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The Riemann Hypothesis Is True If The Zeros Of $\xi(s)$ Have Strict Increasing Imaginary Parts

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Abstract Based on Hadamard product, a new expression of the completed zeta function $\xi(s)$ is obtained, i.e.,

$$\xi(s) = \xi(0) \prod_{i=1}^{\infty} \left(\frac{\beta_i^2}{\alpha_i^2 + \beta_i^2} + \frac{(s - \alpha_i)^2}{\alpha_i^2 + \beta_i^2} \right)^{d_i}$$

where $\rho_i = \alpha_i + j\beta_i$, $\bar{\rho}_i = \alpha_i - j\beta_i$ are complex conjugate zeros of $\xi(s)$, $d_i \geq 1$ (natural numbers) are the multiplicities of ρ_i , $0 < \alpha_i < 1$, $\beta_i \neq 0$, $i \in \mathbb{N}$ are natural numbers from 1 to infinity. Suppose β_i are in order of strict increasing $|\beta_i|$, i.e., $|\beta_1| < |\beta_2| < |\beta_3| < \dots$, or in another word, for each β_i , there are only four (two pairs) possible zeros (regardless of their multiplicities), i.e., $\alpha_i \pm j\beta_i, 1 - \alpha_i \pm j\beta_i$. According to the functional equation $\xi(s) = \xi(1-s)$, We have

$$\xi(0) \prod_{i=1}^{\infty} \left(\frac{\beta_i^2}{\alpha_i^2 + \beta_i^2} + \frac{(s - \alpha_i)^2}{\alpha_i^2 + \beta_i^2} \right)^{d_i} = \xi(0) \prod_{i=1}^{\infty} \left(\frac{\beta_i^2}{\alpha_i^2 + \beta_i^2} + \frac{(1-s - \alpha_i)^2}{\alpha_i^2 + \beta_i^2} \right)^{d_i}$$

i.e.,

$$\prod_{i=1}^{\infty} \left(1 + \frac{(s - \alpha_i)^2}{\beta_i^2} \right)^{d_i} = \prod_{i=1}^{\infty} \left(1 + \frac{(1-s - \alpha_i)^2}{\beta_i^2} \right)^{d_i}$$

which, by Lemma 3, is equivalent to

$$\alpha_i = \frac{1}{2}, i \in \mathbb{N}, \text{ from 1 to infinity.}$$

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Thus, the Riemann Hypothesis is true if we have $|\beta_1| < |\beta_2| < |\beta_3| < \dots$.

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1 Introduction and the problem description

It has been almost 163 years since the Riemann Hypothesis (RH) was proposed in 1859 [1]. Many efforts and achievements have been made towards proving this celebrated hypothesis, but it is still an open problem [2–3].

The Riemann zeta function is the function of complex variable s , defined in the half-plane $\Re(s) > 1$ by the absolutely convergent series [2]

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad (1)$$

Riemann showed how to extend zeta function to the whole complex plane $\mathbb{C}$ by analytic continuation

$$\zeta(s) = \frac{\pi^{s/2}}{\Gamma(s/2)} \left\{ \frac{1}{s(s-1)} + \int_1^{\infty} (x^{\frac{s}{2}-1} + x^{-\frac{s}{2}-\frac{1}{2}}) \cdot \left(\frac{\theta(x)-1}{2} \right) dx \right\} \quad (2)$$

where $\theta(x) = \sum_{-\infty}^{\infty} e^{-n^2 \pi x}$ being the Jacobi theta function, Γ being the Gamma function in the following Weierstrass expression (Meanwhile, there are also Gauss expression, Euler expression, and integral expression of the Gamma function.)

$$\frac{1}{\Gamma(s)} = s \cdot e^{\gamma s} \prod_{n=1}^{\infty} \left(1 + \frac{s}{n} \right) e^{-s/n} \quad (3)$$

where γ is the Euler-Mascheroni constant.

The connection between the Riemann zeta function and prime numbers can be established through the well-known Euler product.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p (1 - p^{-s})^{-1}, \Re(s) > 1 \quad (4)$$

the product being over the prime numbers p .

As shown by Riemann, $\zeta(s)$ extends to $\mathbb{C}$ as a meromorphic function with only a simple pole at $s = 1$, with residue 1, and satisfies the following functional equation

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s) \quad (5)$$

The Riemann zeta function $\zeta(s)$ has zeros at the negative even integers: $-2, -4, -6, -8, \dots$ and one refers to them as the **trivial zeros**. The other

zeros of $\zeta(s)$ are the complex numbers, i.e., **non-trivial zeros** [2].

In 1896, Hadamard [4] and Poussin [5] independently proved that no zeros could lie on the line $\Re(s) = 1$. Together with the functional equation and the fact that there are no zeros with real part greater than 1, this showed that all non-trivial zeros must lie in the interior of the critical strip $0 < \Re(s) < 1$. This was a key step in their first proofs of the famous **Prime Number Theorem**.

Later on, Hardy (1914) [6], Hardy and Littlewood (1921) [7] showed that there are infinitely many zeros on the critical line $\Re(s) = \frac{1}{2}$, which was an astonishing result at that time.

As a summary of the properties of the non-trivial zeros of $\zeta(s)$, we have the following results [8–9].

Lemma 1: Non-trivial zeroes of $\zeta(s)$, noted as $\rho = \alpha + j\beta$, have the following properties

- 1) The number of non-trivial zeroes is infinity;
- 2) $\beta \neq 0$;
- 3) $0 < \alpha < 1$;
- 4) $\rho, \bar{\rho}, 1 - \bar{\rho}, 1 - \rho$ are all non-trivial zeroes.

As further study, a completed zeta function $\xi(s)$ is defined as

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) \quad (6)$$

It is well-known that $\xi(s)$ is an entire function of order 1. This implies $\xi(s)$ is analytic, and can be expressed as infinite polynomial, in the whole complex plane $\mathbb{C}$.

In addition, replacing s with $1-s$ in Eq.(6), and combining Eq.(5), we have the following functional equation

$$\xi(s) = \xi(1-s) \quad (7)$$

Considering the definition of $\xi(s)$, and recalling Eq.(3), the trivial zeros of $\zeta(s)$ are canceled by the poles of $\Gamma(\frac{s}{2})$. The zero of $s-1$ and the pole of $\zeta(s)$ cancel; the zero $s=0$ and the pole of $\Gamma(\frac{s}{2})$ cancel [9–10]. Thus, all the zeros of $\xi(s)$ are exactly the nontrivial zeros of $\zeta(s)$. Then we have the following Lemma 2.

Lemma 2: Zeros of $\xi(s)$ coincide with non-trivial zeros of $\zeta(s)$.

According to Lemma 2, the following two statements for the RH are equivalent.

Statement 1 of the RH: All the non-trivial zeros of $\zeta(s)$ have real part equal to $\frac{1}{2}$.

Statement 2 of the RH: All the zeros of $\xi(s)$ have real part equal to $\frac{1}{2}$.

To prove the RH, the natural thinking is to estimate the number of zeros of $\zeta(s)$ in some certain closed areas according to the Argument Principle. Along this train of thought, there are some famous research works. Let $N(T)$ denote the number of zeros of $\zeta(s)$ inside the rectangle: $0 < \alpha < 1, 0 < \beta \leq T$, and let $N_0(T)$ denote the number of zeros of $\zeta(s)$ on the line $\alpha = \frac{1}{2}, 0 < \beta \leq T$. Selberg proved that there exist positive constants c and T_0 , such that $N_0(T) > cN(T), (T > T_0)$ [11], later on, Levinson proved that $c \geq \frac{1}{3}$ [12], Lou and Yao proved that $c \geq 0.3484$ [13], Conrey proved that $c \geq \frac{2}{5}$ [14], Bui, Conrey and Young proved that $c \geq 0.41$ [15], Feng proved that $c \geq 0.4128$ [16].

On the other hand, many zeros have been calculated by hand or by computers. Among others, Riemann found the first three non-trivial zeros [17]. Gram found the first 15 zeros based on Euler-Maclaurin summation [18]. Titchmarsh calculated the 138th to 195th zeros using the Riemann-Siegel formula [19–20]. Here are the first three zeros: $\frac{1}{2} \pm j14.1347251$; $\frac{1}{2} \pm j21.0220396$; $\frac{1}{2} \pm j25.0108575$. Then we certainly have $\alpha_1 = \frac{1}{2}$.

The idea of this paper is originated from Euler's work on proving that

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{6} \quad (8)$$

This interesting and famous result is deduced from two types of infinite expansions, i.e., infinite polynomial and infinite product as follows

$$\begin{aligned} \frac{\sin x}{x} &= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots \\ &= \left(1 - \frac{x^2}{\pi^2}\right) \left(1 - \frac{x^2}{4\pi^2}\right) \left(1 - \frac{x^2}{9\pi^2}\right) \cdots \end{aligned} \quad (9)$$

This paper is also motivated by the fact that $\xi(s)$ can be expressed by the following Hadamard product, which was first proposed by Riemann. However, it was Hadamard [21] who showed the validity of this infinite product expansion.

$$\xi(s) = \xi(0) \prod_{\rho} \left(1 - \frac{s}{\rho}\right) \quad (10)$$

where ρ are precisely the non-trivial zeros of the Riemann zeta function $\zeta(s)$, or in another word, ρ runs over the zeros of the completed zeta function $\xi(s)$. Besides, it is well-known that $\xi(0) = \frac{1}{2}$.

2 A proof of the RH

This section is planned to present a proof of the Riemann Hypothesis. We first prove that Statement 2 of the RH is true, and then by Lemma 2 we know that

Statement 1 of the RH is also true. For this purpose, we need the following Lemma 3.

Lemma 3: Given two infinite products

$$f(s) = \prod_{i=1}^{\infty} \left(1 + \frac{(s - \alpha_i)^2}{\beta_i^2}\right)^{d_i} \quad (11)$$

and

$$f(1-s) = \prod_{i=1}^{\infty} \left(1 + \frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} \quad (12)$$

where s is a complex variable, $\alpha_i \neq 0, \beta_i \neq 0$ are real numbers, $d_i \geq 1$ are natural numbers, $i \in \mathbb{N}$ are natural numbers from 1 to infinity, β_i are in order of strict increasing $|\beta_i|$, i.e., $|\beta_1| < |\beta_2| < |\beta_3| < \dots$.

Then we have

$$f(s) = f(1-s) \Leftrightarrow \alpha_i = \frac{1}{2}, i \in \mathbb{N}, \text{ from 1 to infinity.}$$

where " $\Leftrightarrow$ " is the equivalent sign.

Proof: The proof is mainly based on Mathematical Induction. We first prove that $P(n)$ holds for all natural numbers $n \in \mathbb{N}$, and then prove that $P(\infty)$ holds. It should be noted that we prefer to use elementary method instead of Transfinite Induction to complete the proof.

First of all, we have the following proposition:

$$\left(1 + \frac{(s - \alpha)^2}{\beta^2}\right)^{d_i} = \left(1 + \frac{(1-s-\alpha)^2}{\beta^2}\right)^{d_i} \Leftrightarrow (s - \alpha)^2 = (1-s-\alpha)^2 \Leftrightarrow \alpha = \frac{1}{2} \quad (13)$$

Let $P(n)$ be:

$$\begin{aligned} \prod_{i=1}^n \left(1 + \frac{(s - \alpha_i)^2}{\beta_i^2}\right)^{d_i} &= \prod_{i=1}^n \left(1 + \frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} \\ \Leftrightarrow & \\ \begin{cases} \left(1 + \frac{(s-\alpha_1)^2}{\beta_1^2}\right)^{d_1} = \left(1 + \frac{(1-s-\alpha_1)^2}{\beta_1^2}\right)^{d_1} \\ \dots \\ \left(1 + \frac{(s-\alpha_n)^2}{\beta_n^2}\right)^{d_n} = \left(1 + \frac{(1-s-\alpha_n)^2}{\beta_n^2}\right)^{d_n} \end{cases} & \quad (14) \\ \Leftrightarrow (\text{by Eq.(13)}) & \\ \alpha_i = \frac{1}{2}, i = 1, 2, 3, \dots, n & \end{aligned}$$

Also by Eq.(13), $P(1)$ is an obvious fact as the base case, i.e.,

$$\begin{aligned}
& \prod_{i=1}^1 \left(1 + \frac{(s - \alpha_i)^2}{\beta_i^2}\right)^{d_1} = \prod_{i=1}^1 \left(1 + \frac{(1 - s - \alpha_i)^2}{\beta_i^2}\right)^{d_1} \\
& \Leftrightarrow \\
& \left(1 + \frac{(s - \alpha_1)^2}{\beta_1^2}\right)^{d_1} = \left(1 + \frac{(1 - s - \alpha_1)^2}{\beta_1^2}\right)^{d_1} \quad (15) \\
& \Leftrightarrow \\
& \alpha_1 = \frac{1}{2}
\end{aligned}$$

For the inductive step, assume that $P(n)$ holds, i.e., Eq.(14) holds, we need to show that $P(n+1)$ holds too. We have

$$\begin{aligned}
& \prod_{i=1}^{n+1} \left(1 + \frac{(s - \alpha_i)^2}{\beta_i^2}\right)^{d_i} = \prod_{i=1}^{n+1} \left(1 + \frac{(1 - s - \alpha_i)^2}{\beta_i^2}\right)^{d_i} \\
& \Leftrightarrow \\
& \prod_{i=1}^n \left(1 + \frac{(s - \alpha_i)^2}{\beta_i^2}\right)^{d_i} \left(1 + \frac{(s - \alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}} = \prod_{i=1}^n \left(1 + \frac{(1 - s - \alpha_i)^2}{\beta_i^2}\right)^{d_i} \left(1 + \frac{(1 - s - \alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}} \\
& \Leftrightarrow \text{(by Lemma 4)} \\
& \left\{ \begin{aligned} \prod_{i=1}^n \left(1 + \frac{(s - \alpha_i)^2}{\beta_i^2}\right)^{d_i} &= \prod_{i=1}^n \left(1 + \frac{(1 - s - \alpha_i)^2}{\beta_i^2}\right)^{d_i} \\ \left(1 + \frac{(s - \alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}} &= \left(1 + \frac{(1 - s - \alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}} \end{aligned} \right. \\
& \Leftrightarrow \text{(by Eq.(14))} \\
& \left\{ \begin{aligned} \left(1 + \frac{(s - \alpha_1)^2}{\beta_1^2}\right)^{d_1} &= \left(1 + \frac{(1 - s - \alpha_1)^2}{\beta_1^2}\right)^{d_1} \\ \dots & \\ \left(1 + \frac{(s - \alpha_n)^2}{\beta_n^2}\right)^{d_n} &= \left(1 + \frac{(1 - s - \alpha_n)^2}{\beta_n^2}\right)^{d_n} \\ \left(1 + \frac{(s - \alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}} &= \left(1 + \frac{(1 - s - \alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}} \end{aligned} \right. \\
& \Leftrightarrow \text{(by Eq.(13))} \\
& \alpha_i = \frac{1}{2}, i = 1, 2, 3, \dots, n, n+1 \quad (16)
\end{aligned}$$

That is to say $P(n+1)$ holds when $P(n)$ is true.

Hence by Mathematical Induction, $P(n)$ is true for all natural numbers n .

Further, from the above process, we see that

$$P(n+1) \Leftrightarrow P(n) \wedge P(1) \Leftrightarrow P(n-1) \wedge P(1) \wedge P(1) \Leftrightarrow \dots \Leftrightarrow \underbrace{P(1) \wedge \dots \wedge P(1)}_{(n+1) \text{ times}} \Leftrightarrow P(1)$$

Then we also have

$$P(\infty) \Leftrightarrow \underbrace{P(1) \wedge \cdots}_{\text{infinity times}} \Leftrightarrow P(1)$$

Recalling Eq.(13), we know that $P(1)$ holds without regard to $\alpha_i, i \in \mathbb{N}$ from 1 to ∞ . Hence we conclude that $P(\infty)$ holds.

That completes the proof of Lemma 3.

Lemma 4: We have the following fact that

$$\begin{aligned} \prod_{i=1}^{n+1} \left(1 + \frac{(s - \alpha_i)^2}{\beta_i^2}\right)^{d_i} &= \prod_{i=1}^{n+1} \left(1 + \frac{(1 - s - \alpha_i)^2}{\beta_i^2}\right)^{d_i} \\ \Leftrightarrow \\ \left\{ \prod_{i=1}^n \left(1 + \frac{(s - \alpha_i)^2}{\beta_i^2}\right)^{d_i} = \prod_{i=1}^n \left(1 + \frac{(1 - s - \alpha_i)^2}{\beta_i^2}\right)^{d_i} \right. \\ \left. \left(1 + \frac{(s - \alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}} = \left(1 + \frac{(1 - s - \alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}} \right\} \end{aligned}$$

where s is a complex variable, $\alpha_i \neq 0, \beta_i \neq 0$ are real numbers, $i, n \geq 1, d \geq 1$ are natural numbers, β_i are in order of strict increasing $|\beta_i|$, i.e., $|\beta_1| < |\beta_2| < |\beta_3| < \cdots$.

Proof: Let's consider

$$\prod_{i=1}^{n+1} \left(1 + \frac{(s - \alpha_i)^2}{\beta_i^2}\right)^{d_i} = \prod_{i=1}^{n+1} \left(1 + \frac{(1 - s - \alpha_i)^2}{\beta_i^2}\right)^{d_i} \quad (17)$$

i.e.,

$$\begin{aligned} &\prod_{i=1}^n \left(1 + \frac{(s - \alpha_i)^2}{\beta_i^2}\right)^{d_i} \left(1 + \frac{(s - \alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}} \\ &= \prod_{i=1}^n \left(1 + \frac{(1 - s - \alpha_i)^2}{\beta_i^2}\right)^{d_i} \left(1 + \frac{(1 - s - \alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}} \end{aligned} \quad (18)$$

Leaving out the trivial cases

$$\begin{aligned} \left(1 + \frac{(s - \alpha_i)^2}{\beta_i^2}\right)^{d_i} &= \left(1 + \frac{(1 - s - \alpha_i)^2}{\beta_i^2}\right)^{d_i} = 0 \\ \Leftrightarrow \\ (s - \alpha_i)^2 &= (1 - s - \alpha_i)^2 = -\beta_i^2, i = 1, 2, 3, \cdots, n, n + 1 \end{aligned} \quad (19)$$

which directly imply Lemma 4 is true, Eq.(18) is then equivalent to the following Eq.(20).

$$\frac{\prod_{i=1}^n \left(1 + \frac{(s - \alpha_i)^2}{\beta_i^2}\right)^{d_i}}{\prod_{i=1}^n \left(1 + \frac{(1 - s - \alpha_i)^2}{\beta_i^2}\right)^{d_i}} = \frac{\left(1 + \frac{(1 - s - \alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}}}{\left(1 + \frac{(s - \alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}}} \quad (20)$$

Another possibility $\frac{\prod_{i=1}^n \left(1 + \frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}}{\left(1 + \frac{(1-s-\alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}}} = \frac{\prod_{i=1}^n \left(1 + \frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}}{\left(1 + \frac{(s-\alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}}}$ is excluded due to that means $\beta_{n+1} = \beta_n, \alpha_{n+1} = 1 - \alpha_n, d_{n+1} = d_n$, which contradicts the given condition $|\beta_1| < |\beta_2| < |\beta_3| < \dots$. Without loss of generality, we set

$$\frac{\prod_{i=1}^n \left(1 + \frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}}{\prod_{i=1}^n \left(1 + \frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}} = \frac{\left(1 + \frac{(1-s-\alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}}}{\left(1 + \frac{(s-\alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}}} = k \neq 0 \quad (21)$$

where k is a constant to be determined.

Therefore Eq.(20) is equivalent to the following Eq.(22)

$$\begin{cases} \prod_{i=1}^n \left(1 + \frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} = k \prod_{i=1}^n \left(1 + \frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} \\ \left(1 + \frac{(1-s-\alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}} = k \left(1 + \frac{(s-\alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}} \end{cases} \quad (22)$$

By comparing the like terms of polynomials in both sides of Eq.(22), we know that Eq.(22) holds if and only if $k = 1$, that means Eq.(20) is equivalent to the following Eq.(23)

$$\begin{cases} \prod_{i=1}^n \left(1 + \frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} = \prod_{i=1}^n \left(1 + \frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} \\ \left(1 + \frac{(1-s-\alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}} = \left(1 + \frac{(s-\alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}} \end{cases} \quad (23)$$

Recalling that Eq.(17) $\Leftrightarrow$ Eq.(18) $\Leftrightarrow$ Eq.(20), then we have

$$\begin{aligned} \prod_{i=1}^{n+1} \left(1 + \frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} &= \prod_{i=1}^{n+1} \left(1 + \frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} \\ &\Leftrightarrow \begin{cases} \prod_{i=1}^n \left(1 + \frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} = \prod_{i=1}^n \left(1 + \frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} \\ \left(1 + \frac{(1-s-\alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}} = \left(1 + \frac{(s-\alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}} \end{cases} \end{aligned} \quad (24)$$

That completes the proof of Lemma 4.

Next, we present the proof of the RH under the assumption that $\beta_1 < \beta_2 < \beta_3, \dots$.

Proof of the RH: The details are delivered in three steps as follows.

Step 1: Based on the Hadamard product in Eq.(10), we have

$$\begin{aligned}
 \xi(s) &= \xi(0) \prod_{\rho} \left(1 - \frac{s}{\rho}\right) \\
 &= \xi(0) \prod_{i=1}^{\infty} \left(1 - \frac{s}{\rho_i}\right) \left(1 - \frac{s}{\bar{\rho}_i}\right) \\
 &= \xi(0) \prod_{i=1}^{\infty} \left(1 - \frac{s}{\alpha_i + j\beta_i}\right) \left(1 - \frac{s}{\alpha_i - j\beta_i}\right) \\
 &= \xi(0) \prod_{i=1}^{\infty} \left(\frac{\beta_i^2}{\alpha_i^2 + \beta_i^2} + \frac{(s - \alpha_i)^2}{\alpha_i^2 + \beta_i^2}\right)
 \end{aligned} \tag{25}$$

The convergence of the infinite product in Eq.(25) in the form

$$\xi(s) = \xi(0) \prod_{\rho} \left(1 - \frac{s}{\rho}\right) \left(1 - \frac{s}{\bar{\rho}}\right) = \xi(0) \prod_{\rho} \left(1 - \frac{s(2\alpha - s)}{|\rho|^2}\right), 0 < \alpha = \Re(\rho) < 1$$

can be obtained by the similar way that was used in Chapter 2.5 of Reference [22] to prove the convergence of

$$\xi(s) = \xi(0) \prod_{\rho} \left(1 - \frac{s}{\rho}\right) \left(1 - \frac{s}{1 - \rho}\right) = \xi(0) \prod_{\rho} \left(1 - \frac{s(1 - s)}{\rho(1 - \rho)}\right)$$

In fact, the convergence of these two infinite products depends on the convergence of infinite series: $\sum_{\rho} \frac{1}{|\rho|^2}$, and $\sum_{\rho} \frac{1}{|\rho(1 - \rho)|}$, respectively. The proof details of the convergence of these two infinite series can be found on pages 42-43 in Ref. [22], as well as on page 156 in Ref. [9].

Further, taking into account the possibility of multiple zeros, we have

$$\xi(s) = \xi(0) \prod_{i=1}^{\infty} \left(\frac{\beta_i^2}{\alpha_i^2 + \beta_i^2} + \frac{(s - \alpha_i)^2}{\alpha_i^2 + \beta_i^2}\right)^{d_i} \tag{26}$$

where $d_i \geq 1$ are natural numbers.

Step 2: Replacing s with $1 - s$ in Eq.(26), we obtain the infinite product expression of $\xi(1 - s)$

$$\xi(1 - s) = \xi(0) \prod_{i=1}^{\infty} \left(\frac{\beta_i^2}{\alpha_i^2 + \beta_i^2} + \frac{(1 - s - \alpha_i)^2}{\alpha_i^2 + \beta_i^2}\right)^{d_i} \tag{27}$$

Step 3: We have by $\xi(s) = \xi(1 - s)$ that

$$\prod_{i=1}^{\infty} \left(\frac{\beta_i^2}{\alpha_i^2 + \beta_i^2} + \frac{(s - \alpha_i)^2}{\alpha_i^2 + \beta_i^2}\right)^{d_i} = \prod_{i=1}^{\infty} \left(\frac{\beta_i^2}{\alpha_i^2 + \beta_i^2} + \frac{(1 - s - \alpha_i)^2}{\alpha_i^2 + \beta_i^2}\right)^{d_i} \tag{28}$$

which is equivalent to

$$\prod_{i=1}^{\infty} \left(1 + \frac{(s - \alpha_i)^2}{\beta_i^2}\right)^{d_i} = \prod_{i=1}^{\infty} \left(1 + \frac{(1 - s - \alpha_i)^2}{\beta_i^2}\right)^{d_i} \quad (29)$$

Next, we suppose that β_i can be arranged in order of strict increasing $|\beta_i|$, i.e., $|\beta_1| < |\beta_2| < |\beta_3| < \dots$, or in another word, for each β_i , there are only four (two pairs) possible zeros (regardless of their multiplicities), i.e., $\alpha_i \pm j\beta_i, 1 - \alpha_i \pm j\beta_i$.

Then according to Lemma 3, Eq.(29) is equivalent to $\alpha_i = \frac{1}{2}, i \in \mathbb{N}$ from 1 to infinity.

That means, under the assumption that for each β_i , there are only four (two pairs) possible zeros, i.e., $\alpha_i \pm j\beta_i, 1 - \alpha_i \pm j\beta_i$, all the zeros of the completed zeta function $\xi(s)$ have real part equal to $\frac{1}{2}$, i.e., Statement 2 of the RH is true; According to Lemma 2, we know that the Statement 1 of the RH is also true, i.e., All the non-trivial zeros of the Riemann zeta function $\zeta(s)$ have real part equal to $\frac{1}{2}$.

That completes the proof of the RH under the assumption that for each β_i , there are only four (two pairs) possible zeros.

3 Conclusion

Under the assumption that the zeros of $\xi(s)$ can be arranged in strict increasing order of imaginary parts, the Riemann Hypothesis is proved based on a new expression of the completed zeta function $\xi(s)$, i.e., the infinite product of quadratic factors determined by complex conjugate roots of $\xi(s) = 0$, i.e., $\rho_i = \alpha_i + j\beta_i, \bar{\rho}_i = \alpha_i - j\beta_i, 0 < \alpha_i < 1, \beta_i \neq 0, i \in \mathbb{N}$, from 1 to infinity.

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