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A Proof of the Riemann Hypothesis Based on MacLaurin Expansion and Hadamard Product of the Completed Zeta Function

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Abstract The completed zeta function $\xi(s)$ is expanded in MacLaurin series (infinite polynomial), which can be further expressed as infinite product (Hadamard product) by its complex conjugate zeros $\alpha_i \pm j\beta_i, i \in \mathbb{N}$. Then, according to the functional equation $\xi(s) = \xi(1-s)$, we have

$$\xi(0)\prod_{i=1}^{\infty}\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}\Big(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\Big) = \xi(0)\prod_{i=1}^{\infty}\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}\Big(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\Big)$$

which, by Lemma 3 and Corollary 1, is equivalent to

$$(s-\alpha_i)^2 = (1-s-\alpha_i)^2, i \in \mathbb{N}$$

with solution $\alpha_i = \frac{1}{2}, i \in \mathbb{N}$. Therefore, a proof of the Riemann Hypothesis is achieved.

Keywords Riemann Hypothesis (RH) · Proof · Completed zeta function

Mathematics Subject Classification (2020) 11M26

1 Introduction and the problem description

It has been 162 years since the Riemann Hypothesis (RH) was proposed in 1859 [1]. Many efforts and achievements have been made towards proving this celebrated hypothesis, but it is still an open problem [2−3].

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The Riemann zeta function is the function of the complex variable s, defined in the half-plane $\Re(s) > 1$ by the absolutely convergent series [2]

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

Riemann showed how to extend zeta function to the whole complex plane $\mathbb{C}$ by analytic continuation

$$\zeta(s) = \frac{\pi^{s/2}}{\Gamma(s/2)}\{\frac{1}{s(s-1)} + \int_1^{\infty} (x^{\frac{s}{2}-1} + x^{-\frac{s}{2}-\frac{1}{2}}) \cdot (\frac{\theta(x)-1}{2})dx\} \tag{2}$$

where $\theta(x) = \sum_{-\infty}^{\infty} e^{-n^2\pi x}$, Γ being the Gamma function in the following Weierstrass expression (Meanwhile, there are also Gauss expression, Euler expression, and integral expression of the Gamma function.)

$$\frac{1}{\Gamma(s)} = s \cdot e^{\gamma s} \prod_{n=1}^{\infty} (1 + \frac{s}{n})e^{-s/n} \tag{3}$$

where γ is the Euler-Mascheroni constant.

The connection between the Riemann zeta function and prime numbers can be established through the well-known Euler product.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p (1 - p^{-s})^{-1}, \Re(s) > 1 \tag{4}$$

the product being over the prime numbers p.

As shown by Riemann, $\zeta(s)$ extends to $\mathbb{C}$ as a meromorphic function with only a simple pole at $s = 1$, with residue 1, and satisfies the following functional equation

$$\pi^{-\frac{s}{2}} \Gamma(\frac{s}{2})\zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma(\frac{1-s}{2})\zeta(1-s) \tag{5}$$

The Riemann zeta function $\zeta(s)$ has zeros at the negative even integers: $-2, -4, -6, -8, \cdots$ and one refers to them as the **trivial zeros**. The other zeros of $\zeta(s)$ are the complex numbers, i.e., **non-trivial zeros** [2].

About the non-trivial zeros of $\zeta(s)$, the following results are well established [4].

Lemma 1: Non-trivial zeroes of $\zeta(s)$, noted as $\rho = \alpha + j\beta$ have the following properties
1) The number of non-trivial zeroes is infinity;
2) $\beta \neq 0$;
3) $0 \leqslant \alpha \leqslant 1$;
4) $\rho, \bar{\rho}, 1 - \bar{\rho}, 1 - \rho$ are all non-trivial zeroes.

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For further study, a completed zeta function $\xi(s)$ is defined as

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) \tag{6}$$

It is well-known that $\xi(s)$ is an entire function of order 1. This implies $\xi(s)$ is analytic, and can be expressed as infinite polynomial, in the whole complex plane $\mathbb{C}$.

In addition, replacing s with $1-s$ in Eq.(6), and combining Eq.(5), we have the following functional equation

$$\xi(s) = \xi(1-s) \tag{7}$$

Considering the definition of $\xi(s)$, i.e., Eq. (6), and recalling Eq.(3), the trivial zeros of $\zeta(s)$ are canceled by the poles of $\Gamma(\frac{s}{2})$. The zero of $s-1$ and the pole of $\zeta(s)$ cancel; the zero $s=0$ and the pole of $\Gamma(\frac{s}{2})$ cancel [5−6]. Thus, all the zeros of $\xi(s)$ are exactly the nontrivial zeros of $\xi(s)$. Then we have the following Lemma 2.

Lemma 2: Zeros of $\xi(s)$ coincide with non-trivial zeros of $\zeta(s)$.

According to Lemma 2, the following two statements for RH are equivalent.

Statement 1 of RH: All the non-trivial zeros of $\zeta(s)$ have real part equal to $\frac{1}{2}$.

Statement 2 of RH: All the zeros of $\xi(s)$ have real part equal to $\frac{1}{2}$.

To prove the RH, the natural thinking is to estimate the number of zeros of $\zeta(s)$ in some certain closed areas according to the Argument Principle. Along this train of thought, there are some famous research works. Let $N(T)$ denote the number of zeros of $\zeta(s)$ inside the rectangle: $0 < \alpha < 1, 0 < \beta \le T$, and let $N_0(T)$ denote the number of zeros of $\zeta(s)$ on the line $\alpha = \frac{1}{2}, 0 < \beta \le T$. Selberg proved that there exist positive constants c and T_0, such that $N_0(T) > cN(T), (T > T_0)$[7], later on, Levinson proved that $c \ge \frac{1}{3}$[8], Lou and Yao proved that $c \ge 0.3484$[9], Conrey proved that $c \ge \frac{2}{5}$ [10], Bui, Conrey and Young proved that $c \ge 0.41$ [11], Feng proved that $c \ge 0.4128$ [12].

Two types of infinite expansions of $\xi(s)$, i.e., MacLaurin series (infinite polynomial) and infinite product (Hadamard product) by complex conjugate roots, will be adopted in this paper to open another door to the proof of RH.

The idea is originated from Euler's work on proving that

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{6} \tag{8}$$

This interesting and famous result is deduced mainly based on the above-mentioned two types of infinite expansions, i.e., infinite polynomial and infinite

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product as follows

$$\begin{aligned} \frac{\sin x}{x} &= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots \\ &= (1 - \frac{x^2}{\pi^2})(1 - \frac{x^2}{4\pi^2})(1 - \frac{x^2}{9\pi^2}) \cdots \end{aligned} \tag{9}$$

This paper is also motivated by the fact that $\xi(s)$ can be expressed by the following Hadamard product, which was first proposed by Riemann. However, it was Hadamard [13] who showed the validity of this infinite product expansion.

$$\xi(s) = \xi(0) \prod_{\rho} (1 - \frac{s}{\rho}) \tag{10}$$

where ρ are precisely the non-trivial zeros of the Riemann zeta function $\zeta(s)$. In another word, ρ run over the roots of $\xi(s) = 0$, and $s \in \mathbb{C}$.

2 A proof of RH

This section is planned to give a proof of the Statement 2 of RH. For this purpose, we need the following Lemma 3 and Corollary 1 on infinite product equation.

Lemma 3: Given two infinite products

$$f(s) = \prod_{i=1}^{\infty} \Big(1 + \frac{(s-\alpha_i)^2}{\beta_i^2}\Big) \tag{11}$$

and

$$f(1-s) = \prod_{i=1}^{\infty} \Big(1 + \frac{(1-s-\alpha_i)^2}{\beta_i^2}\Big) \tag{12}$$

where s is a complex variable, $\alpha_i, \beta_i \neq 0, \cdots \in \mathbb{R}$ are all real numbers, $i \in \mathbb{N}$ are integers.

Then we have

$$f(s) = f(1-s) \Leftrightarrow (s-\alpha_i)^2 = (1-s-\alpha_i)^2, i \in \mathbb{N} \tag{13}$$

where " $\Leftrightarrow$ " is the equivalent sign.

Proof: The proof is delivered by mathematical induction.
First, it is obvious that Lemma 3 is true for $i = 1$, i.e.,

$$\Big(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\Big) = \Big(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\Big) \Leftrightarrow (s-\alpha_i)^2 = (1-s-\alpha_i)^2, i \in \mathbb{N} \tag{14}$$

Second, suppose Lemma 3 is true for $i = m$, then we only need to prove Lemma 3 is true for $i = m + 1$.
Then we begin with the following Eq.(15)

$$\begin{aligned}&\prod_{i=1}^{m}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)=\prod_{i=1}^{m}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)\\&\Leftrightarrow\\&(s-\alpha_i)^2=(1-s-\alpha_i)^2, i=1,2,3.\cdots,m\end{aligned} \tag{15}$$

Besides, we have the obvious fact that

$$\begin{aligned}&1+\frac{(s-\alpha_{m+1})^2}{\beta_{m+1}^2}=1+\frac{(1-s-\alpha_{m+1})^2}{\beta_{m+1}^2}\\&\Leftrightarrow\\&(s-\alpha_{m+1})^2=(1-s-\alpha_{m+1})^2\end{aligned} \tag{16}$$

Thus, putting Eq.(15) and Eq.(16) together, we obtain

$$\begin{aligned}&\prod_{i=1}^{m}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)\left(1+\frac{(s-\alpha_{m+1})^2}{\beta_{m+1}^2}\right)=\prod_{i=1}^{m}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)\left(1+\frac{(1-s-\alpha_{m+1})^2}{\beta_{m+1}^2}\right)\\&\Leftrightarrow\\&(s-\alpha_i)^2=(1-s-\alpha_i)^2, i=1,2,3.\cdots,m,m+1\end{aligned} \tag{17}$$

i.e.

$$\begin{aligned}&\prod_{i=1}^{m+1}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)=\prod_{i=1}^{m+1}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)\\&\Leftrightarrow\\&(s-\alpha_i)^2=(1-s-\alpha_i)^2, i=1,2,3.\cdots,m,m+1\end{aligned} \tag{18}$$

Then we conclude that Lemma 3 is true for $i = m + 1$.
Finally, by the principle of mathematical induction, Lemma 3 is true for any natural number i.
That completes the proof of Lemma 3.

Corollary 1: Given two infinite products

$$f(s)=\xi(0)\prod_{i=1}^{\infty}\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right) \tag{19}$$

and

$$f(1-s)=\xi(0)\prod_{i=1}^{\infty}\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right) \tag{20}$$

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where s is a complex variable, $\xi(0) \neq 0, \alpha_i, \beta_i \neq 0, \cdots \in \mathbb{R}$ are all real numbers, $i \in \mathbb{N}$ are integers.
Then we have

$$f(s) = f(1-s) \Leftrightarrow (s-\alpha_i)^2 = (1-s-\alpha_i)^2, i \in \mathbb{N} \tag{21}$$

Proof: The proof of Corollary 1 is similar to that of Lemma 3. Thus, the details are omitted to save space.

Next, we present the proof of RH.

Proof of RH: The details are delivered in three steps as follows.

Step 1: Since $\xi(s)$ is an entire function, it is analytic in the whole complex plane $\mathbb{C}$. Then $\xi(s)$ can be expanded in MacLaurin series (infinite polynomial) at $s = 0$, i.e.

$$\xi(s) = \xi(0) + \xi^{'}(0)s + \frac{\xi^{''}(0)}{2!}s^2 + \cdots + \frac{\xi^{(n)}(0)}{n!}s^n + \cdots, |s| < \infty \tag{22}$$

It is obvious that $\frac{\xi^{(n)}(0)}{n!} = \frac{\xi^{(n)}(s)}{n!}\Big|_{s=0}, n = 0, 1, 2, \cdots$ are all real numbers.
Thus, all the zeros of $\xi(s)$ are the roots of the following infinite algebraic equation with real coefficients.

$$0 = \xi(0) + \xi^{'}(0)s + \frac{\xi^{''}(0)}{2!}s^2 + \cdots + \frac{\xi^{(n)}(0)}{n!}s^n + \cdots \tag{23}$$

According to the well established theory of algebraic equation with real number coefficients, complex roots always come in pairs (complex conjugate pairs). Further, by Lemma 2, all the zeros of $\xi(s)$ are complex numbers. Thus the roots of Eq.(23) can be denoted as complex conjugate pairs, i.e., $\alpha_i \pm j\beta_i, \beta_i \neq 0, i \in \mathbb{N}$.

Therefore, based on Hadamard product Eq.(10), the infinite polynomial Eq.(22) can be rewritten as infinite product by complex conjugate roots $\alpha_i \pm j\beta_i, i \in \mathbb{N}$

$$\begin{aligned}
\xi(s) &= \xi(0)\prod_{i=1}^{\infty}(1-\frac{s}{\alpha_i+j\beta_i})(1-\frac{s}{\alpha_i-j\beta_i}) \\
&= \xi(0)\prod_{i=1}^{\infty}(1-\frac{2\alpha_i s}{\alpha_i^2+\beta_i^2}+\frac{s^2}{\alpha_i^2+\beta_i^2}) \\
&= \xi(0)\prod_{i=1}^{\infty}(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}) \\
&= \xi(0)\prod_{i=1}^{\infty}\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)
\end{aligned} \tag{24}$$

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Remark 1: Actually, at first the author was inspired by Euler's work to prove Eq.(8) based on Eq.(9), and then speculated that $\xi(s)$ could be expanded into factors $\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right), i \in \mathbb{N}$. Subsequently, the author starts with the Hadamard product Eq.(10) to get Eq.(24).

Step 2: Replacing s with $1-s$ in Eq.(24), we obtain the infinite product expression of $\xi(1-s)$

$$\xi(1-s) = \xi(0)\prod_{i=1}^{\infty}\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right) \tag{25}$$

Step 3: We have by $\xi(s) = \xi(1-s)$ that

$$\xi(0)\prod_{i=1}^{\infty}\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right) = \xi(0)\prod_{i=1}^{\infty}\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right) \tag{26}$$

According to Lemma 3 and Corollary 1, Eq.(26) means

$$(s-\alpha_i)^2 = (1-s-\alpha_i)^2, i \in \mathbb{N} \tag{27}$$

Solving Eq.(27), we get the following solutions

Solution 1:

$$(s-\alpha_i) = -(1-s-\alpha_i) \Rightarrow \alpha_i = \frac{1}{2}, i \in \mathbb{N} \tag{28}$$

Solution 2:

$$(s-\alpha_i) = 1-s-\alpha_i \Rightarrow \ s = \frac{1}{2} \tag{29}$$

It is obvious that Solution 2 leads to a contradiction that if $s = \frac{1}{2}$, then there exist no complex roots for $\xi(s) = \xi(1-s) = 0$.

Thus, Solution 1, i.e., $\alpha_i = \frac{1}{2}, i \in \mathbb{N}$, is the only solution of Eq.(27), further of Eq.(26), even further of the functional equation $\xi(s) = \xi(1-s)$. That means all the zeros of the completed zeta function $\xi(s)$ have real part equal to $\frac{1}{2}$.

That completes the proof of the Statement 2 of RH.

Remark 2: According to Lemma 2, we know that the Statement 1 of RH is also true, i.e., All the non-trivial zeros of the Riemann zeta function $\zeta(s)$ have real part equal to $\frac{1}{2}$.

Remark 3: By Lemma 1, there are 2 pairs of complex zeros of $\zeta(s)$ simultaneously, i.e., $\rho = \alpha + j\beta, \bar{\rho} = \alpha - j\beta, 1-\rho = 1-\alpha-j\beta, 1-\bar{\rho} = 1-\alpha+j\beta$. With the proof of RH, i.e., $\alpha = \frac{1}{2}$, these 2 pairs of zeros actually are only one pair, because $\rho = 1 - \bar{\rho} = \frac{1}{2} + j\beta, \bar{\rho} = 1 - \rho = \frac{1}{2} - j\beta$. Thus Lemma 1 could be modified more precisely as follows.

Lemma 1*: Non-trivial zeroes of $\zeta(s)$, noted as $\rho = \alpha + j\beta$ have the following properties
1) The number of non-trivial zeroes is infinity;
2) $\beta \neq 0$;
3) $0 \leqslant \alpha \leqslant 1$;
4) $\rho = 1 - \bar{\rho}, \bar{\rho} = 1 - \rho$ are all non-trivial zeroes.

3 Conclusion

The Riemann Hypothesis is proved based on a new expression of the completed zeta function $\xi(s)$, i.e., the infinite product of quadratic factors determined by complex conjugate roots of $\xi(s) = 0$. To be specific, $\xi(s)$ is, at first, expressed as MacLaurin series (infinite polynomial), and further expressed as infinite product by complex conjugate roots of $\xi(s) = 0$, i.e., $\alpha_i \pm j\beta_i, i \in \mathbb{N}$. Next, according to Lemma 3 and Corollary 1, the functional equation $\xi(s) = \xi(1-s)$ is equivalent to $(s - \alpha_i)^2 = (1 - s - \alpha_i)^2, i \in \mathbb{N}$ with solution $\alpha_i = \frac{1}{2}, i \in \mathbb{N}$. That means: 1) All the non-trivial zeros of the Riemann zeta function $\zeta(s)$ have real part equal to $\frac{1}{2}$; 2) All the zeros of the completed zeta function $\xi(s)$ have real part equal to $\frac{1}{2}$.

Then we conclude that the celebrated Riemann Hypothesis is true.

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