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Article

A Proof of the Riemann Hypothesis Based on a New Expression of the Completed Zeta Function

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Abstract: The Riemann Hypothesis (RH) is proved based on a new absolutely convergent expression of $\zeta(s)$, which was obtained from the Hadamard product, through paring ρ_i and $\bar{\rho}_i$, and taking the possible multiple zeros into consideration with their real (unique and unchangeable) multiplicities, i.e. $\zeta(s) = \zeta(0) \prod_{\rho} (1 - \frac{s}{\rho}) = \zeta(0) \prod_{i=1}^{\infty} (1 - \frac{s}{\rho_i})(1 - \frac{s}{\bar{\rho}_i}) = \zeta(0) \prod_{i=1}^{\infty} \left(\frac{\beta_i^2}{\alpha_i^2 + \beta_i^2} + \frac{(s - \alpha_i)^2}{\alpha_i^2 + \beta_i^2} \right)^{m_i}$ where $\zeta(0) = \frac{1}{2}$, $\rho_i = \alpha_i + j\beta_i$ and $\bar{\rho}_i = \alpha_i - j\beta_i$ are the complex conjugate zeros of $\zeta(s)$, $0 < \alpha_i < 1$ and $\beta_i \neq 0$ are real numbers, $m_i \geq 1$ is the real multiplicity of ρ_i , $0 < |\beta_1| \leq |\beta_2| \leq |\beta_3| \leq \dots$. Then, according to the functional equation $\zeta(s) = \zeta(1-s)$, we have $\prod_{i=1}^{\infty} \left(1 + \frac{(s - \alpha_i)^2}{\beta_i^2} \right)^{m_i} = \prod_{i=1}^{\infty} \left(1 + \frac{(1-s - \alpha_i)^2}{\beta_i^2} \right)^{m_i}$ which, owing to the uniqueness and unchangeableness of m_i , is finally equivalent to (for more details, see the proof of Lemma 3.) $\left(1 + \frac{(s - \alpha_i)^2}{\beta_i^2} \right)^{m_i} = \left(1 + \frac{(1-s - \alpha_i)^2}{\beta_i^2} \right)^{m_i} \Leftrightarrow \alpha_i = \frac{1}{2}, 0 < |\beta_1| < |\beta_2| < |\beta_3| < \dots$. Thus, we conclude that the RH is true.

Keywords: Riemann Hypothesis; Hadamard product; new expression of the completed zeta function

1. Introduction

The Riemann Hypothesis [1] is one of the most important unsolved problems in mathematics. Although many efforts and achievements have been made towards proving this celebrated hypothesis, it still remains an open problem [2,3]. The Riemann zeta function is originally defined in the half-plane $\Re(s) > 1$ by the absolutely convergent series [2]

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \Re(s) > 1 \quad (1)$$

The connection between the above-defined Riemann zeta function and prime numbers was discovered by Euler, i.e., the famous Euler product

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p (1 - p^{-s})^{-1}, \Re(s) > 1 \quad (2)$$

where p runs over the prime numbers.

Riemann showed in his paper in 1859 how to extend the zeta function to the whole complex plane $\mathbb{C}$ by analytic continuation, i.e.

$$\zeta(s) = \frac{\Gamma(1-s)}{2\pi i} \int_{\infty}^{\infty} \frac{(-x)^s}{e^x - 1} \cdot \frac{dx}{x} \quad (3a)$$

where " $\int_{\infty}^{\infty}$ " is the symbol adopted by Riemann to represent the contour integral from $+\infty$ to $+\infty$ around a domain which includes the value 0 but no other point of discontinuity of the integrand in its interior.

Or equivalently,

$$\zeta(s) = \frac{\pi^{s/2}}{\Gamma(s/2)} \left\{ \frac{1}{s(s-1)} + \int_1^{\infty} (x^{\frac{s}{2}-1} + x^{-\frac{s}{2}-\frac{1}{2}}) \cdot \left(\frac{\theta(x)-1}{2} \right) dx \right\} \quad (3b)$$

where $\theta(x) = \sum_{n=-\infty}^{\infty} e^{-n^2\pi x}$ is the Jacobi theta function, Γ is the Gamma function in the following Weierstrass expression

$$\frac{1}{\Gamma(s)} = s \cdot e^{\gamma s} \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right) e^{-s/n} \quad (4)$$

where γ is the Euler-Mascheroni constant.

As shown by Riemann, $\zeta(s)$ extends to $\mathbb{C}$ as a meromorphic function with only a simple pole at $s = 1$, with residue 1, and satisfies the following functional equation

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s) \quad (5)$$

The Riemann zeta function $\zeta(s)$ has zeros at the negative even integers: $-2, -4, -6, -8, \dots$ and one refers to them as the **trivial zeros**. The other zeros of $\zeta(s)$ are the complex numbers, i.e., **non-trivial zeros** [2].

In 1896, Hadamard [4] and Poussin [5] independently proved that no zeros could lie on the line $\Re(s) = 1$, together with the functional equation $\zeta(s) = \zeta(1-s)$ and the fact that there are no zeros with real part greater than 1, this showed that all non-trivial zeros must lie in the interior of the **critical strip** $0 < \Re(s) < 1$. Later on, Hardy (1914) [6], Hardy and Littlewood (1921) [7] showed that there are infinitely many zeros on the **critical line** $\Re(s) = \frac{1}{2}$.

To give a summary of the related research on the RH, we have the following results on the properties of the non-trivial zeros of $\zeta(s)$ [4–9].

Lemma 1: Non-trivial zeroes of $\zeta(s)$, noted as $\rho = \alpha + j\beta$, have the following properties

- 1) The number of non-trivial zeroes is infinity;
- 2) $\beta \neq 0$;
- 3) $0 < \alpha < 1$;
- 4) $\rho, \bar{\rho}, 1 - \bar{\rho}, 1 - \rho$ are all non-trivial zeroes.

As further study, a completed zeta function $\xi(s)$ is defined as

$$\xi(s) = \frac{1}{2} s(s-1) \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) \quad (6)$$

It is well-known that $\xi(s)$ is an entire function of order 1. This implies $\xi(s)$ is analytic, and can be expressed as infinite polynomial, in the whole complex plane $\mathbb{C}$. In addition, replacing s with $1-s$ in Eq.(6), and combining Eq.(5), we obtain the following functional equation

$$\xi(s) = \xi(1-s) \quad (7)$$

Considering the definition of $\xi(s)$, and recalling Eq.(4), the trivial zeros of $\zeta(s)$ are canceled by the poles of $\Gamma(\frac{s}{2})$. The zero of $s-1$ and the pole of $\zeta(s)$ cancel; the zero $s=0$ and the pole of $\Gamma(\frac{s}{2})$ cancel [9,10]. Thus, all the zeros of $\xi(s)$ are exactly the nontrivial zeros of $\zeta(s)$. Then we have the following Lemma 2.

Lemma 2: The zeros of $\xi(s)$ coincide with the non-trivial zeros of $\zeta(s)$.

Accordingly, the following two statements of the RH are equivalent.

Statement 1: All the non-trivial zeros of $\zeta(s)$ have real part equal to $\frac{1}{2}$.

Statement 2: All the zeros of $\xi(s)$ have real part equal to $\frac{1}{2}$.

To prove the RH, a natural thinking is to estimate the numbers of non-trivial zeros of $\zeta(s)$ inside or outside some certain areas according to Argument Principle. Along this train of thought, there are many research works. Let $N(T)$ denote the number of non-trivial zeros of $\zeta(s)$ inside the rectangle: $0 < \alpha < 1, 0 < \beta \leq T$, and let $N_0(T)$ denote the number of non-trivial zeros of $\zeta(s)$ on the line $\alpha = \frac{1}{2}, 0 < \beta \leq T$. Selberg proved that there exist positive constants c and T_0 , such that $N_0(T) > cN(T)$, ($T > T_0$) [11], later on, Levinson proved that $c \geq \frac{1}{3}$ [12], Lou and Yao proved that

$c \geq 0.3484$ [13], Conrey proved that $c \geq \frac{2}{5}$ [14], Bui, Conrey and Young proved that $c \geq 0.41$ [15], Feng proved that $c \geq 0.4128$ [16], Wu proved that $c \geq 0.4172$ [17].

On the other hand, many non-trivial zeros have been calculated by hand or by computer programs. Among others, Riemann found the first three non-trivial zeros [18]. Gram found the first 15 zeros based on Euler-Maclaurin summation [19]. Titchmarsh calculated the 138th to 195th zeros using the Riemann-Siegel formula [20,21]. Here are the first three (pairs of) non-trivial zeros: $\frac{1}{2} \pm j14.1347251$; $\frac{1}{2} \pm j21.0220396$; $\frac{1}{2} \pm j25.0108575$.

The idea of this paper is originated from Euler's work on proving the following famous equality

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{6} \quad (8)$$

This interesting result is deduced by comparing the like terms of two types of infinite expressions, i.e., infinite polynomial and infinite product, as shown in the following

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots = \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right) \cdots \quad (9)$$

Then the author of this paper conjectured that $\zeta(s)$ should be factored into $\left(1 + \frac{(s-\alpha_i)^2}{\beta_i^2}\right)$ or something like that, which was verified by pairing ρ_i and $\bar{\rho}_i$ in the Hadamard product of $\zeta(s)$, i.e. $\left(1 - \frac{s}{\rho_i}\right)\left(1 - \frac{s}{\bar{\rho}_i}\right) = \frac{\beta_i^2}{\alpha_i^2 + \beta_i^2} \left(1 + \frac{(s-\alpha_i)^2}{\beta_i^2}\right)$

The Hadamard product of $\zeta(s)$ as shown in Eq.(10) was first proposed by Riemann, however, it was Hadamard who showed the validity of this infinite product expansion [22].

$$\zeta(s) = \zeta(0) \prod_{\rho} \left(1 - \frac{s}{\rho}\right) \quad (10)$$

where $\zeta(0) = \frac{1}{2}$, ρ runs over all the zeros of the completed zeta function $\zeta(s)$.

Hadamard pointed out that to ensure the absolute convergence of the infinite product expansion, ρ and $1 - \rho$ are paired. Later in Section 3, we will show that ρ and $\bar{\rho}$ can also be paired to ensure the absolute convergence of the infinite product expansion.

2. Lemmas

In this section, we first explain the concept of the real multiplicity of a zero of $\zeta(s)$. And then we prove Lemma 3 to support the proof of the RH.

Multiple zeros of $\zeta(s)$ and their real multiplicities: As shown in Figure 1, the multiple zeros of $\zeta(s)$ are defined in terms of the quadruplet, i.e., $\rho, \bar{\rho}, 1 - \rho, 1 - \bar{\rho}$.

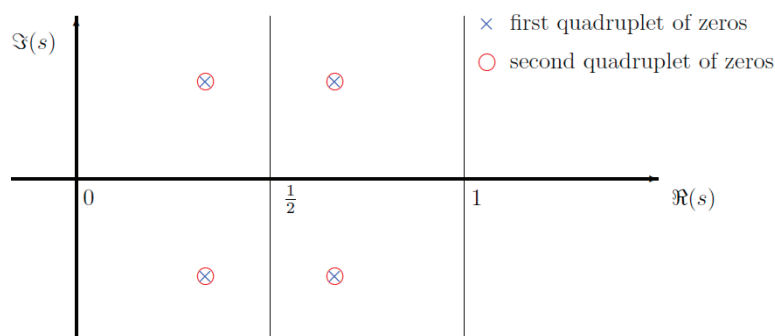


Figure 1. Illustration of the multiple zeros of $\zeta(s)$

There are two different expressions of factors of $\zeta(s)/\zeta(1-s)$ for the multiple zeros in Figure 1, respectively, i.e., $\left(1 + \frac{(s-\alpha_1)^2}{\beta_1^2}\right)^2 / \left(1 + \frac{(1-s-\alpha_1)^2}{\beta_1^2}\right)^2$, or $\left(1 + \frac{(s-\alpha_1)^2}{\beta_1^2}\right) \left(1 + \frac{(s-\alpha_2)^2}{\beta_2^2}\right) / \left(1 + \frac{(1-s-\alpha_1)^2}{\beta_1^2}\right) \left(1 + \frac{(1-s-\alpha_2)^2}{\beta_2^2}\right)$ with $\alpha_1 + \alpha_2 = 1, \beta_1^2 = \beta_2^2$.

To exclude the latter expression, we stipulate that zero ρ_i related factors of $\zeta(s)/\zeta(1-s)$ take the unique form of $\left(1 + \frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{m_i} / \left(1 + \frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{m_i}$, where $m_i \geq 1$ is the real multiplicity of ρ_i , here "real" means unique and unchangeable. In Figure 1, the multiplicity of ρ_1 is 2, i.e., $m_1 = 2$.

Remark: Although the real multiplicity m_i of zero ρ_i is unknown, it is an objective existence, unique, and unchangeable. This is the key point in the proof of Lemma 3.

Lemma 3: Given two absolutely convergent infinite products

$$f(s) = \prod_{i=1}^{\infty} \left(1 + \frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{m_i} \quad (11)$$

and

$$f(1-s) = \prod_{i=1}^{\infty} \left(1 + \frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{m_i} \quad (12)$$

where s is a complex variable, $\rho_i = \alpha_i + j\beta_i$ and $\bar{\rho}_i = \alpha_i - j\beta_i$ are the complex conjugate zeros of $\zeta(s)$, $0 < \alpha_i < 1$ and $\beta_i \neq 0$ are real numbers, $m_i \geq 1$ is the real multiplicity of ρ_i , $0 < |\beta_1| \leq |\beta_2| \leq |\beta_3| \leq \dots$.

Then we have

$$f(s) = f(1-s) \Leftrightarrow \begin{cases} \alpha_i = \frac{1}{2} \\ 0 < |\beta_1| < |\beta_2| < |\beta_3| < \dots \\ i = 1, 2, 3, \dots, \infty \end{cases} \quad (13)$$

where " $\Leftrightarrow$ " is the equivalent sign.

Proof: First of all, we have the following fact:

$$\left(1 + \frac{(s-\alpha)^2}{\beta^2}\right)^m = \left(1 + \frac{(1-s-\alpha)^2}{\beta^2}\right)^m \Leftrightarrow (s-\alpha)^2 = (1-s-\alpha)^2 \Leftrightarrow \alpha = \frac{1}{2} \quad (14)$$

where $m \geq 1$ is positive integer, $\alpha \neq 0$ and $\beta \neq 0$ are real numbers.

Next, the proof is based on the divisibility of infinite products with reference to the divisibility of polynomials. It is obvious that

$$\begin{aligned} f(s) = f(1-s) &\Leftrightarrow \prod_{i=1}^{\infty} \left(1 + \frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{m_i} = \prod_{i=1}^{\infty} \left(1 + \frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{m_i} \\ &\Leftrightarrow \left(1 + \frac{(s-\alpha_l)^2}{\beta_l^2}\right)^{m_l} f_l(s) = \left(1 + \frac{(1-s-\alpha_l)^2}{\beta_l^2}\right)^{m_l} f_l(1-s) \end{aligned} \quad (15)$$

where

$$f_l(s) = \prod_{\substack{i=1 \\ i \neq l}}^{\infty} \left(1 + \frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{m_i} \quad (16)$$

$$f_l(1-s) = \prod_{\substack{i=1 \\ i \neq l}}^{\infty} \left(1 + \frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{m_i} \quad (17)$$

Then we have

$$\begin{aligned}
 \left(1 + \frac{(s - \alpha_l)^2}{\beta_l^2}\right)^{m_l} f_l(s) &= \left(1 + \frac{(1 - s - \alpha_l)^2}{\beta_l^2}\right)^{m_l} f_l(1 - s) \\
 \Rightarrow \\
 \left(1 + \frac{(s - \alpha_l)^2}{\beta_l^2}\right)^{m_l} &\Big| \left(1 + \frac{(1 - s - \alpha_l)^2}{\beta_l^2}\right)^{m_l} f_l(1 - s) \\
 \left(1 + \frac{(1 - s - \alpha_l)^2}{\beta_l^2}\right)^{m_l} &\Big| \left(1 + \frac{(s - \alpha_l)^2}{\beta_l^2}\right)^{m_l} f_l(s)
 \end{aligned} \tag{18}$$

where " $|$ " is the divisible sign.

Considering $\left(1 + \frac{(s - \alpha_l)^2}{\beta_l^2}\right)$, $0 < \alpha_l < 1$, $\beta_l \neq 0$, is irreducible over the field R of real numbers, we know from Eq.(18) that

$$\begin{aligned}
 \left(1 + \frac{(s - \alpha_l)^2}{\beta_l^2}\right)^{m_l} &\Big| f_l(1 - s) \Rightarrow \left(1 + \frac{(s - \alpha_l)^2}{\beta_l^2}\right)^{m_l} \Big| f_l(1 - s) \\
 \Rightarrow (\text{by Lemma 4}) \\
 \left(1 + \frac{(s - \alpha_l)^2}{\beta_l^2}\right)^{m_l} &\Big| \left(1 + \frac{(1 - s - \alpha_i)^2}{\beta_i^2}\right)^{m_i}, i \neq l \\
 \Rightarrow \left(1 + \frac{(1 - s - \alpha_i)^2}{\beta_i^2}\right)^{m_i} &= k \left(1 + \frac{(s - \alpha_l)^2}{\beta_l^2}\right)^{m_l}, i \neq l \\
 \Rightarrow (\text{by comparing the like terms in the above polynomial equation}) \\
 \alpha_i + \alpha_l &= 1, \beta_i^2 = \beta_l^2, k = 1, i \neq l
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 \left(1 + \frac{(1 - s - \alpha_l)^2}{\beta_l^2}\right)^{m_l} &\Big| f_l(s) \Rightarrow \left(1 + \frac{(1 - s - \alpha_l)^2}{\beta_l^2}\right)^{m_l} \Big| f_l(s) \\
 \Rightarrow (\text{by Lemma 4}) \\
 \left(1 + \frac{(1 - s - \alpha_l)^2}{\beta_l^2}\right)^{m_l} &\Big| \left(1 + \frac{(s - \alpha_i)^2}{\beta_i^2}\right)^{m_i}, i \neq l \\
 \Rightarrow \left(1 + \frac{(s - \alpha_i)^2}{\beta_i^2}\right)^{m_i} &= k \left(1 + \frac{(1 - s - \alpha_l)^2}{\beta_l^2}\right)^{m_l}, i \neq l \\
 \Rightarrow (\text{by comparing the like terms in the above polynomial equation}) \\
 \alpha_i + \alpha_l &= 1, \beta_i^2 = \beta_l^2, k = 1, i \neq l
 \end{aligned}$$

As explained in the situation of Figure 1, $\alpha_i + \alpha_l = 1$, $\beta_i^2 = \beta_l^2$, $i \neq l$ means that $\alpha_i \pm j\beta_i$ and $\alpha_l \pm j\beta_l$ are the same zeros in terms of quadruplet, i.e., $\rho, \bar{\rho}, 1 - \rho$, and $1 - \bar{\rho}$, which contradicts the definition of real multiplicities of zeros.

Thus, $(1 + \frac{(s-\alpha_l)^2}{\beta_l^2})^{m_l}$ and $f_l(1-s)$ are relatively prime, $(1 + \frac{(1-s-\alpha_l)^2}{\beta_l^2})^{m_l}$ and $f_l(s)$ are relatively prime. Accordingly, by Lemma 5, we obtain from Eq.(18) the following result.

$$\begin{aligned}
 (1 + \frac{(s-\alpha_l)^2}{\beta_l^2})^{m_l} f_l(s) &= (1 + \frac{(1-s-\alpha_l)^2}{\beta_l^2})^{m_l} f_l(1-s) \\
 \Rightarrow \\
 (1 + \frac{(s-\alpha_l)^2}{\beta_l^2})^{m_l} &\Big| (1 + \frac{(1-s-\alpha_l)^2}{\beta_l^2})^{m_l} \\
 (1 + \frac{(1-s-\alpha_l)^2}{\beta_l^2})^{m_l} &\Big| (1 + \frac{(s-\alpha_l)^2}{\beta_l^2})^{m_l} \\
 \Rightarrow \\
 (1 + \frac{(s-\alpha_l)^2}{\beta_l^2})^{m_l} &= k(1 + \frac{(1-s-\alpha_l)^2}{\beta_l^2})^{m_l} \\
 \Rightarrow (k=1, \text{ by comparing the highest-order terms in the above polynomial equation}) \\
 (1 + \frac{(s-\alpha_l)^2}{\beta_l^2})^{m_l} &= (1 + \frac{(1-s-\alpha_l)^2}{\beta_l^2})^{m_l} \\
 \Rightarrow (\text{by Eq.(14)}) \\
 \alpha_l &= \frac{1}{2}
 \end{aligned} \tag{19}$$

Let l run over from 1 to ∞ , and repeat the above process, we get

$$\begin{aligned}
 \prod_{i=1}^{\infty} (1 + \frac{(s-\alpha_i)^2}{\beta_i^2})^{m_i} &= \prod_{i=1}^{\infty} (1 + \frac{(1-s-\alpha_i)^2}{\beta_i^2})^{m_i} \\
 \Rightarrow \\
 (1 + \frac{(s-\alpha_i)^2}{\beta_i^2})^{m_i} &= (1 + \frac{(1-s-\alpha_i)^2}{\beta_i^2})^{m_i} \\
 \Rightarrow \\
 \alpha_i &= \frac{1}{2}, i = 1, 2, 3, \dots, \infty
 \end{aligned} \tag{20}$$

Also, we have the following obvious fact

$$\begin{aligned}
 \alpha_i &= \frac{1}{2}, i = 1, 2, 3, \dots, \infty \\
 \Rightarrow \\
 (1 + \frac{(s-\alpha_i)^2}{\beta_i^2})^{m_i} &= (1 + \frac{(1-s-\alpha_i)^2}{\beta_i^2})^{m_i} \\
 \Rightarrow \\
 \prod_{i=1}^{\infty} (1 + \frac{(s-\alpha_i)^2}{\beta_i^2})^{m_i} &= \prod_{i=1}^{\infty} (1 + \frac{(1-s-\alpha_i)^2}{\beta_i^2})^{m_i}
 \end{aligned} \tag{21}$$

Further, limiting the imaginary parts β_i of zeros to $0 < |\beta_1| < |\beta_2| < |\beta_3| < \dots$ in order to keep the real multiplicities of zeros unchanged, we finally get

$$\begin{aligned} \prod_{i=1}^{\infty} \left(1 + \frac{(s - \alpha_i)^2}{\beta_i^2}\right)^{m_i} &= \prod_{i=1}^{\infty} \left(1 + \frac{(1-s - \alpha_i)^2}{\beta_i^2}\right)^{m_i} \\ &\Leftrightarrow \\ \begin{cases} \left(1 + \frac{(s - \alpha_i)^2}{\beta_i^2}\right)^{m_i} = \left(1 + \frac{(1-s - \alpha_i)^2}{\beta_i^2}\right)^{m_i} \\ 0 < |\beta_1| < |\beta_2| < |\beta_3| < \dots \\ i = 1, 2, 3, \dots, \infty \end{cases} \\ &\Leftrightarrow \\ \begin{cases} \alpha_i = \frac{1}{2} \\ 0 < |\beta_1| < |\beta_2| < |\beta_3| < \dots \\ i = 1, 2, 3, \dots, \infty \end{cases} \end{aligned}$$

i.e.,

$$f(s) = f(1-s) \Leftrightarrow \begin{cases} \alpha_i = \frac{1}{2} \\ 0 < |\beta_1| < |\beta_2| < |\beta_3| < \dots \\ i = 1, 2, 3, \dots, \infty \end{cases}$$

That completes the proof of Lemma 3.

To support the proof of Lemma 3, we need the following classical results (Lemma 4 and Lemma 5) in Polynomial Algebra over Fields.

Lemma 4: Let F be a field, $m(x), g_1(x), \dots, g_n(x) \in F[x], n \geq 2$. If $m(x)$ is irreducible (prime) and divides the product $g_1(x) \cdots g_n(x)$, then $m(x)$ divides one of the polynomials $g_1(x), \dots, g_n(x)$.

Lemma 5: Let F be a field, $p(x), q(x), m(x) \in F[x]$. If $m(x)$ divides the product $p(x)q(x)$, but $m(x)$ and $p(x)$ are relatively prime, i.e., $\gcd(m, p) = 1$, ($\gcd$: greatest common divisor), then $m(x)$ divides $q(x)$.

Remark: $F[x]$ is defined as the set of all polynomials in x over F :

$$F[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \mid a_i \in F, a_i \neq 0 \text{ for all but a finite number of } i \right\}$$

The set $F[x]$ equipped with the operations $+$ and $\cdot$ is the polynomial ring in x over the field F . In this paper, F is specified as the field R of real numbers.

Remark: The contents of Lemma 4 and Lemma 5 can be found in many textbooks of Linear Algebra or Advanced Algebra, such as, Kenneth M Hoffman, Ray Kunze, Linear Algebra (Second Edition), Prentice-Hall, INC., Englewood Cliffs, New Jersey, 1971; Linda Gilbert, Jimmie Gilbert, Elements of Modern Algebra (Seventh Edition), CENGAGE Learning, Belmont, CA, 2009.

3. A Proof of the RH

This section is planned to present a proof of the Riemann Hypothesis. We first prove that Statement 2 of the RH is true, and then by Lemma 2, Statement 1 of the RH is also true. To be brief, to prove the Riemann Hypothesis, it suffices to show that $\alpha_i = \frac{1}{2}, i = 1, 2, 3, \dots, \infty$ in the new expression of $\zeta(s)$ as shown in Eq.(22).

Proof of the RH: The details are delivered in three steps as follows.

Step 1:

It is well-known that all the zeros of $\zeta(s)$ always come in complex conjugate pairs. Then by pairing $\rho_i = \alpha_i + j\beta_i$ and $\bar{\rho}_i = \alpha_i - j\beta_i$ in the Hadamard product as shown in Eq.(10), we have

$$\begin{aligned}\zeta(s) &= \zeta(0) \prod_{\rho} \left(1 - \frac{s}{\rho}\right) = \zeta(0) \prod_{i=1}^{\infty} \left(1 - \frac{s}{\rho_i}\right) \left(1 - \frac{s}{\bar{\rho}_i}\right) \\ &= \zeta(0) \prod_{i=1}^{\infty} \left(1 - \frac{s}{\alpha_i + j\beta_i}\right) \left(1 - \frac{s}{\alpha_i - j\beta_i}\right) = \zeta(0) \prod_{i=1}^{\infty} \left(\frac{\beta_i^2}{\alpha_i^2 + \beta_i^2} + \frac{(s - \alpha_i)^2}{\alpha_i^2 + \beta_i^2}\right)\end{aligned}\quad (22)$$

where $\zeta(0) = \frac{1}{2}$, $0 < \alpha_i < 1$, $\beta_i \neq 0$.

The absolute convergence of the infinite product in Eq.(22) in the form

$$\zeta(s) = \zeta(0) \prod_{i=1}^{\infty} \left(1 - \frac{s}{\rho_i}\right) \left(1 - \frac{s}{\bar{\rho}_i}\right) = \zeta(0) \prod_{i=1}^{\infty} \left(1 - \frac{s(2\alpha_i - s)}{|\rho_i|^2}\right) \quad (23)$$

depends on the convergence of infinite series $\sum_{i=1}^{\infty} \frac{1}{|\rho_i|^2}$, which is an obvious fact according to Theorem 2 in Section 2, Chapter IV of Ref. [23].

Further, considering the absolute convergence of

$$\zeta(s) = \zeta(0) \prod_{i=1}^{\infty} \left(1 - \frac{s(2\alpha_i - s)}{|\rho_i|^2}\right) = \zeta(0) \prod_{i=1}^{\infty} \left(\frac{\beta_i^2}{\alpha_i^2 + \beta_i^2} + \frac{(s - \alpha_i)^2}{\alpha_i^2 + \beta_i^2}\right) \quad (24)$$

we have the following new expression of $\zeta(s)$ by putting all the ρ_i related multiple factors (zeros) together in the above Eq.(24)

$$\zeta(s) = \zeta(0) \prod_{i=1}^{\infty} \left(\frac{\beta_i^2}{\alpha_i^2 + \beta_i^2} + \frac{(s - \alpha_i)^2}{\alpha_i^2 + \beta_i^2}\right)^{m_i} \quad (25)$$

where $m_i \geq 1$ is the real multiplicity of ρ_i , $i = 1, 2, 3, \dots, \infty$.

Step 2: Replacing s with $1 - s$ in Eq.(25), we obtain the infinite product expression of $\zeta(1 - s)$, i.e.,

$$\zeta(1 - s) = \zeta(0) \prod_{i=1}^{\infty} \left(\frac{\beta_i^2}{\alpha_i^2 + \beta_i^2} + \frac{(1 - s - \alpha_i)^2}{\alpha_i^2 + \beta_i^2}\right)^{m_i} \quad (26)$$

Step 3: According to the functional equation $\zeta(s) = \zeta(1 - s)$, and considering Eq.(25) and Eq.(26), we have

$$\zeta(0) \prod_{i=1}^{\infty} \left(\frac{\beta_i^2}{\alpha_i^2 + \beta_i^2} + \frac{(s - \alpha_i)^2}{\alpha_i^2 + \beta_i^2}\right)^{m_i} = \zeta(0) \prod_{i=1}^{\infty} \left(\frac{\beta_i^2}{\alpha_i^2 + \beta_i^2} + \frac{(1 - s - \alpha_i)^2}{\alpha_i^2 + \beta_i^2}\right)^{m_i} \quad (27)$$

which is equivalent to

$$\prod_{i=1}^{\infty} \left(1 + \frac{(s - \alpha_i)^2}{\beta_i^2}\right)^{m_i} = \prod_{i=1}^{\infty} \left(1 + \frac{(1 - s - \alpha_i)^2}{\beta_i^2}\right)^{m_i} \quad (28)$$

where β_i are in order of increasing $|\beta_i|$, i.e., $0 < |\beta_1| \leq |\beta_2| \leq |\beta_3| \leq \dots$.

To check the absolute convergence of both sides of Eq.(28), it suffices to make a comparison with Eq.(23) without considering multiple zeros in Eq. (28), i.e., to make a comparison between $\prod_{i=1}^{\infty} \left(1 + \frac{(s - \alpha_i)^2}{\beta_i^2}\right)$ and $\zeta(0) \prod_{i=1}^{\infty} \left(1 - \frac{s(2\alpha_i - s)}{|\rho_i|^2}\right)$. It is well-known that the absolute convergence of

$\zeta(0) \prod_{i=1}^{\infty} \left(1 - \frac{s(2\alpha_i - s)}{|\rho_i|^2}\right)$ depends on the convergence of infinite series $\sum_{i=1}^{\infty} \frac{1}{|\rho_i|^2}$ (already proved in Step 1); the absolute convergence of $\prod_{i=1}^{\infty} \left(1 + \frac{(s - \alpha_i)^2}{\beta_i^2}\right)$ depends on the convergence of infinite series $\sum_{i=1}^{\infty} \frac{1}{\beta_i^2}$, which is also an obvious fact because $0 < \alpha_i < 1, |\rho_i| \rightarrow \infty, |\beta_i| \rightarrow \infty$, as $i \rightarrow \infty, \lim_{i \rightarrow \infty} \frac{\beta_i^2}{|\rho_i|^2} = \lim_{i \rightarrow \infty} \frac{\beta_i^2}{\alpha_i^2 + \beta_i^2} = 1$, that means $\sum_{i=1}^{\infty} \frac{1}{\beta_i^2}$ and $\sum_{i=1}^{\infty} \frac{1}{|\rho_i|^2}$ have the same convergence.

Then, according to Lemma 3, Eq.(28) is equivalent to

$$\alpha_i = \frac{1}{2}; 0 < |\beta_1| < |\beta_2| < |\beta_3| < \cdots; i = 1, 2, 3, \cdots, \infty \quad (29)$$

Thus, we conclude that all the zeros of the completed zeta function $\tilde{\zeta}(s)$ have real part equal to $\frac{1}{2}$, i.e., Statement 2 of the RH is true. According to Lemma 2, Statement 1 of the RH is also true, i.e., all the non-trivial zeros of the Riemann zeta function $\zeta(s)$ have real part equal to $\frac{1}{2}$.

That completes the proof of the RH.

4. Retrospection and Discussion

On the simultaneous zeros of $\tilde{\zeta}(s)$

According to Lemma 1, there are two pairs of complex zeros of $\tilde{\zeta}(s)$ simultaneously, i.e., $\rho = \alpha + j\beta, \bar{\rho} = \alpha - j\beta, 1 - \rho = 1 - \alpha - j\beta, 1 - \bar{\rho} = 1 - \alpha + j\beta$. With the proof of the RH, these 2 pairs of zeros are actually only one pair, because $\rho = 1 - \bar{\rho} = \frac{1}{2} + j\beta, \bar{\rho} = 1 - \rho = \frac{1}{2} - j\beta$. Thus Lemma 1 could be modified more precisely as follows.

Lemma 1*: Non-trivial zeroes of $\zeta(s)$, noted as $\rho = \alpha + j\beta$, have the following properties

- 1) The number of non-trivial zeroes is infinity;
- 2) $\beta \neq 0$;
- 3) $0 < \alpha < 1$;
- 4) $\rho = 1 - \bar{\rho}, \bar{\rho} = 1 - \rho$ are all non-trivial zeroes.

On the paring of zeros of $\tilde{\zeta}(s)$

Hadamard pointed out that to ensure the absolute convergence of the Hadamard product, i.e., $\tilde{\zeta}(s) = \zeta(0) \prod_{\rho} \left(1 - \frac{s}{\rho}\right)$, ρ and $1 - \rho$ are paired. In Section 3, the author proved that ρ and $\bar{\rho}$ can also be paired to ensure the absolute convergence of the Hadamard product. And that the paring of conjugate zeros, i.e., ρ and $\bar{\rho}$, is the right way to express the most essential characteristic of $\tilde{\zeta}(s)$ as (infinite) polynomial with real coefficients, whereas $1 - \rho$ and $1 - \bar{\rho}$ are just another pair of conjugate zeros given by $\tilde{\zeta}(s) = \tilde{\zeta}(1 - s)$.

5. Conclusion

This paper presents a proof of the RH based on a new expression of $\tilde{\zeta}(s)$, i.e.,

$$\tilde{\zeta}(s) = \zeta(0) \prod_{i=1}^{\infty} \left(\frac{\beta_i^2}{\alpha_i^2 + \beta_i^2} + \frac{(s - \alpha_i)^2}{\alpha_i^2 + \beta_i^2} \right)^{m_i}$$

where $\zeta(0) = \frac{1}{2}, \rho_i = \alpha_i + j\beta_i$ and $\bar{\rho}_i = \alpha_i - j\beta_i$ are the complex conjugate zeros of $\tilde{\zeta}(s)$, $0 < \alpha_i < 1$ and $\beta_i \neq 0$ are real numbers, $0 < |\beta_1| \leq |\beta_2| \leq |\beta_3| \leq \cdots, m_i \geq 1$ is the real multiplicity of ρ_i .

The proof is conducted according to the divisibility implied in the (infinite) polynomial equation $\tilde{\zeta}(s) = \tilde{\zeta}(1 - s)$. The first key-point is the paring of conjugate zeros ρ and $\bar{\rho}$ to get the new expression of $\tilde{\zeta}(s)$. The second key-point is the use of "real multiplicity" of a zero of $\tilde{\zeta}(s)$. Obviously, the real multiplicity of a zero of $\tilde{\zeta}(s)$ is an objective existence, unique, and unchangeable. As a result, the functional equation $\tilde{\zeta}(s) = \tilde{\zeta}(1 - s)$ finally leads to $\alpha_i = \frac{1}{2}; 0 < |\beta_1| < |\beta_2| < |\beta_3| < \cdots; i = 1, 2, 3, \cdots, \infty$.

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Finally, with this manuscript, the author pays tribute to Bernhard Riemann and other predecessor mathematicians. They are the shining stars in the sky of human civilization.

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