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A Proof of the Riemann Hypothesis Based on a New Expression of the Completed Zeta Function

Weicun Zhang

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Abstract The completed zeta function $\xi(s)$ is expanded in MacLaurin series (infinite polynomial), which can be further expressed as infinite product (Hadamard product) of quadratic factors by its complex conjugate zeros $\rho_i = \alpha_i + j\beta_i, \bar{\rho}_i = \alpha_i - j\beta_i, 0 < \alpha_i < 1, \beta_i \neq 0, i \in \mathbb{N}$ are natural numbers from 1 to infinity, ρ_i are in order of increasing $|\rho_i| = \sqrt{\alpha_i^2 + \beta_i^2}$, i.e., $|\rho_1| < |\rho_2| \leq |\rho_3| \leq |\rho_4|, \cdots$, together with $\beta_1 < \beta_2 \leq \beta_3 \leq \beta_4, \cdots$. Then, according to the functional equation $\xi(s) = \xi(1-s)$, we have

$$\xi(0)\prod_{i\in\mathbb{N}}\left(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}\right)=\xi(0)\prod_{i\in\mathbb{N}}\left(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(1-s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}\right)$$

which, by Lemma 3, is equivalent to

$$(s-\alpha_i)^2 = (1-s-\alpha_i)^2, i \in \mathbb{N}, \text{from 1 to infinity.}$$

with only valid solution $\alpha_i = \frac{1}{2}$ (another solution $s = \frac{1}{2}$ is invalid due to obvious contradiction).
Thus, a proof of the Riemann Hypothesis is achieved.

Keywords Riemann Hypothesis (RH) · Proof · Completed zeta function

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Weicun Zhang
University of Science and Technology Beijing
Beijing 100083, China
Tel.: +86-13601102534
Fax: +86-1062936073
ORCID: 0000-0003-0047-0558
E-mail: weicunzhang@ustb.edu.cn

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1 Introduction and the problem description

It has been 163 years since the Riemann Hypothesis (RH) was proposed in 1859 [1]. Many efforts and achievements have been made towards proving this celebrated hypothesis, but it is still an open problem [2−3].

The Riemann zeta function is the function of the complex variable s, defined in the half-plane $\Re(s) > 1$ by the absolutely convergent series [2]

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

Riemann showed how to extend zeta function to the whole complex plane $\mathbb{C}$ by analytic continuation

$$\zeta(s) = \frac{\pi^{s/2}}{\Gamma(s/2)}\{\frac{1}{s(s-1)} + \int_1^{\infty} (x^{\frac{s}{2}-1} + x^{-\frac{s}{2}-\frac{1}{2}}) \cdot (\frac{\theta(x)-1}{2})dx\} \tag{2}$$

where $\theta(x) = \sum_{-\infty}^{\infty} e^{-n^2\pi x}$ being the Jaccobi theta function, Γ being the Gamma function in the following Weierstrass expression (Meanwhile, there are also Gauss expression, Euler expression, and integral expression of the Gamma function.)

$$\frac{1}{\Gamma(s)} = s \cdot e^{\gamma s} \prod_{n=1}^{\infty} (1 + \frac{s}{n}) e^{-s/n} \tag{3}$$

where γ is the Euler-Mascheroni constant.

The connection between the Riemann zeta function and prime numbers can be established through the well-known Euler product.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p (1 - p^{-s})^{-1}, \Re(s) > 1 \tag{4}$$

the product being over the prime numbers p.

As shown by Riemann, $\zeta(s)$ extends to $\mathbb{C}$ as a meromorphic function with only a simple pole at $s = 1$, with residue 1, and satisfies the following functional equation

$$\pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma(\frac{1-s}{2}) \zeta(1-s) \tag{5}$$

The Riemann zeta function $\zeta(s)$ has zeros at the negative even integers: $-2, -4, -6, -8, \cdots$ and one refers to them as the **trivial zeros**. The other zeros of $\zeta(s)$ are the complex numbers, i.e., **non-trivial zeros** [2].

In 1896, Hadamard [4] and Poussin [5] independently proved that no zeros could lie on the line $\Re(s) = 1$. Together with the functional equation and the fact that there are no zeros with real part greater than 1, this showed that all non-trivial zeros must lie in the interior of the critical strip $0 < \Re(s) < 1$. This was a key step in their first proofs of the famous **Prime Number Theorem**.

Later on, Hardy (1914) [6], Hardy and Littlewood (1921) [7] showed that there are infinitely many zeros on the critical line $\Re(s) = \frac{1}{2}$, which was an astonishing result at that time.

As a summary of the properties of the non-trivial zeros of $\zeta(s)$, we have the following results [8−9].

Lemma 1: Non-trivial zeroes of $\zeta(s)$, noted as $\rho = \alpha + j\beta$, have the following properties
1) The number of non-trivial zeroes is infinity;
2) $\beta \neq 0$;
3) $0 < \alpha < 1$;
4) $\rho, \bar{\rho}, 1-\bar{\rho}, 1-\rho$ are all non-trivial zeroes.

As further study, a completed zeta function $\xi(s)$ is defined as

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) \tag{6}$$

It is well-known that $\xi(s)$ is an entire function of order 1. This implies $\xi(s)$ is analytic, and can be expressed as infinite polynomial, in the whole complex plane $\mathbb{C}$.

In addition, replacing s with $1-s$ in Eq.(6), and combining Eq.(5), we have the following functional equation

$$\xi(s) = \xi(1-s) \tag{7}$$

Considering the definition of $\xi(s)$, and recalling Eq.(3), the trivial zeros of $\zeta(s)$ are canceled by the poles of $\Gamma(\frac{s}{2})$. The zero of $s-1$ and the pole of $\zeta(s)$ cancel; the zero $s = 0$ and the pole of $\Gamma(\frac{s}{2})$ cancel [9−10]. Thus, all the zeros of $\xi(s)$ are exactly the nontrivial zeros of $\zeta(s)$. Then we have the following Lemma 2.

Lemma 2: Zeros of $\xi(s)$ coincide with non-trivial zeros of $\zeta(s)$.

According to Lemma 2, the following two statements for RH are equivalent.

Statement 1 of RH: All the non-trivial zeros of $\zeta(s)$ have real part equal to $\frac{1}{2}$.

Statement 2 of RH: All the zeros of $\xi(s)$ have real part equal to $\frac{1}{2}$.

To prove the RH, the natural thinking is to estimate the number of zeros of $\zeta(s)$ in some certain closed areas according to the Argument Principle. Along this train of thought, there are some famous research works. Let $N(T)$ denote the number of zeros of $\zeta(s)$ inside the rectangle: $0 < \alpha < 1, 0 < \beta \leq T$, and let $N_0(T)$ denote the number of zeros of $\zeta(s)$ on the line $\alpha = \frac{1}{2}, 0 < \beta \leq T$.

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Selberg proved that there exist positive constants c and T_0, such that $N_0(T) > cN(T), (T > T_0)$ [11], later on, Levinson proved that $c \geq \frac{1}{3}$ [12], Lou and Yao proved that $c \geq 0.3484$ [13], Conrey proved that $c \geq \frac{2}{5}$ [14], Bui, Conrey and Young proved that $c \geq 0.41$ [15], Feng proved that $c \geq 0.4128$ [16].

Two types of infinite expansions of $\xi(s)$, i.e., MacLaurin series (infinite polynomial) and infinite product (Hadamard product) by complex conjugate roots, will be adopted in this paper to open another door to the proof of RH.

The idea is originated from Euler's work on proving that

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{6} \tag{8}$$

This interesting and famous result is deduced mainly based on the above-mentioned two types of infinite expansions, i.e., infinite polynomial and infinite product as follows

$$\begin{aligned}\frac{\sin x}{x} &= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots \\ &= (1 - \frac{x^2}{\pi^2})(1 - \frac{x^2}{4\pi^2})(1 - \frac{x^2}{9\pi^2}) \cdots\end{aligned} \tag{9}$$

This paper is also motivated by the fact that $\xi(s)$ can be expressed by the following Hadamard product, which was first proposed by Riemann. However, it was Hadamard [17] who showed the validity of this infinite product expansion.

$$\xi(s) = \xi(0) \prod_{\rho} (1 - \frac{s}{\rho}) \tag{10}$$

where ρ are precisely the non-trivial zeros of the Riemann zeta function $\zeta(s)$, or in another word, ρ runs over the zeros of the completed zeta function $\xi(s)$. Besides, it is well-known that $\xi(0) = \frac{1}{2}$.

2 A proof of RH

This section is planned to give a proof of the Statement 2 and Statement 1 of RH. For this purpose, we need the following Lemma 3.

Lemma 3: Given two infinite products

$$f(s) = \prod_{i \in \mathbb{N}} \Big(\frac{\beta_i^2}{\alpha_i^2 + \beta_i^2} + \frac{(s - \alpha_i)^2}{\alpha_i^2 + \beta_i^2} \Big) \tag{11}$$

and

$$f(1-s) = \prod_{i \in \mathbb{N}} \Big(\frac{\beta_i^2}{\alpha_i^2 + \beta_i^2} + \frac{(1 - s - \alpha_i)^2}{\alpha_i^2 + \beta_i^2} \Big) \tag{12}$$

where s is a complex variable, $0 < \alpha_i < 1$ and $\beta_i \neq 0$ are all real numbers, $i \in \mathbb{N}$ are natural numbers from 1 to infinity, $\sqrt{\alpha_i^2 + \beta_i^2} = |\rho_i|$, and ρ_i are

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in order of increasing $|\rho_i|$, i.e., $|\rho_1| < |\rho_2| \leq |\rho_3| \leq |\rho_4|, \cdots$, together with $\beta_1 < \beta_2 \leq \beta_3 \leq \beta_4, \cdots$.

Then we have

$$f(s) = f(1-s) \Leftrightarrow (s-\alpha_i)^2 = (1-s-\alpha_i)^2, i \in \mathbb{N}, \text{from 1 to infinity.}$$

where " $\Leftrightarrow$ " is the equivalent sign.

Proof: The proof is mainly based on mathematical induction. We first prove that $P(n)$ holds for all natural numbers, and then prove that $P(\infty)$ holds.

Let $P(n)$ be

$$\prod_{i=1}^{n}\Big(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2} + \frac{(s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}\Big) = \prod_{i=1}^{n}\Big(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2} + \frac{(1-s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}\Big)$$
$$\Leftrightarrow$$
$$(s-\alpha_i)^2 = (1-s-\alpha_i)^2, i = 1, 2, 3.\cdots, n$$

For our base case, we need to show $P(1)$ is true. It is obvious that

$$\frac{\beta_1^2}{\alpha_1^2+\beta_1^2} + \frac{(s-\alpha_1)^2}{\alpha_1^2+\beta_1^2} = \frac{\beta_1^2}{\alpha_1^2+\beta_1^2} + \frac{(1-s-\alpha_1)^2}{\alpha_1^2+\beta_1^2} \Leftrightarrow (s-\alpha_1)^2 = (1-s-\alpha_1)^2 \tag{13}$$

To avoid counterexample in next inductive step, we also need to show $P(2)$ is true. Considering $|\rho_1| < |\rho_2|$, together with $\beta_1 < \beta_2$, it is not difficult to see that

$$\Big(\frac{\beta_1^2}{\alpha_1^2+\beta_1^2} + \frac{(s-\alpha_1)^2}{\alpha_1^2+\beta_1^2}\Big)\Big(\frac{\beta_2^2}{\alpha_2^2+\beta_2^2} + \frac{(s-\alpha_2)^2}{\alpha_2^2+\beta_2^2}\Big) = \Big(\frac{\beta_1^2}{\alpha_1^2+\beta_1^2} + \frac{(1-s-\alpha_1)^2}{\alpha_1^2+\beta_1^2}\Big)\Big(\frac{\beta_2^2}{\alpha_2^2+\beta_2^2} + \frac{(1-s-\alpha_2)^2}{\alpha_2^2+\beta_2^2}\Big)$$
$$\Leftrightarrow$$
$$\Big(1+\frac{(s-\alpha_1)^2}{\beta_1^2}\Big)\Big(1+\frac{(s-\alpha_2)^2}{\beta_2^2}\Big) = \Big(1+\frac{(1-s-\alpha_1)^2}{\beta_1^2}\Big)\Big(1+\frac{(1-s-\alpha_2)^2}{\beta_2^2}\Big)$$

$\Leftrightarrow$ (Leaving out the trivial cases: $(s-\alpha_1)^2 = (1-s-\alpha_1)^2 = -\beta_1^2, (s-\alpha_2)^2 = (1-s-\alpha_2)^2 = -\beta_2^2$)

$$\frac{1+\frac{(s-\alpha_1)^2}{\beta_1^2}}{1+\frac{(1-s-\alpha_1)^2}{\beta_1^2}} = \frac{1+\frac{(1-s-\alpha_2)^2}{\beta_2^2}}{1+\frac{(s-\alpha_2)^2}{\beta_2^2}} = k, k \text{ is a constant to be determined as } k = 1.$$

(Another possible case $\frac{1+\frac{(s-\alpha_1)^2}{\beta_1^2}}{1+\frac{(1-s-\alpha_2)^2}{\beta_2^2}} = \frac{1+\frac{(1-s-\alpha_1)^2}{\beta_1^2}}{1+\frac{(s-\alpha_2)^2}{\beta_2^2}} = k$ has no solution of k due to $\beta_1 \neq \beta_2$.)

$$\Leftrightarrow$$
$$(s-\alpha_1)^2 = (1-s-\alpha_1)^2, (s-\alpha_2)^2 = (1-s-\alpha_2)^2 \tag{14}$$

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For the inductive step, assume that for some $n > 2$, $P(n)$ holds, we need to show that $P(n+1)$ holds.
Thus, we begin with the following Eq.(15)

$$\begin{aligned}&\prod_{i=1}^{n}\Big(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}\Big)=\prod_{i=1}^{n}\Big(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(1-s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}\Big)\\&\Leftrightarrow\\&(s-\alpha_i)^2=(1-s-\alpha_i)^2, i=1,2,3.\cdots,n\end{aligned}\tag{15}$$

Let's consider

$$\prod_{i=1}^{n+1}\Big(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}\Big)=\prod_{i=1}^{n+1}\Big(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(1-s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}\Big)\tag{16}$$

i.e.,

$$\begin{aligned}&\prod_{i=1}^{n}\Big(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}\Big)\Big(\frac{\beta_{n+1}^2}{\alpha_{n+1}^2+\beta_{n+1}^2}+\frac{(s-\alpha_{n+1})^2}{\alpha_{n+1}^2+\beta_{n+1}^2}\Big)\\=&\prod_{i=1}^{n}\Big(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(1-s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}\Big)\Big(\frac{\beta_{n+1}^2}{\alpha_{n+1}^2+\beta_{n+1}^2}+\frac{(1-s-\alpha_{n+1})^2}{\alpha_{n+1}^2+\beta_{n+1}^2}\Big)\end{aligned}\tag{17}$$

Leaving out the special (trivial) cases

$$\begin{aligned}&\Big(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}\Big)=\Big(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(1-s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}\Big)=0\\&\Leftrightarrow\\&(s-\alpha_i)^2=(1-s-\alpha_i)^2=-\beta_i^2, i=1,2,3,\cdots,n,n+1\end{aligned}\tag{18}$$

which directly imply $P(n+1)$ is true, Eq.(17) is then equivalent to

$$\frac{\prod_{i=1}^{n}\left(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}\right)}{\prod_{i=1}^{n}\left(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(1-s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}\right)}=\frac{\left(\frac{\beta_{n+1}^2}{\alpha_{n+1}^2+\beta_{n+1}^2}+\frac{(s-\alpha_{n+1})^2}{\alpha_{n+1}^2+\beta_{n+1}^2}\right)}{\left(\frac{\beta_{n+1}^2}{\alpha_{n+1}^2+\beta_{n+1}^2}+\frac{(1-s-\alpha_{n+1})^2}{\alpha_{n+1}^2+\beta_{n+1}^2}\right)}\tag{19}$$

Without loss of generality, set

$$\frac{\prod_{i=1}^{n}\left(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}\right)}{\prod_{i=1}^{n}\left(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(1-s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}\right)}=\frac{\left(\frac{\beta_{n+1}^2}{\alpha_{n+1}^2+\beta_{n+1}^2}+\frac{(s-\alpha_{n+1})^2}{\alpha_{n+1}^2+\beta_{n+1}^2}\right)}{\left(\frac{\beta_{n+1}^2}{\alpha_{n+1}^2+\beta_{n+1}^2}+\frac{(1-s-\alpha_{n+1})^2}{\alpha_{n+1}^2+\beta_{n+1}^2}\right)}=k\neq 0\tag{20}$$

where k is a real or complex number to be determined.
Then Eq.(20) is equivalent to the following Eq.(21)

$$\begin{cases}\prod_{i=1}^{n}\left(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}\right)=k\prod_{i=1}^{n}\left(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(1-s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}\right)\\\left(\frac{\beta_{n+1}^2}{\alpha_{n+1}^2+\beta_{n+1}^2}+\frac{(s-\alpha_{n+1})^2}{\alpha_{n+1}^2+\beta_{n+1}^2}\right)=k\left(\frac{\beta_{n+1}^2}{\alpha_{n+1}^2+\beta_{n+1}^2}+\frac{(1-s-\alpha_{n+1})^2}{\alpha_{n+1}^2+\beta_{n+1}^2}\right)\end{cases}\tag{21}$$

By comparing the like terms of polynomials in both sides of Eq. (21), we know that Eq.(21) holds if and only if $k = 1$, that means Eq.(20) is equivalent to the following Eq.(22)

$$\begin{cases}\prod_{i=1}^{n}\left(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}\right)=\prod_{i=1}^{n}\left(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(1-s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}\right)\\ \left(\frac{\beta_{n+1}^2}{\alpha_{n+1}^2+\beta_{n+1}^2}+\frac{(s-\alpha_{n+1})^2}{\alpha_{n+1}^2+\beta_{n+1}^2}\right)=\left(\frac{\beta_{n+1}^2}{\alpha_{n+1}^2+\beta_{n+1}^2}+\frac{(1-s-\alpha_{n+1})^2}{\alpha_{n+1}^2+\beta_{n+1}^2}\right)\end{cases} \tag{22}$$

Combining Eq.(22) and Eq.(15), we obtain

$$\begin{aligned}&\begin{cases}\prod_{i=1}^{n}\left(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}\right)=\prod_{i=1}^{n}\left(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(1-s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}\right)\\ \left(\frac{\beta_{n+1}^2}{\alpha_{n+1}^2+\beta_{n+1}^2}+\frac{(s-\alpha_{n+1})^2}{\alpha_{n+1}^2+\beta_{n+1}^2}\right)=\left(\frac{\beta_{n+1}^2}{\alpha_{n+1}^2+\beta_{n+1}^2}+\frac{(1-s-\alpha_{n+1})^2}{\alpha_{n+1}^2+\beta_{n+1}^2}\right)\end{cases}\\ &\Leftrightarrow\\ &(s-\alpha_i)^2=(1-s-\alpha_i)^2, i=1,2,3.\cdots,n,n+1\end{aligned} \tag{23}$$

i.e.,

$$\begin{aligned}&\prod_{i=1}^{n+1}\left(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}\right)=\prod_{i=1}^{n+1}\left(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(1-s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}\right)\\ &\Leftrightarrow\\ &(s-\alpha_i)^2=(1-s-\alpha_i)^2, i=1,2,3.\cdots,n,n+1\end{aligned} \tag{24}$$

That is to say $P(n+1)$ holds when $P(n)$ is true, so $P(n)$ is true for all natural numbers n.
Further, from the process of proving: $P(1)$ holds; $P(n+1)$ holds if $P(n)$ is true, we see that $P(n)$ is actually equivalent to $P(1)$ for different $\alpha_i, \beta_i, i = 1, \cdots, n$. It is obvious that $P(1)$ holds without regard to i or n. In fact, $P(1)$ can be rewritten in a general form

$k_1+k_2(s-\alpha)^2 = k_1+k_2(1-s-\alpha)^2 \Leftrightarrow (s-\alpha)^2 = (1-s-\alpha)^2, k_1, k_2 \neq 0, \alpha \neq 0$ are any real numbers.

Thus, we conclude: $P(n)$ is true when n tends to infinity, i.e., $P(\infty)$ holds. That completes the proof of Lemma 3.

Next, we present the proof of RH.

Proof of RH: The details are delivered in three steps as follows.

Step 1: Since $\xi(s)$ is an entire function, it is analytic in the whole complex plane $\mathbb{C}$. Then $\xi(s)$ can be expanded in MacLaurin series (infinite polynomial) at $s = 0$, i.e.

$$\xi(s) = \xi(0) + \xi^{'}(0)s + \frac{\xi^{''}(0)}{2!}s^2 + \cdots + \frac{\xi^{(n)}(0)}{n!}s^n + \cdots, |s| < \infty \tag{25}$$

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where $\frac{\xi^{(n)}(0)}{n!} = \frac{\xi^{(n)}(s)}{n!}\Big|_{s=0}, (n = 0, 1, 2, \cdots)$ are all real numbers.
Thus, all the zeros of $\xi(s)$ are the roots of the following infinite algebraic equation with real coefficients.

$$0 = \xi(0) + \xi^{'}(0)s + \frac{\xi^{''}(0)}{2!}s^2 + \cdots + \frac{\xi^{(n)}(0)}{n!}s^n + \cdots \tag{26}$$

According to the well established theory of algebraic equation with real number coefficients, complex roots always come in pairs (complex conjugate pairs). Further by Lemma 2, all the zeros of $\xi(s)$ are complex numbers, thus the roots of Eq.(26) can be denoted as complex conjugate pairs, i.e., $\rho_i = \alpha_i + j\beta_i, \bar{\rho}_i = \alpha_i - j\beta_i, 0 < \alpha_i < 1, \beta_i \neq 0, i \in \mathbb{N}$, from 1 to infinity.

Therefore, based on the Hadamard product in Eq.(10), the infinite polynomial Eq.(25) can be rewritten as infinite product of quadratic factors by complex conjugate roots, i.e.

$$\begin{aligned}
\xi(s) &= \xi(0)\prod_{\rho}(1 - \frac{s}{\rho}) \\
&= \xi(0)\prod_{\rho}(1 - \frac{s}{\rho})(1 - \frac{s}{\bar{\rho}}) \\
&= \xi(0)\prod_{i\in\mathbb{N}}(1 - \frac{s}{\rho_i})(1 - \frac{s}{\bar{\rho}_i}) \\
&= \xi(0)\prod_{i\in\mathbb{N}}(1 - \frac{s}{\alpha_i + j\beta_i})(1 - \frac{s}{\alpha_i - j\beta_i}) \\
&= \xi(0)\prod_{i\in\mathbb{N}}(1 - \frac{2\alpha_i s}{\alpha_i^2 + \beta_i^2} + \frac{s^2}{\alpha_i^2 + \beta_i^2}) \\
&= \xi(0)\prod_{i\in\mathbb{N}}(\frac{\beta_i^2}{\alpha_i^2 + \beta_i^2} + \frac{(s - \alpha_i)^2}{\alpha_i^2 + \beta_i^2})
\end{aligned} \tag{27}$$

where $\xi(0) = \frac{1}{2}$, ρ_i are in order of increasing $|\rho_i| = \sqrt{\alpha_i^2 + \beta_i^2}$, i.e., $|\rho_1| < |\rho_2| \leq |\rho_3| \leq |\rho_4|, \cdots$, together with $\beta_1 < \beta_2 \leq \beta_3 \leq \beta_4, \cdots$.

The convergence of infinite product in Eq.(27) in the form

$$\xi(s) = \xi(0)\prod_{\rho}(1 - \frac{s}{\rho})(1 - \frac{s}{\bar{\rho}}) = \xi(0)\prod_{\rho}\left(1 - \frac{s(2\alpha - s)}{|\rho|^2}\right), 0 < \alpha = \Re(\rho) < 1$$

can be obtained by the similar way that was used in Chapter 2.5 of Reference [18] to prove the convergence of

$$\xi(s) = \xi(0)\prod_{\rho}(1 - \frac{s}{\rho})(1 - \frac{s}{1 - \rho}) = \xi(0)\prod_{\rho}\left(1 - \frac{s(1 - s)}{\rho(1 - \rho)}\right)$$

In fact, the convergence of these two infinite products depends on the convergence of infinite series: $\sum_{\rho}\frac{1}{|\rho|^2}$, and $\sum_{\rho}\frac{1}{|\rho(1-\rho)|}$, respectively. The proof details of the convergence of these two infinite series can be found on pages 42-43 in Ref. [18] as well as on page 156 in Ref. [9].

Remark 1: The author was inspired at first by Euler's work on proving Eq.(8) based on Eq.(9), and then speculated that $\xi(s)$ could be expanded into quadratic factors $\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)$, $\beta_i \neq 0, i \in \mathbb{N}$. Subsequently, the author started with the Hadamard product Eq.(10) to get Eq.(27).

Step 2: Replacing s with $1-s$ in Eq.(27), we obtain the infinite product expression of $\xi(1-s)$

$$\xi(1-s) = \xi(0)\prod_{i\in\mathbb{N}}\left(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2} + \frac{(1-s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}\right) \tag{28}$$

where $\xi(0) = \frac{1}{2}$, $\sqrt{\alpha_i^2+\beta_i^2} = |\rho_i|$, and ρ_i are in order of increasing $|\rho_i|$, i.e., $|\rho_1| < |\rho_2| \leq |\rho_3| \leq |\rho_4|, \cdots$, together with $\beta_1 < \beta_2 \leq \beta_3 \leq \beta_4, \cdots$.

Step 3: We have by $\xi(s) = \xi(1-s)$ that

$$\xi(0)\prod_{i\in\mathbb{N}}\left(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2} + \frac{(s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}\right) = \xi(0)\prod_{i\in\mathbb{N}}\left(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2} + \frac{(1-s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}\right) \tag{29}$$

where $\xi(0) = \frac{1}{2}$, $\sqrt{\alpha_i^2+\beta_i^2} = |\rho_i|$, and ρ_i are in order of increasing $|\rho_i|$, i.e., $|\rho_1| < |\rho_2| \leq |\rho_3| \leq |\rho_4|, \cdots$, together with $\beta_1 < \beta_2 \leq \beta_3 \leq \beta_4, \cdots$.

Many non-trivial zeros of $\zeta(s)$, i.e., the zeros of $\xi(s)$, have been calculated by hand or by computer. Among others, Riemann found the first three non-trivial zeros [19]. Gram found the first 15 zeros based on Euler-Maclaurin summation [20]. Titchmarsh calculated the 138^{th} to 195^{th} zeros using the Riemann-Siegel formula [21−22]. Here are the first three zeros: $\frac{1}{2} + j14.1347251$, $\frac{1}{2} + j21.0220396$, $\frac{1}{2} + j25.0108575$. Then we certainly have: $|\rho_1| < |\rho_2| \leq |\rho_3| \leq |\rho_4|, \cdots$, together with $\beta_1 < \beta_2 \leq \beta_3 \leq \beta_4, \cdots$.

Next, according to Lemma 3, Eq.(29) is equivalent to

$$(s-\alpha_i)^2 = (1-s-\alpha_i)^2, i \in \mathbb{N}, \text{from 1 to infinity.} \tag{30}$$

Solving Eq.(30), we get the following solutions

Solution 1:

$$(s-\alpha_i) = -(1-s-\alpha_i) \Rightarrow \alpha_i = \frac{1}{2}, i \in \mathbb{N}, \text{from 1 to infinity.} \tag{31}$$

Solution 2:

$$(s-\alpha_i) = 1-s-\alpha_i \Rightarrow \ s = \frac{1}{2} \tag{32}$$

It is obvious that Solution 2 leads to a contradiction that if $s = \frac{1}{2}$, then there exist no complex roots for $\xi(s) = \xi(1-s) = 0$. Therefore, Solution 2 is invalid.

Thus, Solution 1, i.e., $\alpha_i = \frac{1}{2}, i \in \mathbb{N}$, from 1 to infinity, is the only valid solution of Eq.(30), further of Eq.(29), even further of $\xi(s) = \xi(1-s)$. That means all the zeros of the completed zeta function $\xi(s)$ have real part equal to $\frac{1}{2}$, i.e., Statement 2 of RH is true.

According to Lemma 2, we know that the Statement 1 of RH is also true, i.e., All the non-trivial zeros of the Riemann zeta function $\zeta(s)$ have real part equal to $\frac{1}{2}$.

That completes the proof of RH.

Remark 2: By Lemma 1, there are 2 pairs of complex zeros of $\zeta(s)$ simultaneously, i.e., $\rho = \alpha + j\beta, \bar{\rho} = \alpha - j\beta, 1-\rho = 1-\alpha-j\beta, 1-\bar{\rho} = 1-\alpha+j\beta$ are all non-trivial zeroes of $\zeta(s)$. With the proof of RH, i.e., $\alpha = \frac{1}{2}$, these 2 pairs of zeros are actually only one pair, because $\rho = 1-\bar{\rho} = \frac{1}{2}+j\beta, \bar{\rho} = 1-\rho = \frac{1}{2}-j\beta$. Thus Lemma 1 could be modified more precisely as follows.

Lemma 1*: Non-trivial zeroes of $\zeta(s)$, noted as $\rho = \alpha + j\beta$, have the following properties
1) The number of non-trivial zeroes is infinity;
2) $\beta \neq 0$;
3) $0 < \alpha < 1$;
4) $\rho = 1 - \bar{\rho}, \bar{\rho} = 1 - \rho$ are all non-trivial zeroes.

3 Conclusion

The Riemann Hypothesis is proved based on a new expression of the completed zeta function $\xi(s)$, i.e., the infinite product of quadratic factors determined by complex conjugate roots of $\xi(s) = 0$. To be specific, $\xi(s)$ is, at first, expressed as MacLaurin series (infinite polynomial), and further expressed as infinite product by complex conjugate roots of $\xi(s) = 0$, i.e., $\rho_i = \alpha_i + j\beta_i, \bar{\rho}_i = \alpha_i - j\beta_i, 0 < \alpha_i < 1, \beta_i \neq 0, i \in \mathbb{N}$, from 1 to infinity. Next, according to Lemma 3, the functional equation $\xi(s) = \xi(1-s)$ is equivalent to $(s-\alpha_i)^2 = (1-s-\alpha_i)^2, i \in \mathbb{N}$, from 1 to infinity with unique valid solution $\alpha_i = \frac{1}{2}, i \in \mathbb{N}$, from 1 to infinity. That means:

1) All the non-trivial zeros of the Riemann zeta function $\zeta(s)$ have real part equal to $\frac{1}{2}$;

2) All the zeros of the completed zeta function $\xi(s)$ have real part equal to $\frac{1}{2}$.

Then we conclude that the celebrated Riemann Hypothesis is true.

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