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A Proof Of The Riemann Hypothesis Based On A New Expression Of The Completed Zeta Function

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A Proof Of The Riemann Hypothesis Based On A New Expression Of The Completed Zeta Function

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Abstract Based on the Hadamard product $\xi(s) = \xi(0)\prod_{\rho}(1-\frac{s}{\rho})$, a new absolute convergent expression of $\xi(s)$ is obtained by paring ρ_i and $\bar{\rho}_i$, and putting all the ρ_i related multiple zeros together in one factor

$$\xi(s) = \xi(0)\prod_{i=1}^{\infty}\Big(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}\Big)^{d_i}$$

where $\xi(0)=\frac{1}{2}$, $\rho_i=\alpha_i+j\beta_i$ and $\bar{\rho}_i=\alpha_i-j\beta_i$ are the complex conjugate zeros of $\xi(s)$, $0<\alpha_i<1$ and $\beta_i\neq 0$ are real numbers, $d_i\geq 1$ are the real multiplicities of ρ_i, β_i are in order of increasing $|\beta_i|$. Then, by the functional equation $\xi(s)=\xi(1-s)$, we have

$$\xi(0)\prod_{i=1}^{\infty}\Big(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}\Big)^{d_i} = \xi(0)\prod_{i=1}^{\infty}\Big(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(1-s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}\Big)^{d_i}$$

i.e.,

$$\prod_{i=1}^{\infty}\Big(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\Big)^{d_i} = \prod_{i=1}^{\infty}\Big(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\Big)^{d_i}$$

which, by Lemma 3, is equivalent to

$$\begin{cases} \alpha_i = \frac{1}{2}, i = 1, 2, 3, \cdots, \infty \\ \beta_1 < \beta_2 < \beta_3 < \cdots \end{cases}$$

Thus, we conclude that the Riemann Hypothesis is true.

Keywords Riemann Hypothesis (RH) · Proof · Completed zeta function

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1 Introduction

It has been almost 163 years since the Riemann Hypothesis (RH) was proposed in 1859 [1]. Many efforts and achievements have been made towards proving this celebrated hypothesis, but it is still an open problem [2−3].

The Riemann zeta function is the function of the complex variable s, defined in the half-plane $\Re(s) > 1$ by the absolutely convergent series [2]

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \Re(s) > 1 \tag{1}$$

The connection between the Riemann zeta function and prime numbers can be established through the well-known Euler product, i.e.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p} (1 - p^{-s})^{-1}, \Re(s) > 1 \tag{2}$$

where p runs over the prime numbers.

Riemann showed how to extend zeta function to the whole complex plane $\mathbb{C}$ by analytic continuation

$$\zeta(s) = \frac{\pi^{s/2}}{\Gamma(s/2)} \{ \frac{1}{s(s-1)} + \int_1^{\infty} (x^{\frac{s}{2}-1} + x^{-\frac{s}{2}-\frac{1}{2}}) \cdot (\frac{\theta(x)-1}{2}) dx \} \tag{3}$$

where $\theta(x) = \sum_{-\infty}^{\infty} e^{-n^2 \pi x}$ being the Jaccobi theta function, Γ being the Gamma function in the following Weierstrass expression (Meanwhile, there are also Gauss expression, Euler expression, and integral expression of the Gamma function.)

$$\frac{1}{\Gamma(s)} = s \cdot e^{\gamma s} \prod_{n=1}^{\infty} (1 + \frac{s}{n}) e^{-s/n} \tag{4}$$

where γ is the Euler-Mascheroni constant.

As shown by Riemann, $\zeta(s)$ extends to $\mathbb{C}$ as a meromorphic function with only a simple pole at $s = 1$, with residue 1, and satisfies the following functional equation

$$\pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma(\frac{1-s}{2}) \zeta(1-s) \tag{5}$$

The Riemann zeta function $\zeta(s)$ has zeros at the negative even integers: -2, -4, -6, -8, $\cdots$ and one refers to them as the **trivial zeros**. The other zeros of $\zeta(s)$ are the complex numbers, i.e., **non-trivial zeros** [2].

In 1896, Hadamard [4] and Poussin [5] independently proved that no zeros could lie on the line $\Re(s) = 1$. Together with the functional equation and the fact that there are no zeros with real part greater than 1, this showed that all non-trivial zeros must lie in the interior of the **critical strip** $0 < \Re(s) < 1$.

This was a key step in their first proofs of the famous **Prime Number Theorem**.

Later on, Hardy (1914) [6], Hardy and Littlewood (1921) [7] showed that there are infinitely many zeros on the **critical line** $\Re(s) = \frac{1}{2}$, which was an astonishing result at that time.

As a summary, we have the following results on the properties of the non-trivial zeros of $\zeta(s)$ [4−9].
Lemma 1: Non-trivial zeroes of $\zeta(s)$, noted as $\rho = \alpha + j\beta$, have the following properties
1) The number of non-trivial zeroes is infinity;
2) $\beta \neq 0$;
3) $0 < \alpha < 1$;
4) $\rho, \bar{\rho}, 1-\bar{\rho}, 1-\rho$ are all non-trivial zeroes.

As further study, a completed zeta function $\xi(s)$ is defined as

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) \tag{6}$$

It is well-known that $\xi(s)$ is an entire function of order 1. This implies $\xi(s)$ is analytic, and can be expressed as infinite polynomial, in the whole complex plane $\mathbb{C}$.

In addition, replacing s with $1-s$ in Eq.(6), and combining Eq.(5), we have the following functional equation

$$\xi(s) = \xi(1-s) \tag{7}$$

Considering the definition of $\xi(s)$, and recalling Eq.(4), the trivial zeros of $\zeta(s)$ are canceled by the poles of $\Gamma(\frac{s}{2})$. The zero of $s-1$ and the pole of $\zeta(s)$ cancel; the zero $s=0$ and the pole of $\Gamma(\frac{s}{2})$ cancel [9−10]. Thus, all the zeros of $\xi(s)$ are exactly the nontrivial zeros of $\zeta(s)$. Then we have the following Lemma 2.

Lemma 2: The zeros of $\xi(s)$ coincide with the non-trivial zeros of $\zeta(s)$.

According to Lemma 2, the following two statements for the RH are equivalent.

Statement 1 of the RH: All the non-trivial zeros of $\zeta(s)$ have real part equal to $\frac{1}{2}$.

Statement 2 of the RH: All the zeros of $\xi(s)$ have real part equal to $\frac{1}{2}$.

To prove the RH, a natural thinking is to estimate the numbers of non-trivial zeros of $\zeta(s)$ inside or outside some areas according to Argument Principle. Along this train of thought, there are many research works. Let $N(T)$

denote the number of zeros of $\zeta(s)$ inside the rectangle: $0 < \alpha < 1, 0 < \beta \leq T$, and let $N_0(T)$ denote the number of zeros of $\zeta(s)$ on the line $\alpha = \frac{1}{2}, 0 < \beta \leq T$. Selberg proved that there exist positive constants c and T_0, such that $N_0(T) > cN(T), (T > T_0)$ [11], later on, Levinson proved that $c \geq \frac{1}{3}$ [12], Lou and Yao proved that $c \geq 0.3484$ [13], Conrey proved that $c \geq \frac{2}{5}$ [14], Bui, Conrey and Young proved that $c \geq 0.41$ [15], Feng proved that $c \geq 0.4128$ [16].

On the other hand, many zeros have been calculated by hand or by computer programs. Among others, Riemann found the first three non-trivial zeros [17]. Gram found the first 15 zeros based on Euler-Maclaurin summation [18]. Titchmarsh calculated the 138^{th} to 195^{th} zeros using the Riemann-Siegel formula [19−20]. Here are the first three (pairs of) zeros: $\frac{1}{2} \pm j14.1347251$; $\frac{1}{2} \pm j21.0220396$; $\frac{1}{2} \pm j25.0108575$.

The idea of this paper is originated from Euler's work on proving the following famous equality

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{6} \tag{8}$$

This interesting result is deduced by comparing the like terms of two types of infinite expressions, i.e., infinite polynomial and infinite product, as shown in the following

$$\begin{aligned} \frac{\sin x}{x} &= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots \\ &= (1 - \frac{x^2}{\pi^2})(1 - \frac{x^2}{4\pi^2})(1 - \frac{x^2}{9\pi^2}) \cdots \end{aligned} \tag{9}$$

Then it is conjectured that $\xi(s)$ should be factored into $\left(1 + \frac{(s-\alpha_i)^2}{\beta_i^2}\right)$ or something like that, which was verified by paring ρ_i and $\bar{\rho}_i$ in the Hadamard product of $\xi(s)$ to obtain

$$(1 - \frac{s}{\rho_i})(1 - \frac{s}{\bar{\rho}_i}) = \frac{\beta_i^2}{\alpha_i^2 + \beta_i^2}\left(1 + \frac{(s-\alpha_i)^2}{\beta_i^2}\right)$$

The Hadamard product of $\xi(s)$ as shown in Eq.(10) was first proposed by Riemann, however, it was Hadamard [21] who showed the validity of this infinite product expansion.

$$\xi(s) = \xi(0) \prod_{\rho} (1 - \frac{s}{\rho}) \tag{10}$$

where $\xi(0) = \frac{1}{2}$, ρ runs over all the non-trivial zeros of the Riemann zeta function $\zeta(s)$, or in another word, ρ runs over all the zeros of the completed zeta function $\xi(s)$.

To ensure the absolute convergence of the infinite product expansion, ρ and $1 - \rho$ are paired. Later in Section 3, we will show that ρ and $\bar{\rho}$ can also be paired to ensure the absolute convergence of the infinite product expansion.

2 Lemmas

In this section, we first explain the concepts of multiple zeros of $\xi(s)$ with their real multiplicities. And then we give three lemmas to support the proof of the RH, in which Lemma 3 is the key lemma.

Multiple zeros of $\xi(s)$: As shown in Figure 1, the multiple zeros of $\xi(s)$ are defined in terms of the quadruplet, i.e., $\rho, \bar{\rho}, 1-\rho, 1-\bar{\rho}$.

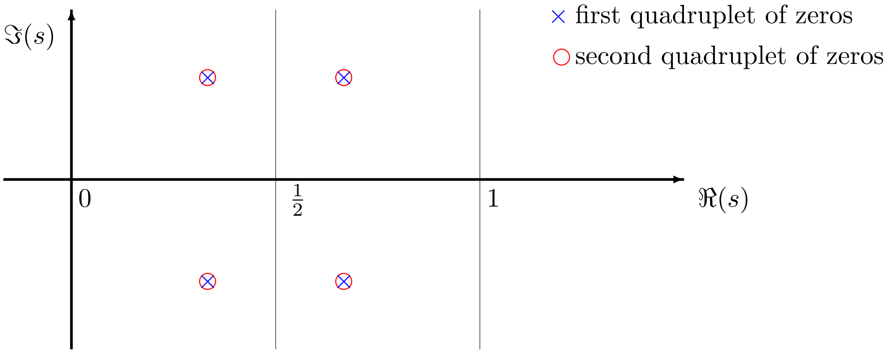

Figure 1 Illustration of the multiple zeros of $\xi(s)$

It should be noticed that the multiple zeros with their real multiplicities of $\xi(s)$ are objective existence, but the expression of the corresponding factors of $\xi(s)$ are optional to some extent. For example, the multiple zeros as shown in Figure 1 have two different expressions as factors of $\xi(s)$ and $\xi(1-s)$, respectively, i.e., $\left[\left(1+\frac{(s-\alpha_1)^2}{\beta_1^2}\right)^2, \left(1+\frac{(1-s-\alpha_1)^2}{\beta_1^2}\right)^2\right]$, or $\left[\left(1+\frac{(s-\alpha_1)^2}{\beta_1^2}\right)\left(1+\frac{(s-\alpha_2)^2}{\beta_2^2}\right), \left(1+\frac{(1-s-\alpha_1)^2}{\beta_1^2}\right)\left(1+\frac{(1-s-\alpha_2)^2}{\beta_2^2}\right), \alpha_1+\alpha_2=1, \beta_1=\beta_2\right]$.

To exclude the latter expression, we stipulate that the multiple zero ρ_i related factor of $\xi(s)$ takes the form of $\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}$, where $d_i \geq 1$ is the real multiplicity of ρ_i.

Lemma 3: Given two infinite products

$$f(s)=\prod_{i=1}^{\infty}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} \tag{11}$$

and

$$f(1-s)=\prod_{i=1}^{\infty}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} \tag{12}$$

where s is a complex variable, $\rho_i=\alpha_i+j\beta_i$ and $\bar{\rho}_i=\alpha_i-j\beta_i$ are the complex conjugate zeros of $\xi(s)$, $0<\alpha_i<1$ and $\beta_i \neq 0$ are real numbers, $d_i \geq 1$ are the real multiplicities of ρ_i, i are natural numbers from 1 to infinity, β_i are in order of increasing $|\beta_i|$, i.e., $|\beta_1| \leq |\beta_2| \leq |\beta_3| \leq \cdots$.

Then we have

$$f(s) = f(1-s) \Leftrightarrow \begin{cases} \alpha_i = \frac{1}{2}, i = 1,2,3,\cdots,\infty \\ \beta_1 < \beta_2 < \beta_3 < \cdots \end{cases} \tag{13}$$

where " $\Leftrightarrow$ " is the equivalent sign.

Proof: First of all, we have the following fact:

$$\begin{aligned} &(1 + \frac{(s-\alpha)^2}{\beta^2})^d = (1 + \frac{(1-s-\alpha)^2}{\beta^2})^d \\ &\Leftrightarrow (s-\alpha)^2 = (1-s-\alpha)^2 \Leftrightarrow \alpha = \frac{1}{2} \end{aligned} \tag{14}$$

where $d \geq 1$ is a natural number, $\alpha \neq 0$ and $\beta \neq 0$ are real numbers.
Next, the proof is based on Transfinite Induction. Let $P(n)$ be:

$$\begin{aligned} &\prod_{i=1}^{n} \left(1 + \frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} = \prod_{i=1}^{n} \left(1 + \frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} \\ &\Leftrightarrow \\ &\begin{cases} (1 + \frac{(s-\alpha_1)^2}{\beta_1^2})^{d_1} = (1 + \frac{(1-s-\alpha_1)^2}{\beta_1^2})^{d_1} \\ \cdots \\ (1 + \frac{(s-\alpha_n)^2}{\beta_n^2})^{d_n} = (1 + \frac{(1-s-\alpha_n)^2}{\beta_n^2})^{d_n} \\ \beta_1 < \beta_2 < \beta_3 < \cdots < \beta_n \end{cases} \\ &\Leftrightarrow \\ &\begin{cases} \alpha_i = \frac{1}{2}, i = 1 \cdots n \\ \beta_1 < \beta_2 < \beta_3 < \cdots < \beta_n \end{cases} \end{aligned} \tag{15}$$

According to Eq.(14), $P(1)$ is an obvious fact as the **Base Case**, i.e.,

$$\begin{aligned} &\prod_{i=1}^{1} \left(1 + \frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} = \prod_{i=1}^{1} \left(1 + \frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} \\ &\Leftrightarrow \left(1 + \frac{(s-\alpha_1)^2}{\beta_1^2}\right)^{d_1} = \left(1 + \frac{(1-s-\alpha_1)^2}{\beta_1^2}\right)^{d_1} \\ &\Leftrightarrow \alpha_1 = \frac{1}{2} \end{aligned} \tag{16}$$

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To be more convincing, let's further check $P(2)$, i.e.,

$$\begin{aligned}
&\prod_{i=1}^{2}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}=\prod_{i=1}^{2}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}\\
&\Leftrightarrow\\
&\begin{cases}
(1+\frac{(s-\alpha_1)^2}{\beta_1^2})^{d_1}=(1+\frac{(1-s-\alpha_1)^2}{\beta_1^2})^{d_1}\\
(1+\frac{(s-\alpha_2)^2}{\beta_2^2})^{d_2}=(1+\frac{(1-s-\alpha_2)^2}{\beta_2^2})^{d_2}\\
\beta_1<\beta_2
\end{cases}\\
&\Leftrightarrow\\
&\begin{cases}
\alpha_1=\alpha_2=\frac{1}{2}\\
\beta_1<\beta_2
\end{cases}
\end{aligned}\tag{17}$$

which is also an obvious fact according to Lemma 5.

As the **Successor Case**, we need to prove $P(n)\Rightarrow P(n+1)$.
Actually, we have

$$\begin{aligned}
&\prod_{i=1}^{n+1}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}=\prod_{i=1}^{n+1}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}\\
&\Leftrightarrow\\
&\prod_{i=1}^{n}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}\left(1+\frac{(s-\alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}}=\prod_{i=1}^{n}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}\left(1+\frac{(1-s-\alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}}\\
&\Leftrightarrow\Big(\text{by Lemma 5}\Big)\\
&\begin{cases}
\prod_{i=1}^{n}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}=\prod_{i=1}^{n}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}\\
(1+\frac{(s-\alpha_{n+1})^2}{\beta_{n+1}^2})^{d_{n+1}}=(1+\frac{(1-s-\alpha_{n+1})^2}{\beta_{n+1}^2})^{d_{n+1}}\\
\beta_1<\beta_2<\beta_3<\cdots<\beta_n<\beta_{n+1}
\end{cases}\\
&\Leftrightarrow\Big(\text{by Eq.(15)}\Big)\\
&\begin{cases}
(1+\frac{(s-\alpha_1)^2}{\beta_1^2})^{d_1}=(1+\frac{(1-s-\alpha_1)^2}{\beta_1^2})^{d_1}\\
\cdots\\
(1+\frac{(s-\alpha_{n+1})^2}{\beta_{n+1}^2})^{d_{n+1}}=(1+\frac{(1-s-\alpha_{n+1})^2}{\beta_{n+1}^2})^{d_{n+1}}\\
\beta_1<\beta_2<\beta_3<\cdots<\beta_n<\beta_{n+1}
\end{cases}\\
&\Leftrightarrow\Big(\text{by Eq.(14)}\Big)\\
&\begin{cases}
\alpha_i=\frac{1}{2},i=1,2,3,\cdots,n,n+1\\
\beta_1<\beta_2<\beta_3<\cdots<\beta_n<\beta_{n+1}
\end{cases}
\end{aligned}\tag{18}$$

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Thus the **Successor Case** is true, i.e., $P(n) \Rightarrow P(n+1)$.

Next, we prove that $P(\infty)$ holds by considering well-ordered ordinal set A indexing the family of statements $P(\gamma : \gamma \in A)$, $A = \mathbb{N}\bigcup\{\omega\}$ with the ordering that $n < \omega$ for all natural numbers n, ω is the first limit ordinal.
It is well-known that $\omega = \bigcup\{\gamma : \gamma < \omega\}$.

To prove that $P(\infty)$ holds, it suffices to prove the **Limit Case**, i.e., $P(\gamma < \omega) \Rightarrow P(\omega)$.
Actually, we have

$$\prod_{i=1}^{\omega}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} = \prod_{i=1}^{\omega}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}$$

$$\Leftrightarrow$$

$$\prod_{i=1}^{\gamma<\omega}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}\left(1+\frac{(s-\alpha_\omega)^2}{\beta_\omega^2}\right)^{d_\omega} = \prod_{i=1}^{\gamma<\omega}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}\left(1+\frac{(1-s-\alpha_\omega)^2}{\beta_\omega^2}\right)^{d_\omega}$$

$$\Leftrightarrow \Big(\text{by Lemma 5}\Big)$$

$$\begin{cases} \prod_{i=1}^{\gamma<\omega}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} = \prod_{i=1}^{\gamma<\omega}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} \\ \left(1+\frac{(s-\alpha_\omega)^2}{\beta_\omega^2}\right)^{d_\omega} = \left(1+\frac{(1-s-\alpha_\omega)^2}{\beta_\omega^2}\right)^{d_\omega} \\ \beta_1 < \beta_2 < \beta_3 < \cdots < \beta_\omega \end{cases}$$

$$\Leftrightarrow \Big(\text{by } P(\gamma < \omega)\Big)$$

$$\begin{cases} \left(1+\frac{(s-\alpha_1)^2}{\beta_1^2}\right)^{d_1} = \left(1+\frac{(1-s-\alpha_1)^2}{\beta_1^2}\right)^{d_1} \\ \cdots \\ \left(1+\frac{(s-\alpha_\omega)^2}{\beta_\omega^2}\right)^{d_\omega} = \left(1+\frac{(1-s-\alpha_\omega)^2}{\beta_\omega^2}\right)^{d_\omega} \\ \beta_1 < \beta_2 < \beta_3 < \cdots < \beta_\omega \end{cases}$$

$$\Leftrightarrow \Big(\text{by Eq.(14)}\Big)$$

$$\begin{cases} \alpha_i = \frac{1}{2}, i = 1, 2, 3, \cdots, \omega \\ \beta_1 < \beta_2 < \beta_3 < \cdots < \beta_\omega \end{cases} \tag{19}$$

Thus the **Limit Case** is true, i.e., $P(\gamma < \omega) \Rightarrow P(\omega)$.

Hence we conclude by Transfinite Induction that $P(\infty)$ holds, i.e.,

$$\begin{aligned}
&\prod_{i=1}^{\infty}\Big(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\Big)^{d_i}=\prod_{i=1}^{\infty}\Big(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\Big)^{d_i}\\
&\Leftrightarrow\\
&\begin{cases}
\big(1+\frac{(s-\alpha_1)^2}{\beta_1^2}\big)^{d_1}=\big(1+\frac{(1-s-\alpha_1)^2}{\beta_1^2}\big)^{d_1}\\
\cdots\\
\big(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\big)^{d_i}=\big(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\big)^{d_i}\\
\cdots\\
\beta_1<\beta_2<\beta_3<\cdots
\end{cases}\\
&\Leftrightarrow\\
&\begin{cases}
\alpha_i=\frac{1}{2}, i=1\cdots\infty\\
\beta_1<\beta_2<\beta_3<\cdots
\end{cases}
\end{aligned}\tag{20}$$

i.e.,

$$f(s)=f(1-s)\Leftrightarrow\begin{cases}
\alpha_i=\frac{1}{2}, i=1,2,3,\cdots,\infty\\
\beta_1<\beta_2<\beta_3<\cdots
\end{cases}\tag{21}$$

That completes the proof of Lemma 3.

Lemma 4: Given

$$\Big(1+\frac{(s-\alpha_1)^2}{\beta_1^2}\Big)\Big(1+\frac{(s-\alpha_2)^2}{\beta_2^2}\Big)=\Big(1+\frac{(1-s-\alpha_1)^2}{\beta_1^2}\Big)\Big(1+\frac{(1-s-\alpha_2)^2}{\beta_2^2}\Big)\tag{22}$$

where s is a complex variable, $0<\alpha_i<1$ and $\beta_i\neq 0$ are real numbers, $|\beta_1|\leq|\beta_2|$.
Then we have

$$\begin{aligned}
&\Big(1+\frac{(s-\alpha_1)^2}{\beta_1^2}\Big)\Big(1+\frac{(s-\alpha_2)^2}{\beta_2^2}\Big)=\Big(1+\frac{(1-s-\alpha_1)^2}{\beta_1^2}\Big)\Big(1+\frac{(1-s-\alpha_2)^2}{\beta_2^2}\Big)\\
&\Leftrightarrow\\
&\begin{cases}
\alpha_1=\alpha_2=\frac{1}{2}, & \beta_1\neq\beta_2\\
\alpha_1+\alpha_2=1, & \beta_1=\beta_2
\end{cases}\\
&\Leftrightarrow\\
&\begin{cases}
\big(1+\frac{(s-\alpha_1)^2}{\beta_1^2}\big)=\big(1+\frac{(1-s-\alpha_1)^2}{\beta_1^2}\big),\big(1+\frac{(s-\alpha_2)^2}{\beta_2^2}\big)=\big(1+\frac{(1-s-\alpha_2)^2}{\beta_2^2}\big), & \beta_1\neq\beta_2\\
\big(1+\frac{(s-\alpha_1)^2}{\beta_1^2}\big)=\big(1+\frac{(1-s-\alpha_2)^2}{\beta_2^2}\big),\big(1+\frac{(s-\alpha_2)^2}{\beta_2^2}\big)=\big(1+\frac{(1-s-\alpha_1)^2}{\beta_1^2}\big), & \beta_1=\beta_2
\end{cases}
\end{aligned}\tag{23}$$

Proof: Expanding both sides of Eq.(22), and comparing the coefficients of like terms, we obtain (details are omitted to save space)

$$\begin{aligned}&\Big(1+\frac{(s-\alpha_1)^2}{\beta_1^2}\Big)\Big(1+\frac{(s-\alpha_2)^2}{\beta_2^2}\Big)=\Big(1+\frac{(1-s-\alpha_1)^2}{\beta_1^2}\Big)\Big(1+\frac{(1-s-\alpha_2)^2}{\beta_2^2}\Big)\\&\Rightarrow\\&\begin{cases}\alpha_1=\alpha_2=\frac{1}{2}, & \beta_1\neq\beta_2\\ \alpha_1+\alpha_2=1, & \beta_1=\beta_2\end{cases}\end{aligned}\tag{24}$$

The inverse inference of Eq.(24) is also an obvious fact. i.e.,

$$\begin{aligned}&\begin{cases}\alpha_1=\alpha_2=\frac{1}{2}, & \beta_1\neq\beta_2\\ \alpha_1+\alpha_2=1, & \beta_1=\beta_2\end{cases}\\&\Rightarrow\\&\Big(1+\frac{(s-\alpha_1)^2}{\beta_1^2}\Big)\Big(1+\frac{(s-\alpha_2)^2}{\beta_2^2}\Big)=\Big(1+\frac{(1-s-\alpha_1)^2}{\beta_1^2}\Big)\Big(1+\frac{(1-s-\alpha_2)^2}{\beta_2^2}\Big)\end{aligned}$$

Then we have

$$\begin{aligned}&\Big(1+\frac{(s-\alpha_1)^2}{\beta_1^2}\Big)\Big(1+\frac{(s-\alpha_2)^2}{\beta_2^2}\Big)=\Big(1+\frac{(1-s-\alpha_1)^2}{\beta_1^2}\Big)\Big(1+\frac{(1-s-\alpha_2)^2}{\beta_2^2}\Big)\\&\Leftrightarrow\\&\begin{cases}\alpha_1=\alpha_2=\frac{1}{2}, & \beta_1\neq\beta_2\\ \alpha_1+\alpha_2=1, & \beta_1=\beta_2\end{cases}\end{aligned}\tag{25}$$

Further, according to Eq.(14), i.e.,

$$(1+\frac{(s-\alpha)^2}{\beta^2})^d=(1+\frac{(1-s-\alpha)^2}{\beta^2})^d\Leftrightarrow\alpha=\frac{1}{2}$$

and the following similar facts

$$(1+\frac{(s-\alpha_1)^2}{\beta_1^2})=(1+\frac{(1-s-\alpha_2)^2}{\beta_2^2})\Leftrightarrow\alpha_1+\alpha_2=1,\beta_1=\beta_2$$

$$(1+\frac{(s-\alpha_2)^2}{\beta_2^2})=(1+\frac{(1-s-\alpha_1)^2}{\beta_1^2})\Leftrightarrow\alpha_1+\alpha_2=1,\beta_1=\beta_2$$

we have

$$\Big(1+\frac{(s-\alpha_1)^2}{\beta_1^2}\Big)\Big(1+\frac{(s-\alpha_2)^2}{\beta_2^2}\Big)=\Big(1+\frac{(1-s-\alpha_1)^2}{\beta_1^2}\Big)\Big(1+\frac{(1-s-\alpha_2)^2}{\beta_2^2}\Big)$$
$$\Leftrightarrow$$
$$\begin{cases}\alpha_1=\alpha_2=\frac{1}{2}, & \beta_1\neq\beta_2\\ \alpha_1+\alpha_2=1, & \beta_1=\beta_2\end{cases}$$
$$\Leftrightarrow$$
$$\begin{cases}(1+\frac{(s-\alpha_1)^2}{\beta_1^2})=(1+\frac{(1-s-\alpha_1)^2}{\beta_1^2}),(1+\frac{(s-\alpha_2)^2}{\beta_2^2})=(1+\frac{(1-s-\alpha_2)^2}{\beta_2^2}), & \beta_1\neq\beta_2\\ (1+\frac{(s-\alpha_1)^2}{\beta_1^2})=(1+\frac{(1-s-\alpha_2)^2}{\beta_2^2}),(1+\frac{(s-\alpha_2)^2}{\beta_2^2})=(1+\frac{(1-s-\alpha_1)^2}{\beta_1^2}), & \beta_1=\beta_2\end{cases} \tag{26}$$

That completes the proof of Lemma 4.

Lemma 5: Given

$$\Big(1+\frac{(s-\alpha_1)^2}{\beta_1^2}\Big)^{d_1}\Big(1+\frac{(s-\alpha_2)^2}{\beta_2^2}\Big)^{d_2}=\Big(1+\frac{(1-s-\alpha_1)^2}{\beta_1^2}\Big)^{d_1}\Big(1+\frac{(1-s-\alpha_2)^2}{\beta_2^2}\Big)^{d_2} \tag{27}$$

where s is a complex variable, $0<\alpha_1<1, 0<\alpha_2<1$ and $\beta_1\neq 0, \beta_2\neq 0$ are real numbers, $d_1\geq 1, d_2\geq 1$ are natural numbers, denoting the real multiplicities of $\rho_1=\alpha_1+j\beta_1$ and $\rho_2=\alpha_2+j\beta_2$, respectively, $|\beta_1|\leq|\beta_2|$.

Then we have

$$\Big(1+\frac{(s-\alpha_1)^2}{\beta_1^2}\Big)^{d_1}\Big(1+\frac{(s-\alpha_2)^2}{\beta_2^2}\Big)^{d_2}=\Big(1+\frac{(1-s-\alpha_1)^2}{\beta_1^2}\Big)^{d_1}\Big(1+\frac{(1-s-\alpha_2)^2}{\beta_2^2}\Big)^{d_2}$$
$$\Leftrightarrow$$
$$\begin{cases}\alpha_1=\alpha_2=\frac{1}{2}\\ \beta_1<\beta_2\end{cases}$$
$$\Leftrightarrow$$
$$\begin{cases}(1+\frac{(s-\alpha_1)^2}{\beta_1^2})^{d_1}=(1+\frac{(1-s-\alpha_1)^2}{\beta_1^2})^{d_1}\\ (1+\frac{(s-\alpha_2)^2}{\beta_2^2})^{d_2}=(1+\frac{(1-s-\alpha_2)^2}{\beta_2^2})^{d_2}\\ \beta_1<\beta_2\end{cases} \tag{28}$$

Proof: Based on Lemma 4, and considering additional possibilities that $\Big(1+\frac{(s-\alpha_1)^2}{\beta_1^2}\Big)^{d_1} \mid \Big(1+\frac{(1-s-\alpha_2)^2}{\beta_2^2}\Big)^{d_2}, \Big(1+\frac{(1-s-\alpha_1)^2}{\beta_1^2}\Big)^{d_1} \mid \Big(1+\frac{(s-\alpha_2)^2}{\beta_2^2}\Big)^{d_2}$(where

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"|" is the divisible sign), or vice versa, we have

$$\left(1+\frac{(s-\alpha_1)^2}{\beta_1^2}\right)^{d_1}\left(1+\frac{(s-\alpha_2)^2}{\beta_2^2}\right)^{d_2}=\left(1+\frac{(1-s-\alpha_1)^2}{\beta_1^2}\right)^{d_1}\left(1+\frac{(1-s-\alpha_2)^2}{\beta_2^2}\right)^{d_2}$$

$\Leftrightarrow$

$$\begin{cases}(1+\frac{(s-\alpha_1)^2}{\beta_1^2})^{d_1}=(1+\frac{(1-s-\alpha_1)^2}{\beta_1^2})^{d_1},(1+\frac{(s-\alpha_2)^2}{\beta_2^2})^{d_2}=(1+\frac{(1-s-\alpha_2)^2}{\beta_2^2})^{d_2}, & \beta_1<\beta_2\\ \begin{cases}(1+\frac{(s-\alpha_1)^2}{\beta_1^2})^{d_1}\mid(1+\frac{(1-s-\alpha_2)^2}{\beta_2^2})^{d_2},(1+\frac{(1-s-\alpha_1)^2}{\beta_1^2})^{d_1}\mid(1+\frac{(s-\alpha_2)^2}{\beta_2^2})^{d_2}\\ (1+\frac{(s-\alpha_2)^2}{\beta_2^2})^{d_2-d_1}=(1+\frac{(1-s-\alpha_2)^2}{\beta_2^2})^{d_2-d_1}\end{cases}, & \beta_1=\beta_2,d_1<d_2\\ \begin{cases}(1+\frac{(1-s-\alpha_2)^2}{\beta_2^2})^{d_2}\mid(1+\frac{(s-\alpha_1)^2}{\beta_1^2})^{d_1},(1+\frac{(s-\alpha_2)^2}{\beta_2^2})^{d_2}\mid(1+\frac{(1-s-\alpha_1)^2}{\beta_1^2})^{d_1}\\ (1+\frac{(s-\alpha_1)^2}{\beta_1^2})^{d_1-d_2}=(1+\frac{(1-s-\alpha_1)^2}{\beta_1^2})^{d_1-d_2}\end{cases}, & \beta_1=\beta_2,d_1>d_2\\ (1+\frac{(s-\alpha_1)^2}{\beta_1^2})^{d_1}=(1+\frac{(1-s-\alpha_2)^2}{\beta_2^2})^{d_2},(1+\frac{(1-s-\alpha_1)^2}{\beta_1^2})^{d_1}=(1+\frac{(s-\alpha_2)^2}{\beta_2^2})^{d_2}, & \beta_1=\beta_2,d_1=d_2\end{cases}$$

$\Leftrightarrow$

$$\begin{cases}\alpha_1=\alpha_2=\frac{1}{2}, & \beta_1<\beta_2\\ \alpha_1=\alpha_2=\frac{1}{2}, & \beta_1=\beta_2,d_1<d_2 & \star\\ \alpha_1=\alpha_2=\frac{1}{2}, & \beta_1=\beta_2,d_1>d_2 & \star\\ \alpha_1+\alpha_2=1, & \beta_1=\beta_2,d_1=d_2 & \star\end{cases}$$

$\Leftrightarrow$

(To exclude the duplication situation $\star$ between the first quadruplet of zeros represented by $\rho_1=\alpha_1+j\beta_1$ and the second quadruplet of zeros represented by $\rho_2=\alpha_2+j\beta_2$, which contradicts the given condition that the real multiplicities of ρ_1 and ρ_2 are d_1 and d_2, respectively.)

$$\begin{cases}\alpha_1=\alpha_2=\frac{1}{2}\\ \beta_1<\beta_2\end{cases}$$

$\Leftrightarrow \left(\text{by Eq.(14)}\right)$

$$\begin{cases}(1+\frac{(s-\alpha_1)^2}{\beta_1^2})^{d_1}=(1+\frac{(1-s-\alpha_1)^2}{\beta_1^2})^{d_1}\\ (1+\frac{(s-\alpha_2)^2}{\beta_2^2})^{d_2}=(1+\frac{(1-s-\alpha_2)^2}{\beta_2^2})^{d_2}\\ \beta_1<\beta_2\end{cases} \tag{29}$$

That completes the proof of Lemma 5.

3 A Proof of the RH

This section is planned to present a proof of the Riemann Hypothesis. We first prove that Statement 2 of the RH is true, and then by Lemma 2, Statement 1 of the RH is also true.

Proof of the RH: The details are delivered in three steps as follows.

Step 1: It is well-known that all the zeros of $\xi(s)$ always come in complex conjugate pairs. Then by pairing $\rho_i = \alpha_i + j\beta_i$ and $\bar{\rho}_i = \alpha_i - j\beta_i$ in the Hadamard product as shown in Eq.(10), we have

$$\begin{aligned}\xi(s) &= \xi(0)\prod_{\rho}(1-\frac{s}{\rho}) \\ &= \xi(0)\prod_{i=1}^{\infty}(1-\frac{s}{\rho_i})(1-\frac{s}{\bar{\rho}_i}) \\ &= \xi(0)\prod_{i=1}^{\infty}(1-\frac{s}{\alpha_i+j\beta_i})(1-\frac{s}{\alpha_i-j\beta_i}) \\ &= \xi(0)\prod_{i=1}^{\infty}(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(s-\alpha_i)^2}{\alpha_i^2+\beta_i^2})\end{aligned} \tag{30}$$

where $\xi(0) = \frac{1}{2}, 0 < \alpha_i < 1, \beta_i \neq 0$.
The absolute convergence of the infinite product in Eq.(30) in the form

$$\xi(s) = \xi(0)\prod_{i=1}^{\infty}(1-\frac{s}{\rho_i})(1-\frac{s}{\bar{\rho}_i}) = \xi(0)\prod_{i=1}^{\infty}\left(1-\frac{s(2\alpha_i - s)}{|\rho_i|^2}\right) \tag{31}$$

depends on the convergence of infinite series $\sum_{i=1}^{\infty}\frac{1}{|\rho_i|^2}$, or equivalently, $\sum_{\rho}\frac{1}{|\rho|^2}$, which is an obvious fact according to Theorem 2 in Section 2, Chapter IV of Ref.[22], as shown in the following.

Theorem 2.[22] The function $\xi(s)$ is an entire function of order one that has infinitely many zeros ρ_n such that $0 \leq \mathbf{Re}\ \rho_n \leq 1$. The series $\sum|\rho_n|^{-1}$ diverges, but the series $\sum|\rho_n|^{-1-\varepsilon}$ converges for any $\varepsilon > 0$. The zeros of $\xi(s)$ are the nontrivial zeros of $\zeta(s)$.

Remark: In the Theorem 2 of Ref.[22], $\mathbf{Re}(\cdot)$ is identical to $\Re(\cdot)$ in this paper, both $\mathbf{Re}(\cdot)$ and $\Re(\cdot)$ mean the real part of any complex number.

Further, considering the absolute convergence of

$$\xi(s) = \xi(0)\prod_{i=1}^{\infty}\left(1-\frac{s(2\alpha_i - s)}{|\rho_i|^2}\right) = \xi(0)\prod_{i=1}^{\infty}(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}) \tag{32}$$

we have the following new expression of $\xi(s)$ by putting all the ρ_i related multiple factors (zeros) together in the above Eq.(32)

$$\xi(s) = \xi(0)\prod_{i=1}^{\infty}(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(s-\alpha_i)^2}{\alpha_i^2+\beta_i^2})^{d_i} \tag{33}$$

where $d_i \geq 1$ are the real multiplicities of ρ_i, i are natural numbers from 1 to infinity.

Step 2: Replacing s with $1-s$ in Eq.(33), we obtain the infinite product expression of $\xi(1-s)$, i.e.,

$$\xi(1-s) = \xi(0)\prod_{i=1}^{\infty}(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2} + \frac{(1-s-\alpha_i)^2}{\alpha_i^2+\beta_i^2})^{d_i} \tag{34}$$

Remark: According to the new expressions of $\xi(s)$ and $\xi(1-s)$, i.e., Eq.(33) and Eq.(34), we may conclude that all ρ_i and $1-\rho_i$ related multiple zeros, i.e., $(\alpha_i \pm j\beta_i)^{d_i}, (1-\alpha_i \pm j\beta_i)^{d_i}$ are included in the i^{th} group of factors, $\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}$ and $\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}$, respectively, or in another word, before or after the i^{th} group of factors of $\xi(s)$ and $\xi(1-s)$, there are no ρ_i and $1-\rho_i$ related multiple zeros.

Actually, with such arrangement of ρ_i and $1-\rho_i$ related multiple factors of $\xi(s)$ and $\xi(1-s)$, we "assigned" a reason for excluding, in the proof of Lemma 5 and Lemma 3, the "abnormal" situation, i.e., the successor factor and its predecessor factor represent the same quadruplet of zeros.

Step 3: According to the functional equation $\xi(s) = \xi(1-s)$, and considering Eq.(33) and Eq.(34), we have

$$\xi(0)\prod_{i=1}^{\infty}(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2} + \frac{(s-\alpha_i)^2}{\alpha_i^2+\beta_i^2})^{d_i} = \xi(0)\prod_{i=1}^{\infty}(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2} + \frac{(1-s-\alpha_i)^2}{\alpha_i^2+\beta_i^2})^{d_i} \tag{35}$$

which is equivalent to

$$\prod_{i=1}^{\infty}(1+\frac{(s-\alpha_i)^2}{\beta_i^2})^{d_i} = \prod_{i=1}^{\infty}(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2})^{d_i} \tag{36}$$

And that β_i can be certainly arranged in order of increasing $|\beta_i|$, i.e., $|\beta_1| \leq |\beta_2| \leq |\beta_3| \leq \cdots$.
Then, according to Lemma 3, Eq.(36) is equivalent to

$$\begin{cases} \alpha_i = \frac{1}{2}, i = 1, 2, 3, \cdots, \infty \\ \beta_1 < \beta_2 < \beta_3 < \cdots \end{cases}$$

Thus, we conclude that all the zeros of the completed zeta function $\xi(s)$ have real part equal to $\frac{1}{2}$, i.e., Statement 2 of the RH is true. According to Lemma 2, Statement 1 of the RH is also true, i.e., all the non-trivial zeros of the Riemann zeta function $\zeta(s)$ have real part equal to $\frac{1}{2}$.

That completes the proof of the RH.

Remark: By Lemma 1, there are 2 pairs of complex zeros of $\zeta(s)$ simultaneously, i.e., $\rho=\alpha+j\beta, \bar{\rho}=\alpha-j\beta, 1-\rho=1-\alpha-j\beta, 1-\bar{\rho}=1-\alpha+j\beta$ are all the non-trivial zeroes of $\zeta(s)$. With the proof of the RH, i.e., $\alpha=\frac{1}{2}$, these 2 pairs of zeros are actually only one pair, because $\rho=1-\bar{\rho}=\frac{1}{2}+j\beta, \bar{\rho}=1-\rho=\frac{1}{2}-j\beta$. Thus Lemma 1 could be modified more precisely as follows.

Lemma 1*: Non-trivial zeroes of $\zeta(s)$, noted as $\rho=\alpha+j\beta$, have the following properties
1) The number of non-trivial zeroes is infinity;
2) $\beta \neq 0$;
3) $0<\alpha<1$;
4) $\rho=1-\bar{\rho}, \bar{\rho}=1-\rho$ are all non-trivial zeroes.

4 Conclusion

The celebrated Riemann Hypothesis is proved to be true based on a new expression of the completed zeta function $\xi(s)$, i.e.,

$$\xi(s)=\xi(0)\prod_{i=1}^{\infty}\left(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}\right)^{d_i}$$

where $\xi(0)=\frac{1}{2}, \rho_i=\alpha_i+j\beta_i$ and $\bar{\rho}_i=\alpha_i-j\beta_i$ are the complex conjugate zeros of $\xi(s)$, $0<\alpha_i<1$ and $\beta_i \neq 0$ are real numbers, $d_i \geq 1$ are the real multiplicities of ρ_i, i are natural numbers from 1 to infinity, β_i are in order of increasing $|\beta_i|$, i.e., $|\beta_1| \leq |\beta_2| \leq |\beta_3| \leq \cdots$.

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