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A Proof of Riemann Hypothesis Based on Maclaurin Expansion of the Completed Zeta Function

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Abstract: The basic idea is to expand the completed zeta function $\xi(s)$ in Maclaurin series. Thus, by $\xi(s) = \xi(1-s)$, we have the following polynomial equation

$$\begin{aligned}&\xi(0)+\xi'(0)s+\frac{\xi''(0)}{2!}s^2+\cdots+\frac{\xi^{(n)}(0)}{n!}s^n+\cdots\\=&\xi(0)+\xi'(0)(1-s)+\frac{\xi''(0)}{2!}(1-s)^2+\cdots+\frac{\xi^{(n)}(0)}{n!}(1-s)^n+\cdots\end{aligned}$$

which finally leads to $s = 1-s, s = \alpha \pm j\beta, \beta \neq 0$, then a proof of RH can be achieved.

1 The problem description

It has been 162 years since Riemann Hypothesis (RH) is proposed in 1859 [1]. Many efforts and achievements have been made towards proving the hypothesis, but it is still an open problem [2−3].

The Riemann zeta function is the function of the complex variable s, defined in the half-plane $\Re(s) > 1$ by the absolutely convergent series [2]

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

Riemann showed how to extend zeta function to the whole complex plane $\mathbb{C}$ by analytic continuation

$$\zeta(s) = \frac{\pi^{s/2}}{\Gamma(s/2)}\{\frac{1}{s(s-1)} + \int_1^{\infty}(x^{\frac{s}{2}-1} + x^{-\frac{s}{2}-\frac{1}{2}})\cdot(\frac{\theta(x)-1}{2})dx\} \tag{2}$$

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where $\theta(x) = \sum_{-\infty}^{\infty} e^{-n^2\pi x}$, Γ being the Gamma function in the following equivalent form

$$\frac{1}{\Gamma(s)} = s \cdot e^{\gamma s} \prod_{n=1}^{\infty} (1 + \frac{s}{n}) e^{-s/n} \tag{3}$$

where γ is Euler's constant.

The connection between the zeta function and prime numbers can be established through the well-known Euler product.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p} (1 - p^{-s})^{-1}, \Re(s) > 1 \tag{4}$$

the product being over the prime numbers p.

As shown by Riemann, $\zeta(s)$ extends to $\mathbb{C}$ as a meromorphic function with only a simple pole at $s = 1$, with residue 1, and satisfies the following functional equation

$$\pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma(\frac{1-s}{2}) \zeta(1-s) \tag{5}$$

The Riemann zeta function $\zeta(s)$ has zeros at the negative even integers $-2, -4, -6, -8, \cdots$ and one refers to them as the **trivial zeros**. The other zeros of $\zeta(s)$ are the complex numbers, i.e., **non-trivial zeros** [2].

About the non-trivial zeros of $\zeta(s)$, the following results are well established [4].

Lemma 1: Non-trivial zeroes of $\zeta(s)$, noted as $\rho = \alpha + j\beta$ have the following properties
1) The number of non-trivial zeroes is infinity;
2) $\beta \neq 0$;
3) $0 \leqslant \alpha \leqslant 1$;
4) $\rho, \bar{\rho}, 1 - \bar{\rho}, 1 - \rho$ are all non-trivial zeroes.

For further study, a completed zeta function $\xi(s)$ is defined as

$$\xi(s) = \frac{1}{2} s(1-s) \pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) \zeta(s) \tag{6}$$

It is well-known that $\xi(s)$ is an entire function of order 1.

Replacing s with $1 - s$ in Eq.(6), and considering Eq.(5), we have the following functional equation

$$\xi(s) = \xi(1-s) \tag{7}$$

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Considering the definition of $\xi(s)$, i.e., Eq. (6), and recalling Eq.(3), the trivial zeros of $\zeta(s)$ are canceled by the poles of $\Gamma(\frac{s}{2})$. The zero of $s-1$ and the pole of $\zeta(s)$ cancel; the zero $s=0$ and the pole of $\Gamma(\frac{s}{2})$ cancel [5−6]. Thus, all the zeros of $\xi(s)$ must be the nontrivial zeros of $\xi(s)$, and vice versa. That means the following result, i.e., Lemma 2.

Lemma 2: Zeros of $\xi(s)$ coincide with nontrivial zeros of $\zeta(s)$.

It is well-known that $\xi(s)$ can be expressed by the following infinite product, which was first proposed by Riemann. However, it was Hadamard who showed the validity of this infinite product expansion [7]

$$\xi(s) = \xi(0)\prod_{\rho}(1-\frac{s}{\rho}) \tag{8}$$

where ρ are precisely the roots of the Riemann zeta function $\zeta(s)$, the factor ρ and $1-\rho$ are paired.

This paper will use another infinite expansion of $\xi(s)$, i.e., Maclaurin series to open the door to the proof of RH.
The following two statements for Riemann Hypothesis are equivalent.

Statement 1 of Riemann Hypothesis (in terms of $\zeta(s)$):
The non-trivial zeros of $\zeta(s)$ have real part equal to $\frac{1}{2}$.

Statement 2 of Riemann Hypothesis (in terms of $\xi(s)$):
All the zeros of $\xi(s)$ have real part equal to $\frac{1}{2}$.

2 A proof of RH

This section is planned to give a proof of the Statement 2 of RH. For this purpose, we need the following result on polynomial equation.
Lemma 3: Given two infinite polynomials

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots \tag{9}$$

and

$$f(y) = a_0 + a_1y + a_2y^2 + \cdots + a_ny^n + \cdots \tag{10}$$

where x and y are variables(either real or complex variable), and $a_0, a_1, a_2, \cdots, a_n, \cdots$ are all arbitrary non-zero real constants.

Then we have

$$f(x) = f(y) \Leftrightarrow x = y \tag{11}$$

Proof: The proof is delivered by mathematical induction.

First, it is obvious that Lemma 3 is true for $n = 1$, i.e.,

$$a_0 + a_1 x = a_0 + a_1 y \Leftrightarrow x = y \tag{12}$$

Second, suppose Lemma 3 is true for $n = m$, then we only need to prove Lemma 3 is true for $n = m + 1$.

Thus we begin with Eq.(12) and the following Eq.(13)

$$a_0 + a_1 x + \cdots + a_m x^m = a_0 + a_1 y + \cdots + a_m y^m \Leftrightarrow x = y \tag{13}$$

It is obvious that if $x = y$ then

$$a_0 + a_1 x + \cdots + a_{m+1} x^{m+1} = a_0 + a_1 y + \cdots + a_{m+1} y^{m+1} \tag{14}$$

Now we only need to deduce $x = y$ from Eq.(14) which means

$$a_1 x + \cdots + a_{m+1} x^{m+1} = a_1 y + \cdots + a_{m+1} y^{m+1} \tag{15}$$

i.e.,

$$x(a_1 + a_2 x + \cdots + a_{m+1} x^m) = y(a_1 + a_2 y + \cdots + a_{m+1} y^m) \tag{16}$$

Next we proceed the proof by contradiction, suppose $x \neq y$ in Eq.(16), then by Eq.(13) we have

$$a_1 + a_2 x + \cdots + a_{m+1} x^m \neq a_1 + a_2 y + \cdots + a_{m+1} y^m \tag{17}$$

That means(considering the arbitrarity of the coefficients in Eq.(17))

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m = a_0 + a_1 y + a_2 y^2 + \cdots + a_m y^m$$

has no solution.
This is a contradiction with Eq.(13).
Then we conclude that

$$a_0 + a_1 x + \cdots + a_{m+1} x^{m+1} = a_0 + a_1 y + \cdots + a_{m+1} y^{m+1} \Leftrightarrow x = y \tag{18}$$

i.e., Lemma 3 is true for $n = m + 1$.
Thus, Lemma 3 is true for any natural number n.
That completes the proof of Lemma 3.

Proof of RH: The details are delivered in four steps as follows.

Step 1: Since $\xi(s)$ is an entire function, it is analytic in the whole complex plane $\mathbb{C}$. Then $\xi(s)$ can be expanded in an infinite Maclaurin series at $s = 0$, i.e.

$$\xi(s) = \xi(0) + \xi^{'}(0)s + \frac{\xi^{''}(0)}{2!}s^2 + \cdots + \frac{\xi^{(n)}(0)}{n!}s^n + \cdots, |s| < \infty \tag{19}$$

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It is obvious that

$$\frac{\xi^{(n)}(0)}{n!} = \left.\frac{\xi^{(n)}(s)}{n!}\right|_{s=0}, n = 0, 1, 2, \cdots$$

are all real numbers.
Step2: In Eq. (19), replace s with $1-s$, we obtain

$$\xi(1-s) = \xi(0)+\xi^{'}(0)(1-s)+\frac{\xi^{''}(0)}{2!}(1-s)^2+\cdots+\frac{\xi^{(n)}(0)}{n!}(1-s)^n+\cdots, |s| < \infty \tag{20}$$

Step 3: Due to $\xi(s) = \xi(1-s)$, thus we have

$$\begin{aligned}&\xi(0) + \xi^{'}(0)s + \frac{\xi^{''}(0)}{2!}s^2 + \cdots + \frac{\xi^{(n)}(0)}{n!}s^n + \cdots \\ =&\xi(0) + \xi^{'}(0)(1-s) + \frac{\xi^{''}(0)}{2!}(1-s)^2 + \cdots + \frac{\xi^{(n)}(0)}{n!}(1-s)^n + \cdots\end{aligned} \tag{21}$$

By Lemma 3, we have

$$s = 1 - s \tag{22}$$

According to Lemma 2, without loss of generality, set $s = \alpha \pm j\beta, \beta \neq 0$, it follows that

$$\alpha \pm j\beta = 1 - \alpha \pm j\beta \quad \Rightarrow \quad \alpha = 1 - \alpha \quad \Rightarrow \quad \alpha = \frac{1}{2} \tag{23}$$

That means the polynomial equation $\xi(s) = \xi(1-s)$ limits the solution to be $s = \frac{1}{2} \pm j\beta$, i.e., on the line $\Re(s) = \frac{1}{2}$.

Step 4: All the zeros of $\xi(s)$ are determined by $\xi(s) = \xi(1-s) = 0$, which is obvious a special case of $\xi(s) = \xi(1-s)$, then we conclude that all the zeros of $\xi(s)$ have real part equal to $\frac{1}{2}$.
That completes the proof of RH (in terms of $\xi(s)$).

Remark: According to Lemma 2, we know that RH (in terms of $\zeta(s)$) is also true, i.e., The non-trivial zeros of $\zeta(s)$ have real part equal to $\frac{1}{2}$.

3 Conclusion

A proof of Riemann Hypothesis is presented based on a new road map: First, the completed zeta function $\xi(s)$ is expressed as Maclaurin series; Second, by $\xi(s) = \xi(1-s)$, there exists a polynomial equation, both sides with the same real number coefficients and infinite degree, which leads to

$$s = 1 - s \Rightarrow \alpha \pm j\beta = 1 - \alpha \pm j\beta \Rightarrow \alpha = \frac{1}{2}$$

Then we conclude that the celebrated Riemann Hypothesis is true.

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