots$.

Then we have

$$f(s) = f(1-s) \Leftrightarrow \begin{cases} \alpha_i = \frac{1}{2} \\ 0 < |\beta_1| < |\beta_2| < |\beta_3| < \cdots \\ i = 1,2,3,\cdots,\infty \end{cases} \tag{13}$$

where $'' \Leftrightarrow ''$ is the equivalent sign.

Proof: First of all, we have the following fact:

$$(1+\frac{(s-\alpha)^2}{\beta^2})^m = (1+\frac{(1-s-\alpha)^2}{\beta^2})^m \Leftrightarrow (s-\alpha)^2 = (1-s-\alpha)^2 \Leftrightarrow \alpha = \frac{1}{2} \tag{14}$$

where $m \geq 1$ is positive integer, $\alpha \neq 0$ and $\beta \neq 0$ are real numbers.

Next, the proof is based on the divisibility of infinite products with reference to the divisibility of polynomials. It is obvious that

$$\begin{aligned} f(s) = f(1-s) &\Leftrightarrow \prod_{i=1}^{\infty}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{m_i} = \prod_{i=1}^{\infty}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{m_i} \\ &\Leftrightarrow (1+\frac{(s-\alpha_l)^2}{\beta_l^2})^{m_l} f_l(s) = (1+\frac{(1-s-\alpha_l)^2}{\beta_l^2})^{m_l} f_l(1-s) \end{aligned} \tag{15}$$

where

$$f_l(s) = \prod_{i\in\mathbb{I}\setminus\{l\}}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{m_i} \tag{16}$$

$$f_l(1-s) = \prod_{i\in\mathbb{I}\setminus\{l\}}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{m_i} \tag{17}$$

with $\mathbb{I} = \{1,2,3,\cdots,\infty\}$, and "$l$" is an arbitrary element of set $\mathbb{I}$. In brief, $i \in \mathbb{I}\setminus\{l\}$ means that i runs over the elements of $\mathbb{I}$ excluding "l".

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Then we have

$$
\begin{aligned}
&(1+\frac{(s-\alpha_l)^2}{\beta_l^2})^{m_l} f_l(s) = (1+\frac{(1-s-\alpha_l)^2}{\beta_l^2})^{m_l} f_l(1-s) \\
&\Rightarrow \\
&\begin{cases} (1+\frac{(s-\alpha_l)^2}{\beta_l^2})^{m_l} \Big| (1+\frac{(1-s-\alpha_l)^2}{\beta_l^2})^{m_l} f_l(1-s) \\ (1+\frac{(1-s-\alpha_l)^2}{\beta_l^2})^{m_l} \Big| (1+\frac{(s-\alpha_l)^2}{\beta_l^2})^{m_l} f_l(s) \end{cases}
\end{aligned} \tag{18}
$$

where "|" is the divisible sign.

Next, we exclude the possibility of $(1+\frac{(s-\alpha_l)^2}{\beta_l^2})^{m_l} \Big| f_l(1-s)$ and $(1+\frac{(1-s-\alpha_l)^2}{\beta_l^2})^{m_l} \Big| f_l(s)$ in Eq.(18) with the help of the real multiplicities of zeros of $\xi(s)$.

Considering $(1+\frac{(s-\alpha_l)^2}{\beta_l^2}), 0<\alpha_l<1, \beta_l \neq 0$, is irreducible over the field R of real numbers, we know that

$$
\begin{aligned}
&(1+\frac{(s-\alpha_l)^2}{\beta_l^2})^{m_l} \Big| f_l(1-s) \Rightarrow (1+\frac{(s-\alpha_l)^2}{\beta_l^2}) \Big| f_l(1-s) \\
&\Rightarrow \text{(by Lemma 6)} \\
&(1+\frac{(s-\alpha_l)^2}{\beta_l^2}) \Big| (1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}), i \neq l \\
&\Rightarrow (1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}) = k(1+\frac{(s-\alpha_l)^2}{\beta_l^2}), i \neq l \\
&\Rightarrow \text{(by comparing the like terms in the above polynomial equation)} \\
&\alpha_i+\alpha_l = 1, \beta_i^2 = \beta_l^2, k = 1, i \neq l
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&(1+\frac{(1-s-\alpha_l)^2}{\beta_l^2})^{m_l} \Big| f_l(s) \Rightarrow (1+\frac{(1-s-\alpha_l)^2}{\beta_l^2}) \Big| f_l(s) \\
&\Rightarrow \text{(by Lemma 6)} \\
&(1+\frac{(1-s-\alpha_l)^2}{\beta_l^2}) \Big| (1+\frac{(s-\alpha_i)^2}{\beta_i^2}), i \neq l \\
&\Rightarrow (1+\frac{(s-\alpha_i)^2}{\beta_i^2}) = k(1+\frac{(1-s-\alpha_l)^2}{\beta_l^2}), i \neq l \\
&\Rightarrow \text{(by comparing the like terms in the above polynomial equation)} \\
&\alpha_i+\alpha_l = 1, \beta_i^2 = \beta_l^2, k = 1, i \neq l
\end{aligned}
$$

As explained in the situation of Figure 1, $\alpha_i+\alpha_l = 1, \beta_i^2 = \beta_l^2, i \neq l$ means that $\alpha_i \pm j\beta_i$ and $\alpha_l \pm j\beta_l$ are the same zeros in terms of quadruplet, i.e., $\rho, \bar{\rho}, 1-\rho$, and $1-\bar{\rho}$, which contradicts the definition of real multiplicities of zeros of $\xi(s)$.

Thus, in order to keep the real multiplicities of zeros of $\xi(s)$ unchanged, $(1+\frac{(s-\alpha_l)^2}{\beta_l^2})^{m_l}$ can not divide $f_l(1-s)$, $(1+\frac{(1-s-\alpha_l)^2}{\beta_l^2})^{m_l}$ can not divides $f_l(s)$. In addition, $(1+\frac{(s-\alpha_l)^2}{\beta_l^2})$ is irreducible,

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then we know that $(1+\frac{(s-\alpha_l)^2}{\beta_l^2})^{m_l}$ and $f_l(1-s)$ are relatively prime, $(1+\frac{(1-s-\alpha_l)^2}{\beta_l^2})^{m_l}$ and $f_l(s)$ are relatively prime. Consequently, by Lemma 7, we obtain from Eq.(18) the following result.

$$
\begin{aligned}
&(1+\frac{(s-\alpha_l)^2}{\beta_l^2})^{m_l} f_l(s) = (1+\frac{(1-s-\alpha_l)^2}{\beta_l^2})^{m_l} f_l(1-s)\\
&\Rightarrow\\
&(1+\frac{(s-\alpha_l)^2}{\beta_l^2})^{m_l} \Big| (1+\frac{(1-s-\alpha_l)^2}{\beta_l^2})^{m_l}\\
&(1+\frac{(1-s-\alpha_l)^2}{\beta_l^2})^{m_l} \Big| (1+\frac{(s-\alpha_l)^2}{\beta_l^2})^{m_l}\\
&\Rightarrow\\
&(1+\frac{(s-\alpha_l)^2}{\beta_l^2})^{m_l} = k(1+\frac{(1-s-\alpha_l)^2}{\beta_l^2})^{m_l}\\
&\Rightarrow (k=1, \text{by comparing the highest-order terms in the above polynomial equation})\\
&(1+\frac{(s-\alpha_l)^2}{\beta_l^2})^{m_l} = (1+\frac{(1-s-\alpha_l)^2}{\beta_l^2})^{m_l}\\
&\Rightarrow (\text{by Eq.(14)})\\
&\alpha_l = \frac{1}{2}
\end{aligned}
\tag{19}
$$

Let l run over from 1 to ∞, and repeat the above process, we get

$$
\begin{aligned}
&\prod_{i=1}^{\infty}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{m_i} = \prod_{i=1}^{\infty}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{m_i}\\
&\Rightarrow\\
&(1+\frac{(s-\alpha_i)^2}{\beta_i^2})^{m_i} = (1+\frac{(1-s-\alpha_i)^2}{\beta_i^2})^{m_i}\\
&\Rightarrow\\
&\alpha_i = \frac{1}{2}, i = 1,2,3,\cdots,\infty
\end{aligned}
\tag{20}
$$

Also, we have the following obvious fact

$$
\begin{aligned}
&\alpha_i = \frac{1}{2}, i = 1,2,3,\cdots,\infty\\
&\Rightarrow\\
&(1+\frac{(s-\alpha_i)^2}{\beta_i^2})^{m_i} = (1+\frac{(1-s-\alpha_i)^2}{\beta_i^2})^{m_i}\\
&\Rightarrow\\
&\prod_{i=1}^{\infty}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{m_i} = \prod_{i=1}^{\infty}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{m_i}
\end{aligned}
\tag{21}
$$

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Further, limiting the imaginary parts β_i of zeros to $0 < |\beta_1| < |\beta_2| < |\beta_3| < \cdots$ in order to keep the real multiplicities of zeros unchanged, we finally get

$$\prod_{i=1}^{\infty}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{m_i} = \prod_{i=1}^{\infty}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{m_i}$$
$$\Leftrightarrow$$
$$\begin{cases} (1+\frac{(s-\alpha_i)^2}{\beta_i^2})^{m_i} = (1+\frac{(1-s-\alpha_i)^2}{\beta_i^2})^{m_i} \\ 0 < |\beta_1| < |\beta_2| < |\beta_3| < \cdots \\ i = 1,2,3,\cdots,\infty \end{cases}$$
$$\Leftrightarrow$$
$$\begin{cases} \alpha_i = \frac{1}{2} \\ 0 < |\beta_1| < |\beta_2| < |\beta_3| < \cdots \\ i = 1,2,3,\cdots,\infty \end{cases}$$

i.e.,

$$f(s) = f(1-s) \Leftrightarrow \begin{cases} \alpha_i = \frac{1}{2} \\ 0 < |\beta_1| < |\beta_2| < |\beta_3| < \cdots \\ i = 1,2,3,\cdots,\infty \end{cases}$$

That completes the proof of Lemma 3.

To support the proof of Lemma 3, we need the following classical results (Lemma 4 and Lemma 5) in Polynomial Algebra over Fields, with extension to infinite product (Lemma 6 and Lemma 7).

Lemma 4: Let F be a field, $m(x), g_1(x), ..., g_n(x) \in F[x], n \geq 2$. If $m(x)$ is irreducible (prime) and divides the product $g_1(x)\cdots g_n(x)$, then $m(x)$ divides one of the polynomials $g_1(x), ..., g_n(x)$.

Lemma 5: Let F be a field, $f(x), m(x) \in F[x]$. If $m(x)$ is irreducible and $f(x)$ is any polynomial, then either $m(x)$ divides $f(x)$ or $gcd(m, f) = 1$, (gcd: greatest common divisor).

Lemma 6: Let F be a field, $m(x), g_1(x), ..., g_\infty(x) \in F[x]$. If $m(x)$ is irreducible and divides the product $g_1(x)\cdots g_\infty(x)$, then $m(x)$ divides one of the polynomials $g_1(x), ..., g_\infty(x)$.

Lemma 7: Let F be a field, $p_1(x), ..., p_\infty(x), q(x), m(x) \in F[x], p(x) = p_1(x)\cdots p_\infty(x)$. If $m(x)$ is irreducible and divides the product $p(x)q(x)$, but $m(x)$ and $p(x)$ are relatively prime, then $m(x)$ divides $q(x)$.

Remark: $F[x]$ is defined as the set of all polynomials in x over F:

$$F[x] = \{\sum_{i=0}^{\infty} a_i x^i | a_i \in F, a_i \neq 0 \text{ for all but a finite number of } i\}$$

The set $F[x]$ equipped with the operations $+$ and $\cdot$ is the polynomial ring in x over the field F. In this paper, F is specified as the field R of real numbers.

Remark: The contents of Lemma 4 and Lemma 5 can be found in many textbooks of Linear Algebra or Advanced Algebra. Then we need only give the proofs of Lemma 6 and Lemma 7.

Proof of Lemma 6: The proof is conducted by Transfinite Induction.

Let $P(\gamma)$ be the statement of Lemma 4, i.e.

"$m(x), g_1(x), ..., g_n(x) \in F[x], n \geq 2$. If $m(x)$ is irreducible and divides the product $g_1(x)\cdots g_n(x)$, then $m(x)$ divides one of the polynomials $g_1(x), ..., g_n(x)$" with n replaced by γ, where $\gamma \in A$, $A = \mathbb{N}\bigcup\{\omega\}$ with the ordering that $n < \omega$ for all natural numbers n, ω is the smallest limit ordinal other than 0.

Lemma 4 actually can be proved by **Mathematical Induction**, which includes the **Base Case:** $P(2)$ and the **Successor Case:** $P(n) \Rightarrow P(n+1)$ or $P(\gamma) \Rightarrow P(\gamma+1)$, of this proof.

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Next we prove the **Limit Case:** $P(\gamma < \lambda) \Rightarrow P(\lambda)$, λ is any limit ordinal other than 0. For convenience, we first prove $P(\gamma < \omega) \Rightarrow P(\omega)$.

For the sake of contradiction, assume that $P(\gamma < \omega) \nRightarrow P(\omega)$. Then, considering $m(x)$ is irreducible with the property stated in Lemma 5, we have

$m(x)|g_1(x)\cdots g_\gamma(x) \Rightarrow m(x)|g_1(x)\cdots g_\gamma \cdots g_\omega(x) \Rightarrow gcd(m(x), g_i(x)) = 1, i \in \mathbb{N}\cup\{\omega\} \Rightarrow gcd(m(x), g_i(x)) = 1, i \in \mathbb{N}$, which contradicts $P(\gamma < \omega) : m(x)|g_1(x)\cdots g_\gamma(x) \Rightarrow m(x)$ divides one of the polynomials $g_1(x), ..., g_\gamma(x)$.

Thus, we know that the assumption $P(\gamma < \omega) \nRightarrow P(\omega)$ does not hold.

Then $P(\gamma < \omega) \Rightarrow P(\omega)$ is true.

Since $\lambda \geq \omega$, then we obviously have

$m(x)|g_1(x)\cdots g_\gamma(x) \Rightarrow m(x)|g_1(x)\cdots g_\gamma \cdots g_\lambda(x) \Rightarrow gcd(m(x), g_i(x)) = 1, i \in \mathbb{N}\cup\{\omega, \cdots, \lambda\} \Rightarrow gcd(m(x), g_i(x)) = 1, i \in \mathbb{N}$, which contradicts $P(\gamma < \lambda) : m(x)|g_1(x)\cdots g_\gamma(x) \Rightarrow m(x)$ divides one of the polynomials $g_1(x), ..., g_\gamma(x)$.

Then the **Limit Case:** $P(\gamma < \lambda) \Rightarrow P(\lambda)$ is true.

That completes the proof of Lemma 6.

Proof of Lemma 7: If $m(x)$ is irreducible and divides the product $p(x)q(x) = p_1(x)\cdots p_\infty q(x)$, then, according to Lemma 6, $m(x)$ divides one of the polynomials $p_1(x), ..., p_\infty(x), q(x)$. Further, if $m(x)$ and $p(x)$ are relatively prime, then $m(x)$ does not divides any factor $p_i(x), i = 1, \cdots, \infty$ of $p(x)$ (otherwise $m(x)$ divides $p(x)$, which contradicts the condition "$m(x)$ and $p(x)$ are relatively prime"). Thus, $m(x)$ must divides $q(x)$.

That completes the proof of Lemma 7.

3. A Proof of the RH

This section is planned to present a proof of the Riemann Hypothesis. We first prove that Statement 2 of the RH is true, and then by Lemma 2, Statement 1 of the RH is also true. To be brief, to prove the Riemann Hypothesis, it suffices to show that $\alpha_i = \frac{1}{2}, i = 1, 2, 3, \cdots, \infty$ in the new expression of $\xi(s)$ as shown in Eq.(22).

Proof of the RH: The details are delivered in three steps as follows.

Step 1:

It is well-known that all the zeros of $\xi(s)$ always come in complex conjugate pairs. Then by pairing $\rho_i = \alpha_i + j\beta_i$ and $\bar{\rho}_i = \alpha_i - j\beta_i$ in the Hadamard product as shown in Eq.(10), we have

$$\begin{aligned}\xi(s) &= \xi(0)\prod_{\rho}(1-\frac{s}{\rho}) = \xi(0)\prod_{i=1}^{\infty}(1-\frac{s}{\rho_i})(1-\frac{s}{\bar{\rho}_i}) \\ &= \xi(0)\prod_{i=1}^{\infty}(1-\frac{s}{\alpha_i+j\beta_i})(1-\frac{s}{\alpha_i-j\beta_i}) = \xi(0)\prod_{i=1}^{\infty}(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(s-\alpha_i)^2}{\alpha_i^2+\beta_i^2})\end{aligned} \tag{22}$$

where $\xi(0) = \frac{1}{2}, 0 < \alpha_i < 1, \beta_i \neq 0$.

The absolute convergence of the infinite product in Eq.(22) in the form

$$\xi(s) = \xi(0)\prod_{i=1}^{\infty}(1-\frac{s}{\rho_i})(1-\frac{s}{\bar{\rho}_i}) = \xi(0)\prod_{i=1}^{\infty}\left(1-\frac{s(2\alpha_i-s)}{|\rho_i|^2}\right) \tag{23}$$

depends on the convergence of infinite series $\sum_{i=1}^{\infty}\frac{1}{|\rho_i|^2}$, which is an obvious fact according to Theorem 2 in Section 2, Chapter IV of Ref.[23].

Further, considering the absolute convergence of

$$\xi(s) = \xi(0)\prod_{i=1}^{\infty}\left(1-\frac{s(2\alpha_i-s)}{|\rho_i|^2}\right) = \xi(0)\prod_{i=1}^{\infty}(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}) \tag{24}$$

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we have the following new expression of $\xi(s)$ by putting all the ρ_i related multiple factors (zeros) together in the above Eq.(24)

$$\xi(s) = \xi(0)\prod_{i=1}^{\infty}(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2} + \frac{(s-\alpha_i)^2}{\alpha_i^2+\beta_i^2})^{m_i} \tag{25}$$

where $m_i \geq 1$ is the real multiplicity of ρ_i, $i = 1,2,3,\cdots,\infty$.

Step 2: Replacing s with $1-s$ in Eq.(25), we obtain the infinite product expression of $\xi(1-s)$, i.e.,

$$\xi(1-s) = \xi(0)\prod_{i=1}^{\infty}(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2} + \frac{(1-s-\alpha_i)^2}{\alpha_i^2+\beta_i^2})^{m_i} \tag{26}$$

Step 3: According to the functional equation $\xi(s) = \xi(1-s)$, and considering Eq.(25) and Eq.(26), we have

$$\xi(0)\prod_{i=1}^{\infty}(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2} + \frac{(s-\alpha_i)^2}{\alpha_i^2+\beta_i^2})^{m_i} = \xi(0)\prod_{i=1}^{\infty}(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2} + \frac{(1-s-\alpha_i)^2}{\alpha_i^2+\beta_i^2})^{m_i} \tag{27}$$

which is equivalent to

$$\prod_{i=1}^{\infty}(1+\frac{(s-\alpha_i)^2}{\beta_i^2})^{m_i} = \prod_{i=1}^{\infty}(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2})^{m_i} \tag{28}$$

where β_i are in order of increasing $|\beta_i|$, i.e., $0 < |\beta_1| \leq |\beta_2| \leq |\beta_3| \leq \cdots$.

To check the absolute convergence of both sides of Eq.(28), it suffices to make a comparison with Eq.(23) without considering multiple zeros in Eq. (28), i.e., to make a comparison between $\prod_{i=1}^{\infty}(1+\frac{(s-\alpha_i)^2}{\beta_i^2})$ and $\xi(0)\prod_{i=1}^{\infty}\left(1-\frac{s(2\alpha_i-s)}{|\rho_i|^2}\right)$. It is well-known that the absolute convergence of $\xi(0)\prod_{i=1}^{\infty}\left(1-\frac{s(2\alpha_i-s)}{|\rho_i|^2}\right)$ depends on the convergence of infinite series $\sum_{i=1}^{\infty}\frac{1}{|\rho_i|^2}$ (already proved in Step 1); the absolute convergence of $\prod_{i=1}^{\infty}(1+\frac{(s-\alpha_i)^2}{\beta_i^2})$ depends on the convergence of infinite series $\sum_{i=1}^{\infty}\frac{1}{\beta_i^2}$, which is also an obvious fact because $0<\alpha_i<1, |\rho_i| \to \infty, |\beta_i| \to \infty$, as $i \to \infty$, $\lim_{i\to\infty}\frac{\beta_i^2}{|\rho_i|^2} = \lim_{i\to\infty}\frac{\beta_i^2}{\alpha_i^2+\beta_i^2} = 1$, that means $\sum_{i=1}^{\infty}\frac{1}{\beta_i^2}$ and $\sum_{i=1}^{\infty}\frac{1}{|\rho_i|^2}$ have the same convergence.

Then, according to Lemma 3, Eq.(28) is equivalent to

$$\alpha_i = \frac{1}{2};\ 0 < |\beta_1| < |\beta_2| < |\beta_3| < \cdots;\ i = 1,2,3,\cdots,\infty \tag{29}$$

Thus, we conclude that all the zeros of the completed zeta function $\xi(s)$ have real part equal to $\frac{1}{2}$, i.e., Statement 2 of the RH is true. According to Lemma 2, Statement 1 of the RH is also true, i.e., all the non-trivial zeros of the Riemann zeta function $\zeta(s)$ have real part equal to $\frac{1}{2}$.

That completes the proof of the RH.

4. Retrospection and Discussion

On the simultaneous zeros of $\xi(s)$

According to Lemma 1, there are two pairs of complex zeros of $\zeta(s)$ simultaneously, i.e., $\rho = \alpha + j\beta, \bar{\rho} = \alpha - j\beta, 1-\rho = 1-\alpha-j\beta, 1-\bar{\rho} = 1-\alpha+j\beta$. With the proof of the RH, these 2 pairs of zeros are actually only one pair, because $\rho = 1-\bar{\rho} = \frac{1}{2}+j\beta, \bar{\rho} = 1-\rho = \frac{1}{2}-j\beta$. Thus Lemma 1 could be modified more precisely as follows.

Lemma 1*: Non-trivial zeroes of $\zeta(s)$, noted as $\rho = \alpha + j\beta$, have the following properties

1) The number of non-trivial zeroes is infinity;

2) $\beta \neq 0$;

3) $0 < \alpha < 1$;
4) $\rho = 1 - \bar{\rho}, \bar{\rho} = 1 - \rho$ are all non-trivial zeroes.

On the paring of zeros of $\xi(s)$

Hadamard pointed out that to ensure the absolute convergence of the Hadamard product, i.e., $\xi(s) = \xi(0)\prod_\rho(1 - \frac{s}{\rho})$, ρ and $1-\rho$ are paired. In Section 3, the author proved that ρ and $\bar{\rho}$ can also be paired to ensure the absolute convergence of the Hadamard product. And that the paring of conjugate zeros, i.e., ρ and $\bar{\rho}$, is the right way to express the most essential characteristic of $\xi(s)$ as (infinite) polynomial with real coefficients, whereas $1-\rho$ and $1-\bar{\rho}$ are just another pair of conjugate zeros given by $\xi(s) = \xi(1-s)$.

5. Conclusion

This paper presents a proof of the RH based on a new expression of $\xi(s)$, i.e.,

$$\xi(s) = \xi(0)\prod_{i=1}^{\infty}\left(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2} + \frac{(s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}\right)^{m_i}$$

where $\xi(0) = \frac{1}{2}, \rho_i = \alpha_i + j\beta_i$ and $\bar{\rho}_i = \alpha_i - j\beta_i$ are the complex conjugate zeros of $\xi(s)$, $0 < \alpha_i < 1$ and $\beta_i \neq 0$ are real numbers, $0 < |\beta_1| \le |\beta_2| \le |\beta_3| \le \cdots$, $m_i \ge 1$ is the real multiplicity of ρ_i.

The proof is conducted according to the divisibility implied in the (infinite) polynomial equation $\xi(s) = \xi(1-s)$. The first key-point is the paring of conjugate zeros ρ and $\bar{\rho}$ to get the new expression of $\xi(s)$. The second key-point is the use of "real multiplicity" of a zero of $\xi(s)$. Obviously, the real multiplicity of a zero of $\xi(s)$ is an objective existence, unique, and unchangeable. As a result, the functional equation $\xi(s) = \xi(1-s)$ finally leads to $\alpha_i = \frac{1}{2}$; $0 < |\beta_1| < |\beta_2| < |\beta_3| < \cdots$; $i = 1, 2, 3, \cdots, \infty$.

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