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A Proof Of The Riemann Hypothesis Based On A New Expression Of The Completed Zeta Function

Weicun Zhang

Abstract Based on the Hadamard product $\xi(s) = \xi(0)\prod_\rho(1 - \frac{s}{\rho})$, a new expression of $\xi(s)$ is obtained by paring ρ and $\bar{\rho}$

$$\xi(s) = \xi(0)\prod_{i=1}^{\infty}\Big(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2} + \frac{(s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}\Big)^{d_i}$$

where $\xi(0) = \frac{1}{2}$, $\rho_i = \alpha_i + j\beta_i$ and $\bar{\rho}_i = \alpha_i - j\beta_i$ are complex conjugate zeros of $\xi(s)$, $0 < \alpha_i < 1$ and $\beta_i \neq 0$ are real numbers, $d_i \geq 1$ are the multiplicities of ρ_i, β_i are in order of increasing $|\beta_i|$, i.e., $|\beta_1| \leq |\beta_2| \leq |\beta_3| \leq \cdots$. Then we have, by the functional equation $\xi(s) = \xi(1-s)$, that

$$\xi(0)\prod_{i=1}^{\infty}\Big(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2} + \frac{(s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}\Big)^{d_i} = \xi(0)\prod_{i=1}^{\infty}\Big(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2} + \frac{(1-s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}\Big)^{d_i}$$

i.e.,

$$\prod_{i=1}^{\infty}\Big(1 + \frac{(s-\alpha_i)^2}{\beta_i^2}\Big)^{d_i} = \prod_{i=1}^{\infty}\Big(1 + \frac{(1-s-\alpha_i)^2}{\beta_i^2}\Big)^{d_i}$$

which, by Lemma 3, is equivalent to

$$\alpha_i = \frac{1}{2}, i = 1, 2, 3, \cdots, \infty$$

Thus, we conclude that the Riemann Hypothesis is true.

Keywords Riemann Hypothesis (RH) · Proof · Completed zeta function

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Weicun Zhang
University of Science and Technology Beijing, Beijing 100083, China
ORCID: 0000-0003-0047-0558
E-mail: weicunzhang@ustb.edu.cn

1 Introduction

It has been almost 163 years since the Riemann Hypothesis (RH) was proposed in 1859 [1]. Many efforts and achievements have been made towards proving this celebrated hypothesis, but it is still an open problem [2−3].

The Riemann zeta function is the function of complex variable s, defined in the half-plane $\Re(s) > 1$ by the absolutely convergent series [2]

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \Re(s) > 1 \tag{1}$$

The connection between the Riemann zeta function and prime numbers can be established through the well-known Euler product, i.e.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p} (1 - p^{-s})^{-1}, \Re(s) > 1 \tag{2}$$

the product being over the prime numbers p.

Riemann showed how to extend zeta function to the whole complex plane $\mathbb{C}$ by analytic continuation

$$\zeta(s) = \frac{\pi^{s/2}}{\Gamma(s/2)} \{ \frac{1}{s(s-1)} + \int_1^{\infty} (x^{\frac{s}{2}-1} + x^{-\frac{s}{2}-\frac{1}{2}}) \cdot (\frac{\theta(x)-1}{2}) dx \} \tag{3}$$

where $\theta(x) = \sum_{-\infty}^{\infty} e^{-n^2 \pi x}$ being the Jaccobi theta function, Γ being the Gamma function in the following Weierstrass expression (Meanwhile, there are also Gauss expression, Euler expression, and integral expression of the Gamma function.)

$$\frac{1}{\Gamma(s)} = s \cdot e^{\gamma s} \prod_{n=1}^{\infty} (1 + \frac{s}{n}) e^{-s/n} \tag{4}$$

where γ is the Euler-Mascheroni constant.

As shown by Riemann, $\zeta(s)$ extends to $\mathbb{C}$ as a meromorphic function with only a simple pole at $s = 1$, with residue 1, and satisfies the following functional equation

$$\pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma(\frac{1-s}{2}) \zeta(1-s) \tag{5}$$

The Riemann zeta function $\zeta(s)$ has zeros at the negative even integers: -2, -4, -6, -8, $\cdots$ and one refers to them as the **trivial zeros**. The other zeros of $\zeta(s)$ are the complex numbers, i.e., **non-trivial zeros** [2].

In 1896, Hadamard [4] and Poussin [5] independently proved that no zeros could lie on the line $\Re(s) = 1$. Together with the functional equation and the fact that there are no zeros with real part greater than 1, this showed that all non-trivial zeros must lie in the interior of the **critical strip** $0 < \Re(s) < 1$.

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This was a key step in their first proofs of the famous **Prime Number Theorem**.

Later on, Hardy (1914) [6], Hardy and Littlewood (1921) [7] showed that there are infinitely many zeros on the **critical line** $\Re(s) = \frac{1}{2}$, which was an astonishing result at that time.

As a summary, we have the following results on the properties of the non-trivial zeros of $\zeta(s)$ [4−9].

Lemma 1: Non-trivial zeroes of $\zeta(s)$, noted as $\rho = \alpha + j\beta$, have the following properties

1) The number of non-trivial zeroes is infinity;
2) $\beta \neq 0$;
3) $0 < \alpha < 1$;
4) $\rho, \bar{\rho}, 1-\bar{\rho}, 1-\rho$ are all non-trivial zeroes.

As further study, a completed zeta function $\xi(s)$ is defined as

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) \tag{6}$$

It is well-known that $\xi(s)$ is an entire function of order 1. This implies $\xi(s)$ is analytic, and can be expressed as infinite polynomial, in the whole complex plane $\mathbb{C}$.

In addition, replacing s with $1-s$ in Eq.(6), and combining Eq.(5), we have the following functional equation

$$\xi(s) = \xi(1-s) \tag{7}$$

Considering the definition of $\xi(s)$, and recalling Eq.(4), the trivial zeros of $\zeta(s)$ are canceled by the poles of $\Gamma(\frac{s}{2})$. The zero of $s-1$ and the pole of $\zeta(s)$ cancel; the zero $s=0$ and the pole of $\Gamma(\frac{s}{2})$ cancel [9−10]. Thus, all the zeros of $\xi(s)$ are exactly the nontrivial zeros of $\zeta(s)$. Then we have the following Lemma 2.

Lemma 2: Zeros of $\xi(s)$ coincide with non-trivial zeros of $\zeta(s)$.

According to Lemma 2, the following two statements for the RH are equivalent.

Statement 1 of the RH: All the non-trivial zeros of $\zeta(s)$ have real part equal to $\frac{1}{2}$.

Statement 2 of the RH: All the zeros of $\xi(s)$ have real part equal to $\frac{1}{2}$.

To prove the RH, the natural thinking is to estimate the number of zeros of $\zeta(s)$ in some certain closed areas according to the Argument Principle. Along this train of thought, there are some famous research works. Let $N(T)$ denote the number of zeros of $\zeta(s)$ inside the rectangle: $0 < \alpha < 1, 0 < \beta \leq T$, and let $N_0(T)$ denote the number of zeros of $\zeta(s)$ on the line $\alpha = \frac{1}{2}, 0 < \beta \leq T$. Selberg proved that there exist positive constants c and T_0, such that $N_0(T) > cN(T), (T > T_0)$ [11], later on, Levinson proved that $c \geq \frac{1}{3}$ [12], Lou and Yao proved that $c \geq 0.3484$ [13], Conrey proved that $c \geq \frac{2}{5}$ [14], Bui, Conrey and Young proved that $c \geq 0.41$ [15], Feng proved that $c \geq 0.4128$ [16].

On the other hand, many zeros have been calculated by hand or by computer programs. Among others, Riemann found the first three non-trivial zeros [17]. Gram found the first 15 zeros based on Euler-Maclaurin summation [18]. Titchmarsh calculated the 138^{th} to 195^{th} zeros using the Riemann-Siegel formula [19−20]. Here are the first three (pairs of) zeros: $\frac{1}{2} \pm j14.1347251$; $\frac{1}{2} \pm j21.0220396$; $\frac{1}{2} \pm j25.0108575$.

The idea of this paper is originated from Euler's work on proving that

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{6} \tag{8}$$

This interesting and famous result is deduced from two types of infinite expansions, i.e., infinite polynomial and infinite product as follows

$$\begin{aligned} \frac{\sin x}{x} &= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots \\ &= (1 - \frac{x^2}{\pi^2})(1 - \frac{x^2}{4\pi^2})(1 - \frac{x^2}{9\pi^2}) \cdots \end{aligned} \tag{9}$$

This paper is also motivated by the fact that $\xi(s)$ can be expressed by the following Hadamard product, which was first proposed by Riemann. However, it was Hadamard [21] who showed the validity of this infinite product expansion.

$$\xi(s) = \xi(0) \prod_{\rho} (1 - \frac{s}{\rho}) \tag{10}$$

where $\xi(0) = \frac{1}{2}$, ρ runs over the non-trivial zeros of the Riemann zeta function $\zeta(s)$, or in another word, ρ runs over the zeros of the completed zeta function $\xi(s)$. To ensure the absolute convergence of the infinite product expansion, ρ and $1 - \rho$ are paired. Later in Section 2, we will show that ρ and $\bar{\rho}$ can also be paired to ensure the absolute convergence of the infinite product expansion.

2 A Proof of the RH

This section is planned to present a proof of the Riemann Hypothesis. We first prove that Statement 2 of the RH is true, and then by Lemma 2 we know that Statement 1 of the RH is also true. For this purpose, we need the following

Lemma 3.

Lemma 3: Given two infinite products

$$f(s)=\prod_{i=1}^{\infty}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} \tag{11}$$

and

$$f(1-s)=\prod_{i=1}^{\infty}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} \tag{12}$$

where s is a complex variable, $\rho_i=\alpha_i+j\beta_i,\bar{\rho}_i=\alpha_i-j\beta_i$ are complex conjugate zeros of $\xi(s)$, $0<\alpha_i<1$ and $\beta_i\neq 0$ are real numbers, $d_i\geq 1$ are the multiplicities of ρ_i, i are natural numbers from 1 to infinity, β_i are in order of increasing $|\beta_i|$, i.e., $|\beta_1|\leq|\beta_2|\leq|\beta_3|\leq\cdots$.

Then we have that

$$f(s)=f(1-s)\Leftrightarrow\alpha_i=\frac{1}{2},i=1,2,3,\cdots,\infty \tag{13}$$

where " $\Leftrightarrow$ " is the equivalent sign.

Proof: First of all, we have the following fact:

$$\begin{aligned}
&(1+\frac{(s-\alpha)^2}{\beta^2})^d=(1+\frac{(1-s-\alpha)^2}{\beta^2})^d\\
&\Leftrightarrow\\
&(s-\alpha)^2=(1-s-\alpha)^2\\
&\Leftrightarrow\alpha=\frac{1}{2}
\end{aligned} \tag{14}$$

where $d\geq 1$ is a natural number, $\alpha\neq 0$ and $\beta\neq 0$ are real numbers.

Next, the proof will be conducted in two steps:
Step 1 to prove that $f(s)=f(1-s)\Rightarrow\alpha_i=\frac{1}{2},i=1,2,3,\cdots,\infty$;
Step 2 to prove that $\alpha_i=\frac{1}{2}\Rightarrow f(s)=f(1-s),i=1,2,3,\cdots,\infty$.

Step 1: This part of proof is based on Transfinite Induction.
Let $P(n)$ be:

$$\begin{aligned}
&\prod_{i=1}^{n}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}=\prod_{i=1}^{n}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}\\
&\Rightarrow\\
&\alpha_i=\frac{1}{2},i=1,2,3,\cdots,n
\end{aligned} \tag{15}$$

According to Eq.(14), $P(1)$ is an obvious fact as the **Base Case**, i.e.,

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$$\begin{aligned}&\prod_{i=1}^{1}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_1}=\prod_{i=1}^{1}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_1}\\&\Rightarrow\\&\alpha_1=\frac{1}{2}\end{aligned}\tag{16}$$

As the **Successor Case**, we prove that $P(n) \Rightarrow P(n+1)$ by contradiction.

Assume, for the sake of contradiction, that $P(n)$ holds, but $P(n+1)$ does not hold, then by $\prod_{i=1}^{n+1}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}=\prod_{i=1}^{n+1}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}$, there exist at least two pairs of zeros off the critical line (otherwise the above equality can not hold), in which there must exist one pair of zeros with subscript not greater than n, i.e., $\alpha_l \neq \frac{1}{2}, l \leq n$, which contradicts that $P(n)$ holds, i.e.

$$\prod_{i=1}^{n}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}=\prod_{i=1}^{n}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}\not\Rightarrow\alpha_i=\frac{1}{2}, i=1,\cdots,n$$

Thus, the assumption that $P(n)$ holds, but $P(n+1)$ does not hold, is false. Then the **Successor Case** is true, i.e., $P(n) \Rightarrow P(n+1)$.

Next, we prove that $P(\infty)$ holds by considering well-ordered ordinal set A indexing the family of statements $P(\gamma : \gamma \in A)$, $A = \mathbb{N}\bigcup\{\omega\}$ with the ordering that $n < \omega$ for all natural numbers n, ω is the first limit ordinal.
It is well-known that $\omega = \bigcup\{\gamma : \gamma < \omega\}$.
To prove that $P(\infty)$ holds, it suffices to prove the **Limit Case**, i.e., $P(\gamma < \omega) \Rightarrow P(\omega)$.
In the following contents, we will prove the **Limit Case** by contradiction.

Assume, for the sake of contradiction, that $P(\gamma < \omega)$ holds, but $P(\omega)$ does not hold, then by $\prod_{i=1}^{\omega}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}=\prod_{i=1}^{\omega}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}$, there exist at least two pairs of zeros off the critical line (otherwise the above equality can not hold), in which there must exist one pair of zeros with limited subscript, i.e., $\alpha_l \neq \frac{1}{2}, l < \omega$, which contradicts that $P(\gamma < \omega)$ holds, i.e.

$$\prod_{i=1}^{\gamma<\omega}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}=\prod_{i=1}^{\gamma<\omega}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}\not\Rightarrow\alpha_i=\frac{1}{2}, i<\omega$$

Thus, the assumption that $P(\gamma < \omega)$ holds, but $P(\omega)$ does not hold, is false. Then the **Limit Case** is true, i.e., $P(\gamma < \omega) \Rightarrow P(\omega)$.
Hence we conclude by Transfinite Induction that $P(\infty)$ holds, i.e., $f(s) = f(1-s) \Rightarrow \alpha_i = \frac{1}{2}, i = 1, 2, 3, \cdots, \infty$.

Step 2: Based on Eq.(14), we have

$$
\begin{aligned}
\alpha_i = \frac{1}{2} &\Rightarrow (s-\alpha_i)^2 = (1-s-\alpha_i)^2 \\
&\Rightarrow \left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} = \left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} \\
&\Rightarrow f(s) = f(1-s)
\end{aligned}
$$

where $\beta_i \neq 0, i = 1, 2, 3, \cdots, \infty$. i.e.,

$$\alpha_i = \frac{1}{2} \Rightarrow f(s) = f(1-s), i = 1, 2, 3, \cdots, \infty$$

That completes the proof of Lemma 3.

Proof of the RH: The details are delivered in three steps as follows.

Step 1: By pairing $\rho_i = \alpha_i + j\beta_i$ and $\bar{\rho}_i = \alpha_i - j\beta_i$ in the Hadamard product in Eq.(10), we have that

$$
\begin{aligned}
\xi(s) &= \xi(0)\prod_{\rho}(1-\frac{s}{\rho}) \\
&= \xi(0)\prod_{i=1}^{\infty}(1-\frac{s}{\rho_i})(1-\frac{s}{\bar{\rho}_i}) \\
&= \xi(0)\prod_{i=1}^{\infty}(1-\frac{s}{\alpha_i+j\beta_i})(1-\frac{s}{\alpha_i-j\beta_i}) \\
&= \xi(0)\prod_{i=1}^{\infty}(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(s-\alpha_i)^2}{\alpha_i^2+\beta_i^2})
\end{aligned}
\tag{17}
$$

where $\xi(0) = \frac{1}{2}$.
The absolute convergence of the infinite product in Eq.(17) in the form

$$\xi(s) = \xi(0)\prod_{\rho}(1-\frac{s}{\rho})(1-\frac{s}{\bar{\rho}}) = \xi(0)\prod_{\rho}\left(1-\frac{s(2\alpha-s)}{|\rho|^2}\right), 0 < \alpha = \Re(\rho) < 1$$

depends on the convergence of infinite series $\sum_{\rho}\frac{1}{|\rho|^2}$, which is an obvious fact according to Theorem 2 in Section 2, Chapter IV of Ref.[22], i.e.,
Theorem 2.[22] The function $\xi(s)$ is an entire function of order one that has infinitely many zeros ρ_n such that $0 \leq \textbf{Re}\ \rho_n \leq 1$. The series $\sum|\rho_n|^{-1}$ diverges, but the series $\sum|\rho_n|^{-1-\varepsilon}$ converges for any $\varepsilon > 0$. The zeros of $\xi(s)$ are the nontrivial zeros of $\zeta(s)$.

Remarks: In Theorem 2 of Ref.[22], $\textbf{Re}(\cdot)$ is identical to $\Re(\cdot)$ in this paper, both $\textbf{Re}(\cdot)$ and $\Re(\cdot)$ mean the real part of any complex number.

Further, taking into account the possibility of multiple zeros in Eq.(17), we have

$$\xi(s) = \xi(0) \prod_{i=1}^{\infty} \left(\frac{\beta_i^2}{\alpha_i^2 + \beta_i^2} + \frac{(s-\alpha_i)^2}{\alpha_i^2 + \beta_i^2} \right)^{d_i} \tag{18}$$

where $d_i \geq 1$ are natural numbers, i are natural numbers from 1 to infinity.

Step 2: Replacing s with $1-s$ in Eq.(18), we obtain the infinite product expression of $\xi(1-s)$

$$\xi(1-s) = \xi(0) \prod_{i=1}^{\infty} \left(\frac{\beta_i^2}{\alpha_i^2 + \beta_i^2} + \frac{(1-s-\alpha_i)^2}{\alpha_i^2 + \beta_i^2} \right)^{d_i} \tag{19}$$

Step 3: We have by $\xi(s) = \xi(1-s)$ that

$$\xi(0) \prod_{i=1}^{\infty} \left(\frac{\beta_i^2}{\alpha_i^2 + \beta_i^2} + \frac{(s-\alpha_i)^2}{\alpha_i^2 + \beta_i^2} \right)^{d_i} = \xi(0) \prod_{i=1}^{\infty} \left(\frac{\beta_i^2}{\alpha_i^2 + \beta_i^2} + \frac{(1-s-\alpha_i)^2}{\alpha_i^2 + \beta_i^2} \right)^{d_i} \tag{20}$$

which is equivalent to

$$\prod_{i=1}^{\infty} \left(1 + \frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} = \prod_{i=1}^{\infty} \left(1 + \frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} \tag{21}$$

And that β_i can be certainly arranged in order of increasing $|\beta_i|$, i.e., $|\beta_1| \leq |\beta_2| \leq |\beta_3| \leq \cdots$.
Then according to Lemma 3, Eq.(21) is equivalent to $\alpha_i = \frac{1}{2}$, with i from 1 to infinity.
Thus, we conclude that all the zeros of the completed zeta function $\xi(s)$ have real part equal to $\frac{1}{2}$, i.e., Statement 2 of the RH is true. According to Lemma 2, Statement 1 of the RH is also true, i.e., All the non-trivial zeros of the Riemann zeta function $\zeta(s)$ have real part equal to $\frac{1}{2}$.

That completes the proof of the RH.

Remarks: By Lemma 1, there are 2 pairs of complex zeros of $\zeta(s)$ simultaneously, i.e., $\rho = \alpha + j\beta, \bar{\rho} = \alpha - j\beta, 1-\rho = 1-\alpha-j\beta, 1-\bar{\rho} = 1-\alpha+j\beta$ are all non-trivial zeroes of $\zeta(s)$. With the proof of the RH, i.e., $\alpha = \frac{1}{2}$, these 2 pairs of zeros are actually only one pair, because $\rho = 1-\bar{\rho} = \frac{1}{2}+j\beta, \bar{\rho} = 1-\rho = \frac{1}{2}-j\beta$. Thus Lemma 1 could be modified more precisely as follows.

Lemma 1*: Non-trivial zeroes of $\zeta(s)$, noted as $\rho = \alpha + j\beta$, have the following properties
1) The number of non-trivial zeroes is infinity;
2) $\beta \neq 0$;
3) $0 < \alpha < 1$;
4) $\rho = 1 - \bar{\rho}, \bar{\rho} = 1 - \rho$ are all non-trivial zeroes.

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3 Conclusion

The celebrated Riemann Hypothesis is proved to be true based on a new expression of the completed zeta function $\xi(s)$, i.e.,

$$\xi(s) = \xi(0)\prod_{i=1}^{\infty}\Big(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2} + \frac{(s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}\Big)^{d_i}$$

where $\xi(0) = \frac{1}{2}, \rho_i = \alpha_i + j\beta_i, \bar{\rho}_i = \alpha_i - j\beta_i$ are complex conjugate zeros of $\xi(s)$, $0 < \alpha_i < 1$ and $\beta_i \neq 0$ are real numbers, $d_i \geq 1$ are the multiplicities of ρ_i, i are natural numbers from 1 to infinity, β_i are in order of increasing $|\beta_i|$, i.e., $|\beta_1| \leq |\beta_2| \leq |\beta_3| \leq \cdots$.

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