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Time-Delay Synchronization and anti-Synchronization of Variable-Order Fractional Discrete-Time of Chen-Rossler Chaotics Systems Using Variable-Order Fractional-Discrete Time PID Control

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Abstract In this research article we solve the problem of synchronization and anti-synchronization of chaotic systems described by discrete and time-delayed variable fractional order differential equations. To guarantee the synchronization and anti-synchronization of these systems, we use the well-known PID control theory and the Lyapunov-Krasovskii stability theory for discrete systems of variable fractional order.

We illustrate the results obtained through simulation with examples, in which it can be seen that our results are satisfactory, thus achieving synchronization and anti-synchronization of chaotic systems of variable fractional order with discrete time delay.

Keywords Variable-order fractional-discrete time systems; Synchronization and Anti-Synchronization; Lyapunov-Krasovskii Stability; Fractional Order Caputo Derivative; Time-Delay Fractional-Discrete Systems; Fractional Order Discrete Time PID Control

1. Introduction

We present in this research article, the solution to the problem of synchronization [1] and anti-synchronization [2] of discrete chaotic systems described by systems of differential equations of variable fractional order [3] and with time delay [4], where this analysis is carried out for non-linear systems in the sense of the derivative of Caputo for systems of variable fractional order [3].

The dynamics of systems is a branch of mathematics, which studies the performance of physical phenomena in time, which are mathematically modeled by means of differential equations or finite differences, depending on whether the system is in continuous or discrete time, respectively.

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In 1963 E. Lorentz, studying the behavior of the climate, proposed a mathematical model which bears his name, the Lorentz chaotic attractor, which is sensitive to initial conditions and variations in its parameters. This system drastically changed its behavior, so predicting the climate with this mathematical model was impossible, currently there are a great variety of chaotic systems such as the attractor of Chua, Chen, Rossler, Duffing, Lu and Bhalekar-Gejji, etc. These chaotic systems have been extensively studied.

For example, in the pioneering works of Pecora and Carroll, they synchronized two identical chaotic attractors with different initial conditions, this was the first time this study was carried out. At present, chaotic systems have attracted a large number of researchers and the results obtained have a wide range of applications, for example, in encryption, synchronization, anti-synchronization, secure information transfer by electronic means, etc. Lately, the study of these chaotic systems described by first-order differential equations has become generalized to systems of differential equations of variable fractional order, discrete with time delay, which is our case study, and the systems are not the same, where one system is Chen's chaotic system, which we will refer to as the master system, a term widely used in synchronization, and the other chaotic system, is the Rossler system, which we refer to as the slave system.

We refer, in this paper, as the master-slave system, even though the results obtained are for these two systems, the methodology can be used for other non-linear discrete time systems of fractional order variable with time delay in the Caputo sense.

In this investigation, the Rossler system is forced to follow (synchronize) and anti-synchronize with the chaotic Chen system, both systems described, as mentioned above by means of discrete and variable fractional order differential equations with time delay, synchronization and Anti-synchronization are obtained by discrete fractional PID control laws [5] and using the stability theory by Lyapunov Krasovskii [6], as can be seen in the illustrations, the results are satisfactory and the analytical results agree with the results obtained by means of simulation Via Simulink-MatLab.

In this article, we do not discretize the systems, we work with the non-linear system, under the conditions indicated on variable-order fractional [7] discrete-time non-linear systems [8].

This article is organized as follows:

In section 2, the problem of synchronization of the aforementioned systems is raised.

In section 3, the problem of anti-synchronization of the systems, also mentioned above, is raised.

In section 4, the synchronization of the aforementioned chaotic systems is analyzed and a control law is obtained by Lyapunov-Krasovskii stability analysis and a fractional order discrete PID control law.

In section 5, the anti-synchronization of the aforementioned chaotic systems is analyzed and a control law is obtained by means of the Lyapunov-Krasovskii stability analysis and a fractional order discrete PID control law.

In section 6, examples of synchronization of the chaotic systems of Chen (Master) are presented, see (10) and the chaotic system of Rossler (Slave) see (11), where the simulations were carried out in Simulink-MatLab.

In section 7, examples of anti-synchronization of the Chen chaotic systems (Master) are presented, see (10) and the Rossler chaotic system (Slave) see (11), where the simulations were carried out in Simulink-MatLab.

2. Statement of the problem for time-delay synchronization of variable-order fractional discrete-time chaotic system

In this section we present the problem of synchronization between two different chaotic systems and the part (4) we solve the problem of synchronization, the system of Chen, which we will refer to as the master system, which is described by:

$$\nabla^{\alpha_i} x(k+1) = [Px_m + f(x_m)]\Delta\ , \quad \text{where} \quad P = \begin{pmatrix} -35 & 35 & 0 \\ -7 & 28 & 0 \\ 0 & 0 & -3 \end{pmatrix}; \quad X_m = (x_1, x_2, x_3)^T$$

and $f(x_m) = (0, -x_1x_3, x_1x_2)^T$,

and Rossler's system as the slave system, his equations are in the form of a time-delayed discrete variable fractional order.

$$\nabla^{\alpha_i} y(k+1) = [Qy_s(t-\tau) + g(y_s(t-\tau)) + U]\Delta\ , \quad \text{where} \quad Q = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0.2 & 0 \\ 0 & 0 & -5.7 \end{pmatrix};$$

$Y_s(t-\tau) = (y_1(t-\tau), y_2(t-\tau), y_3(t-\tau))^T$ and $g(Y_s) = (0, 0,\ 0.2 + y_1y_2)^T$, and $\tau > 0$.

Consider the following, that the variable-order fractional derivatives are variable with constant values [9], and [10], a chaotic system as a drive system having state vector $X_m \in \mathbb{R}^n$ and $P \in \mathbb{R}^{nxn}$, with $n = 3$, is the master system matrix, given by:

$$\nabla^{\alpha_i} x(k+1) = f\big(x(k)\big) - x(k)$$

$$X_m(k+1) - X_m(k) = [P(X_m) + f(X_m)]\Delta \qquad (1)$$

Consider another chaotic system as a slave system having state vector $Y_s \in \mathbb{R}^n$, and $Q \in \mathbb{R}^{nxn}$, $n = 3$, is the slave system matrix given as:

$$\nabla^{\alpha_i} y(k+1) = g\big(y(k)\big) - y(k)$$

$$Y_s(k+1) - Y_s(k) = [Q(Y_s(t-\tau)) + g(Y_s(t-\tau)) + U]\Delta \qquad (2)$$

Where g is nonlinear part of the slave system as in (11), and U is nonlinear active controller added in (2) for synchronization action. Synchronization error $e \in \mathbb{R}^n$ between X_m and Y_s is defined as:

$$e = Y_s - X_m \qquad (3)$$

Substituting (1) and (2) in the dynamics of the synchronization error (3) we obtain:

:

$$\nabla^{\alpha_i} e(k+1) = f\big(e(k)\big) - e(k)$$

$$\nabla^{\alpha_i} e = \nabla^{\alpha_i} Y_s - \nabla^{\alpha_i} X_m$$

$$[Y_s(k+1) - Y_s(k)] - [X_m(k+1) - X_m(k)] = \qquad (4)$$

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$$\{[Q(Y_s(t-\tau)) + g(Y_s(t-\tau)) + U] - [\,P(X_m) + f(X_m)]\}\Delta$$

Therefore, the synchronization problem is to determine the nonlinear controller U, so that:

$$\lim_{t\to\infty}\|e(t)\| = 0 \qquad (5)$$

To demonstrate the above, we consider a positive definite Lyapunov function as:

$$V(e) = \frac{1}{2}\sum_{k=1}^{n} e_k^2 \qquad (6)$$

Where e_k is the $(k-th)$ error of the state, and our objective is to determine a control action U such that the Lyapunov-Krasovskii derivative $\Delta(V(e)) < 0$, is negative definite, with which it is guaranteed that the synchronization error tends to zero when t tends to infinity and therefore the systems are globally asymptotically synchronized.

We use the derivative function, given in definition 2.1.3, page 104, of Louis Leithold's seventh edition, are given as:

$$\dot{\boldsymbol{f}} = \boldsymbol{f}'(x) = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}, \text{ if this limit exists.}$$

Assuming that, the first partial time derivative of e_k exist, then

$$\Delta(V(e_k)) = \sum_{k=1}^{n}\frac{1}{2}\lim_{\Delta\to 0}\frac{(e_k+\Delta)^2 - {e_k}^2}{\Delta} =$$

$$\sum_{k=1}^{n}\left[\frac{1}{2}\lim_{\Delta\to 0}\frac{{e_k}^2 + 2e_k(\Delta) + (\Delta)^2 - {e_k}^2}{\Delta}\right] =$$

$$\sum_{k=1}^{n}\left[\frac{1}{2}\lim_{\Delta\to 0}\frac{2e_k(\Delta) + (\Delta)^2}{\Delta}\right]$$

Adding and subtracting e_k we have

$$\Delta(V(e_k)) =$$

$$\sum_{k=1}^{n}\left[\frac{1}{2}\lim_{\Delta\to 0}\frac{2e_k[(e_k+(\Delta)-e_k] + (\Delta)^2}{\Delta}\right]$$

$$\Delta(V(e_k)) =$$

$$\sum_{k=1}^{n}\left[\frac{2e_k}{2}\lim_{\Delta\to 0}\frac{[(e_k+(\Delta))-e_k]}{\Delta}\right] +$$

$$\sum_{k=1}^{n}\left[\lim_{\Delta\to 0}\frac{[\Delta^2]}{\Delta}\right] = \sum_{k=1}^{n} e_k\dot{e}_k$$

For our purpose, we can use, in this paper the next inequality widely used in fractional order control systems:

$$\frac{1}{2}\,{}_{t_0}^{C}D_t^{\alpha} e_{\mathrm{k}}^{2}(t) \leq e(t)\,{}_{t_0}^{C}D_t^{\alpha} e(t), \tag{7}$$

$\forall\, \alpha \in (0,1)$, you can see the references [8], [11], [12].

We will find U such that $\Delta(\mathrm{V}(\mathrm{e_k})) < 0$, is negative definite and since $V(e,t) \to \infty$ as $e(t) \to \infty$, then the error is globally asymptotically stable. The states drive and response system, are globally asymptotically synchronized.

In the next section, the anti-synchronization problem is discussed for the chaotic system.

3. Problem statement for time-delay anti-synchronization of variable-order fractional discrete-time chaotic system

In this section, we will denote the anti-synchronization error of the of the aforementioned systems by $e_{as} \in \mathbb{R}^n$, in our case, $n = 3$, and in the section 5 we solve the problem of anti-synchronization, between their states X_m and Y_s, this error is defined for the states system (1) and response system (2) by:

$$e_{as} = Y_s(t-\tau) + X_m \tag{8}$$

Substituting (1) and (2) in the dynamics of the anti-synchronization error (8) we obtain:

$$\nabla^{\alpha_i}\, x(k+1) = f(x(k)) - x(k)$$

$$X_m(k+1) - X_m(k) = [P(X_m) + f(X_m)]\Delta$$

$$\nabla^{\alpha_i}\, y(k+1) = f(y(k)) - y(k)$$

$$Y_s(k+1) - Y_s(k) = [Q(Y_s(t-\tau)) + g(Y_s(t-\tau)) + U]\Delta$$

From (8) we get the fractional variable order derivative:

$$\nabla^{\alpha_i} e_{as} = \nabla^{\alpha_i}\, Y_s(t-\tau) + \nabla^{\alpha_i} X_m$$

$$[Y_s(k+1) - Y_s(k)] + [X_m(k+1) - X_m(k)] = \tag{9}$$

$$\{[Q(Y_s(t-\tau)) + g(Y(t-\tau)_s) + U] + [\,P(X_m) + f(X_m)]\}\Delta$$

The anti-synchronization problem is to determine the nonlinear control U, satisfies $\lim_{t\to\infty}\|e(t)\| = 0,\ \forall\, e_{as}(t) \in \mathbb{R}^3$.

To achieve the goal that the anti-synchronization error tends to zero, we define the following positive definite Lyapunov-Krasovskii function:

$$V(\mathrm{e_{as}}) = \frac{1}{2}\sum_{\mathrm{k}=1}^{\mathrm{n}} e_{as_k}^2$$

With the assumption that the parameters of drive and response systems are known and the states are measurable. The problem is to find U such that the derivative of $V(\mathrm{e_{as}})$ exist and will be negative definite, and using the inequality (7) we have:

$$\Delta(V(\mathrm{e_{as}})) = \left[\sum_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{e}_{\mathrm{as}_\mathrm{k}} \dot{\mathrm{e}}_{\mathrm{as}_\mathrm{k}}\right] < 0$$

We will find U such that $\Delta(\mathrm{V}(\mathrm{e}_{\mathrm{as}})) < 0$, is negative definite and since $V(\mathrm{e}_{\mathrm{as}}, t) \to \infty$ as $\mathrm{e}_{\mathrm{as}}(t) \to \infty$, then the error is globally asymptotically stable. The states drive and response system, are globally asymptotically anti-synchronized.

In the next section we will determine the control law$U$, which is obtained by means of the Lyapunov-Krasovskii function, previously defined.

4. Time-Delay Variable-Order Fractional Discrete-Time chaotic systems for Synchronization of Chen and Rossler

In this section we solve the problem of synchronization of the discrete-time Chen system and discrete-time Rossler systems are considered as master and slave respectively. The discrete-time Chen system dynamics is given as:

$$\nabla^{\alpha_i} x(k+1) = f(x(k)) - x(k),$$
$$\alpha_i = 1, 2, 3$$
$$\alpha_1 = 0.9, \alpha_2 = 0.8, \alpha_3 = 0.7$$
$$x_1(k+1) - x_1(k) = [35(x_2 - \mathrm{x}_1)]\Delta$$
$$x_2(k+1) - x_2(k) = (-7x_1 - \mathrm{x}_1\mathrm{x}_3 + 28x_2)\Delta \quad (10)$$
$$x_3(k+1) - x_3(k) = (-3x_3 + \mathrm{x}_1\mathrm{x}_2)\Delta$$

Where x_1, x_2, x_3 are the states (10). The phase portrait for the chaotic Chen system is given in the Fig. 1.

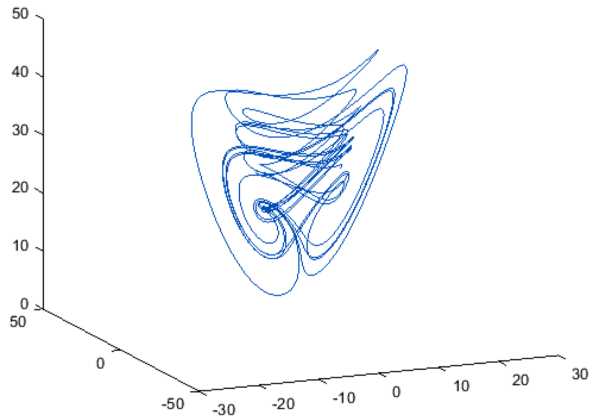

Figure 1. Phase portrait for the discrete-time of Chen chaotic system.

The discrete-time slave system is chosen as a Time-Delay discrete-time Rossler chaotic system. The dynamics of the discrete-time Rossler chaotic system is given as:

$$\nabla^{\alpha_i} y(k+1) = f(y(k)) - y(k),$$
$$\alpha_i = 1, 2, 3$$
$$\alpha_1 = 0.9, \alpha_2 = 0.8, \alpha_3 = 0.7$$

$$y_1(k+1) - y_1(k) =$$
$$[(-y_2(\mathrm{t}-\tau) - y_3(\mathrm{t}-\tau) + P_1 I_1 D_1 + \mathrm{u}_1)]\Delta$$

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$$y_2(k+1) - y_2(k) =$$
$$[(y_1(\mathrm{t}-\tau) + 0.2\mathrm{y}_2(\mathrm{t}-\tau)) + P_2I_2D_2 \ + \mathrm{u}_2]\Delta \quad (11)$$
$$y_3(k+1) - y_3(k) =$$
$$[0.2 - 5.7y_3(\mathrm{t}-\tau) + \mathrm{y}_1(\mathrm{t}-\tau)\mathrm{y}_3(\mathrm{t}-\tau) + P_3I_3D_3 \ + \mathrm{u}_3]\Delta$$

Where y_1, y_2, y_3 are the states of (11). The phase portait for system (11) with $u_i = 0$, and $P_iI_iD_i = 0, \ \forall\, i$ is given in Fig. 2.

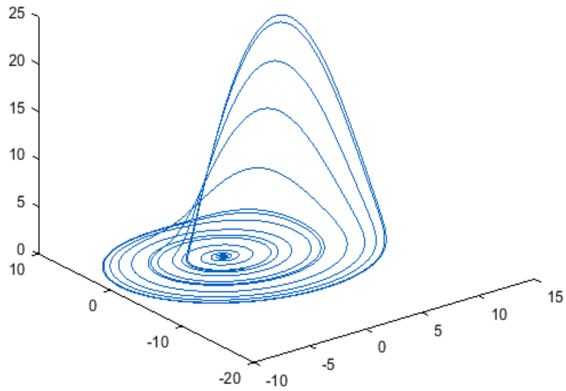

Figure 2. The phase portrait for the discrete-time chaotic Rossler system.

The synchronization error $e \in \mathbb{R}^3$ is defined as:

$$e_i = y_i(\mathrm{t}-\tau) - x_i\,, i = 1,2,3 \quad (12)$$

The error dynamics equations are obtained as follows:

$$\nabla^{\alpha_i} e(k+1) = f\big(e(k)\big) - e(k),$$
$$\alpha_1 = 1, 2,3$$
$$\alpha_i = 0.9, \alpha_2 = 0.8, \alpha_3 = 0.7$$

In this paper we use the Discrete-Time Fractional-Order PID Controller [13].

Where $P_iI_iD_i, i = 1,2,3,$ for each control $u_i, i = 1,2,3$

$$u_1, u_2, u_3$$

$$PID = K_p + K_d \sum_{k=0}^{M} f_k(\mu) z^{-k} + K_i \frac{1+z^{-1}}{1+z^{-1}} \sum_{k=0}^{M} f_k(1-\lambda) z^{-k}$$

And $K_p = k_p, K_d = k_d \alpha^{\mu}, K_i = k_i \alpha^{-\lambda}$

$$e_1(k+1) - e_1(k) =$$
$$y_1(k) + [(-y_2(\mathrm{t}-\tau) - y_3(\mathrm{t}-\tau) + \mathrm{u}_1 + P_1I_1D_1)]\Delta - \{\mathrm{x}_1(k) + [35(x_2 - \mathrm{x}_1)]\Delta\} =$$
$$[-(y_1(\mathrm{t}-\tau) - x_1) + (y_1(\mathrm{t}-\tau) - x_1) - y_2(\mathrm{t}-\tau) - y_3(\mathrm{t}-\tau) - 35x_2 + 35x_1 + u_1 + P_1I_1D_1]\Delta +$$
$$y_1(k) - x_1(k) =$$
$$[-e_1 + (y_1(\mathrm{t}-\tau) - x_1) - y_2(\mathrm{t}-\tau) - y_3(\mathrm{t}-\tau) - 35x_2 + 35x_1 + u_1 + P_1I_1D_1\,]\Delta + y_1(k) - x_1(k) \quad (13)$$

$$e_2(k+1) - e_2(k) =$$
$$y_2(k) + [\mathrm{y}_1(\mathrm{t}-\tau) + 0.2\mathrm{y}_2(\mathrm{t}-\tau) + P_2I_2D_2 + \mathrm{u}_2]\Delta - \{x_2(k) + [-7x_1 - \mathrm{x}_1\mathrm{x}_3 + 28x_2]\Delta\} =$$
$$\{-(y_2(\mathrm{t}-\tau) - x_2) + (y_2(\mathrm{t}-\tau) - x_2) + [y_1(\mathrm{t}-\tau) + 0.2\mathrm{y}_2(\mathrm{t}-\tau) + P_2I_2D_2 + u_2 + 7x_1 + \mathrm{x}_1\mathrm{x}_3 - 28x_2]\}\Delta +$$
$$y_2(k) - x_2(k) =$$
$$[-e_2 + y_2(\mathrm{t}-\tau) - x_2 \ + y_1(\mathrm{t}-\tau) + 0.2\mathrm{y}_2(\mathrm{t}-\tau) + +P_2I_2D_2 \ + u_2 + 7x_1 + \mathrm{x}_1\mathrm{x}_3 - 28x_2]\Delta + y_2(k) - x_2(k)$$

$$e_3(k+1)-e_3(k)=$$
$$y_3(k)+[0.2-5.7y_3(\mathrm{t}-\tau)+\mathrm{y}_1(\mathrm{t}-\tau)\mathrm{y}_3(\mathrm{t}-\tau)+P_3I_3D_3+\mathrm{u}_3]\Delta-[\mathrm{x}_3(\mathrm{k})+(-3x_3+\mathrm{x}_1\mathrm{x}_2)\Delta]=$$
$$\{-(y_3(\mathrm{t}-\tau)-x_3)+(y_3(\mathrm{t}-\tau)-x_3)+$$
$$0.2-5.7y_3(\mathrm{t}-\tau)+\mathrm{y}_1(\mathrm{t}-\tau)\mathrm{y}_3(\mathrm{t}-\tau)+P_3I_3D_3+\mathrm{u}_3+3x_3-\mathrm{x}_1\mathrm{x}_2\}\Delta+y_3(k)-x_3(k)=$$
$$[-e_3+(y_3(\mathrm{t}-\tau)-x_3)+$$
$$0.2-5.7y_3(\mathrm{t}-\tau)+\mathrm{y}_1(\mathrm{t}-\tau)\mathrm{y}_3(\mathrm{t}-\tau)+P_3I_3D_3+\mathrm{u}_3+3x_3-\mathrm{x}_1\mathrm{x}_2\}\Delta+y_3(k)-x_3(k)$$

We need to find the nonlinear active control law for u_i, $\forall\, i$ in such a manner that the error dynamics of (13) is globally asymptotically stable. Let

$$\begin{aligned}u_1&=(-y_1(\mathrm{t}-\tau)+x_1)+y_2(\mathrm{t}-\tau)+y_3(\mathrm{t}-\tau)+35x_2-35x_1-P_1I_1D_1\\ u_2&=-y_2(\mathrm{t}-\tau)+x_2-y_1(\mathrm{t}-\tau)-0.2\mathrm{y}_2(\mathrm{t}-\tau)-P_2I_2D_2-7x_1-\mathrm{x}_1\mathrm{x}_3+28x_2\\ u_3&=(-y_3(\mathrm{t}-\tau)+x_3)-0.2+5.7y_3(\mathrm{t}-\tau)-\mathrm{y}_1(\mathrm{t}-\tau)\mathrm{y}_3(\mathrm{t}-\tau)-P_3I_3D_3-3x_3+\mathrm{x}_1\mathrm{x}_2\end{aligned} \quad (14)$$

Substituting the controller dynamics (14) in error dynamics (13), we have error dynamics as:

$$\begin{aligned}e_1(k+1)-e_1(k)&=[-e_1]\Delta+y_1(k)-x_1(k)\\ e_2(k+1)-e_2(k)&=[-e_2]\Delta+y_2(k)-x_2(k)\\ e_3(k+1)-e_3(k)&=[-e_3]\Delta+y_3(k)-x_3(k)\end{aligned}$$

$$\begin{aligned}e_1(k+1)-e_1(k)&=[-e_1]\Delta+e_1\\ e_2(k+1)-e_2(k)&=[-e_2]\Delta+e_2\\ e_3(k+1)-e_3(k)&=[-e_3]\Delta+e_3\end{aligned} \quad (15)$$

The synchronization problem is to determine the nonlinear controller U, so that:

$$\lim_{t\to\infty}\|e(t)\|=0$$

To show that the previous limit is satisfied, we make use of the following positive definite Lyapunov-Krasovskii function as [14], [15] and [16]:

$$V(e)=\frac{1}{2}(e_1^2+e_2^2+e_3^2)+\int_{t-\tau}^{t}g(x(\zeta))d\zeta \quad (16)$$

This Lyapunov-Krasovskii function is defined for systems that are continuous in time, and for discrete systems, which is our case study, the integral in (16) is replaced by the following summation, the function thus obtained is called the function of Lyapunov-Krasovskii for discrete systems in time.

$$V_1(e_t)=\sum_{i=t-h}^{t-1}e^T(i)Qe(i)$$

Here we use the following known inequality in fractional order systems

$$\frac{1}{2}\,{}_{t_0}^{C}D_t^{\alpha}\mathrm{e_k}^2(t)\le e(t)\,{}_{t_0}^{C}D_t^{\alpha}e(t),\ \ \forall\,\alpha\in(0,1),$$

Assuming first order partial derivatives of (16) exists, we obtained, using the procedure in (7) we have

$$\Delta\big(V(e)\big)=e_1\dot{e}_1+e_2\dot{e}_2+e_3\dot{e}_3+e^T(t)Qe(t)-e^T(t-h)Qe(t-h) \quad (17)$$

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Substituting (15) in (17), we obtain

$$\Delta\big(V(e)\big) = -e_1^2\Delta + \mathrm{e}_1^2 - e_2^2\Delta + \mathrm{e}_2^2 - e_3^2\Delta + \mathrm{e}_3^2 \quad + e^T(t)Qe(t) - e^T(t-h)Qe(t-h)$$

$$\Delta\big(V(e)\big) = -e_1^2\Delta + \|\mathrm{e}_1\| - e_2^2\Delta + \|\mathrm{e}_2\| - e_3^2\Delta + \|\mathrm{e}_3\| + \|e(t-h)\| \tag{18}$$

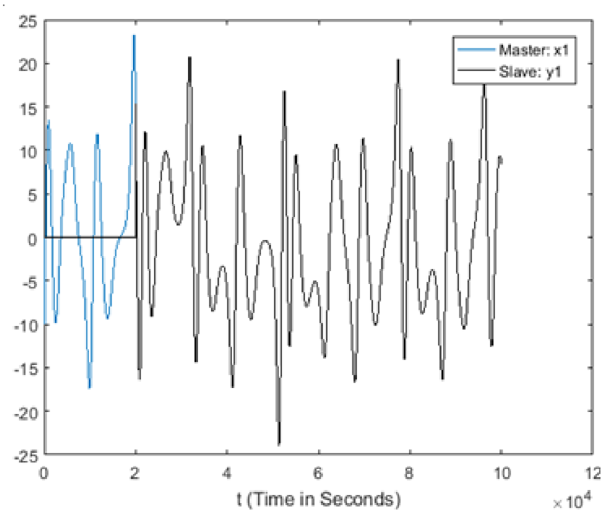
$$\Delta\big(V(e)\big) = -e_1^2\Delta - e_2^2\Delta - e_3^2\Delta < 0$$

Since $\Delta\big(V(e)\big)$ is negative definite. For the Lyapunov stability theory, the error dynamics (15) is globally asymptotically stable and the error dynamics will converge to zero as $t \to \infty$ with the control law in (14). The chaotic systems Chen (10) and Rossler (11) are globally asymptotically synchronized for any initial condition.

The analytical results obtained through examples developed via simulation are illustrated below for sinchronization:

Chen and Rossler systems are simulated in simulink matlab using the control law $\boldsymbol{U}$ (14) for synchronization, The initial conditions for these systems are $\boldsymbol{x}(\mathbf{0}) = [-\mathbf{10}, \mathbf{0}, \mathbf{37}]^{\boldsymbol{T}}$ and $\boldsymbol{y}(\mathbf{0}) = [\mathbf{0.1}, \mathbf{0}, \mathbf{0}]^{\boldsymbol{T}}$, Respectively for simulation:

Time evolution of the states of the Chen and Rossler systems for synchronization with time delayed:



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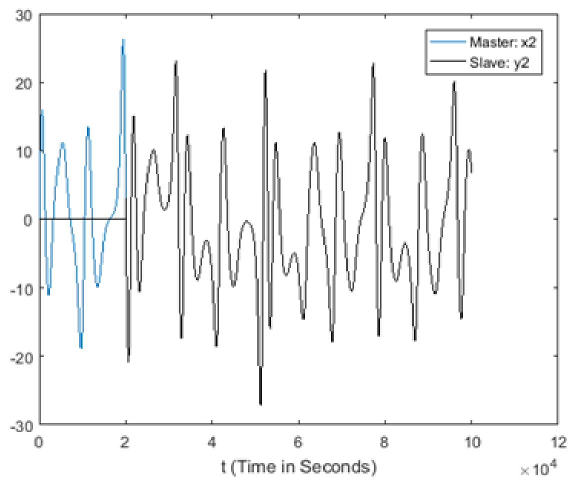

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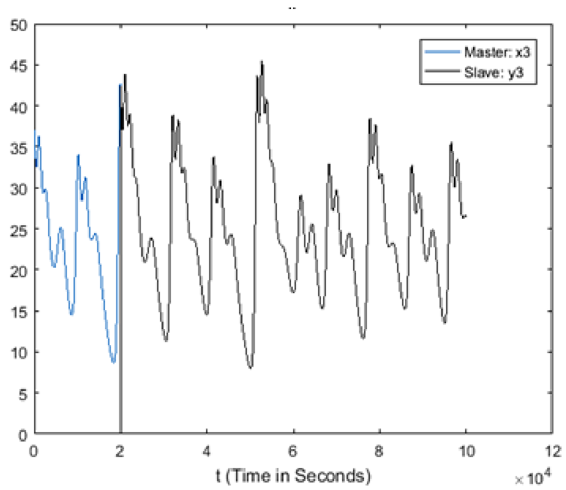

c

Figure 3. Time response of synchronized states of master and slave.

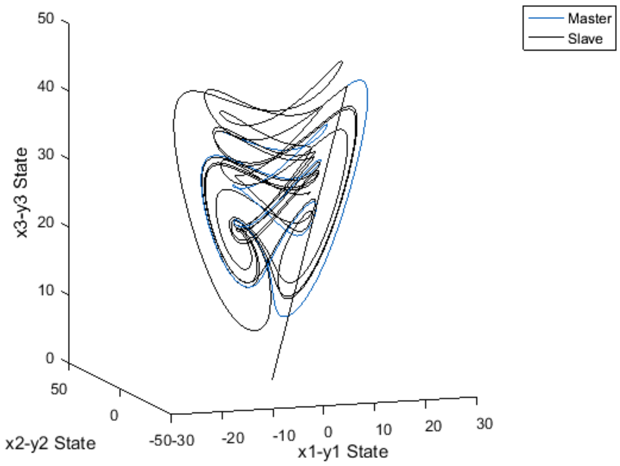

Figure 4. Phase space of the synchronization of the original master slave systems.

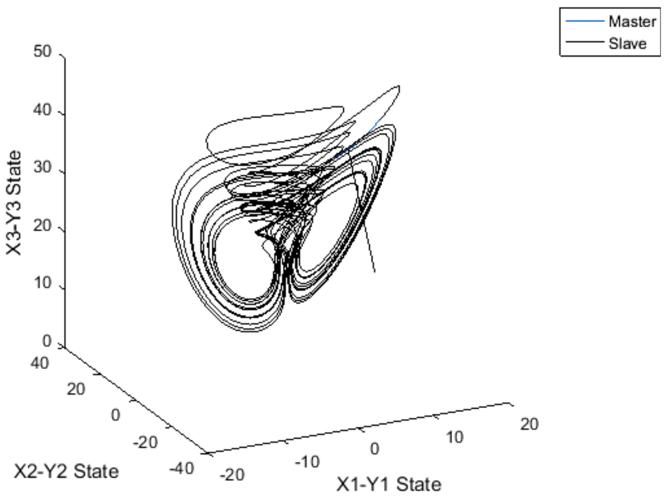

Figure 5. Phase space of the synchronization of the master-slave system whit fractional order given by $c = 0.9$, $c1 = 0.8$, $c2 = 0.7$.

For these simulations we use a $\Delta = 0.001$, and $\tau = 20\ sec$. Synchronization errors of states are shown in the **figure 6**.

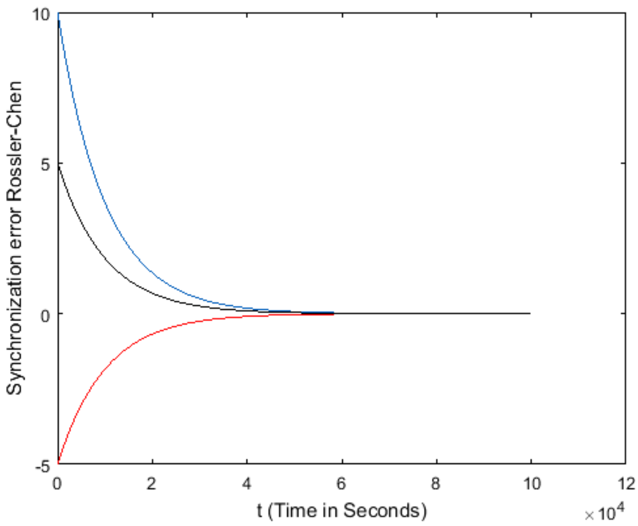

Figure 6. Synchronization errors with time-delay between the states of master and slave systems of variable fractional order derivadtve.

5. Variable-Order Fractional Discrete-time chaotic system for Anti- synchronization of Chen and Rossler

In this section we solve the problem of anti-synchronization of the discrete-time Chen system and discrete-time Rossler systems are considered as master and slave respectively. The discrete-time Chen system dynamics is given as:

The discrete-time anti-synchronization error $e_{as} \in \mathbb{R}^3$ is defined as:

$$e_{as_i} = Y_s + X_m, \quad i = 1, 2, 3 \tag{19}$$

From (1), (2) and (19), the dynamics of the error is:

$$\begin{gathered}\dot{e}_{as} = \dot{Y}_s + \dot{X}_m = \\ [Y_s(k+1) - Y_s(k)] + [X_m(k+1) - X_m(k)] = \\ \{[Q(Y_s) + g(Y_s) + U] + [P(X_m) + f(X_m)]\}\Delta\end{gathered}$$

The anti-synchronization problem is to determine the non.linear control U, satisfies $\lim_{t\to\infty}\|e(t)\| = 0, \ \forall\, e_{as_i}(t) \in \mathbb{R}^n$.

Consider a positive definite Lyapunov function,

$$V(\mathrm{e}_{\mathrm{as_i}}) = \frac{1}{2}\sum_{\mathrm{k=1}}^{\mathrm{n}} e_{as_k}^2$$

and using the procedure in (7) we have:

$$\Delta(V(\mathrm{e_{as}})) = [\textstyle\sum_{\mathrm{k=1}}^{\mathrm{n}} \mathrm{e}_{\mathrm{as_k}}\dot{\mathrm{e}}_{\mathrm{as_k}}]$$

$$\frac{1}{2}\,{}_{t_0}^{C}D_t^{\alpha}\,\mathrm{e}_{\mathrm{as_k}}{}^2(t) \le e(t)\,{}_{t_0}^{C}D_t^{\alpha}\,\mathrm{e}_{\mathrm{as_k}}{}^2(t), \ \forall\, \alpha \in (0,1),$$

With $V(\mathrm{e_{as}}) \to \infty$ as $\|\mathrm{e_{as}}(\mathrm{t})\| \to \infty$, then e_{as} is globally asymptotically stable, the states and response systems are globally asymptotically synchronized.

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The discrete-time Chen system and discrete-time Rossler systems are considered as master and slave respectively. The discrete-time Chen system dynamics is given as:

$$\nabla^{\alpha_i} x(k+1) = f\big(x(k)\big) - x(k),$$
$$\alpha_i = 1, 2,3$$
$$\alpha_1 = 0.9, \alpha_2 = 0.8, \alpha_3 = 0.7$$

$$x_1(k+1) - x_1(k) = [35(x_2 - \mathrm{x}_1)]\Delta$$
$$x_2(k+1) - x_2(k) = (-7x_1 - \mathrm{x}_1\mathrm{x}_3 + 28x_2)\Delta$$
$$x_3(k+1) - x_3(k) = (-3x_3 + \mathrm{x}_1\mathrm{x}_2)\Delta$$

Where $x_1,\ x_2,\ x_3$ are the states (10).

The discrete-time slave system is chosen as a discrete-time Rossler chaotic system. The dynamics of the discrete-time Rossler chaotic system is given as:

$$\nabla^{\alpha_i} y(k+1) = f\big(y(k)\big) - y(k),$$
$$\alpha_i = 1, 2,3$$
$$\alpha_1 = 0.9, \alpha_2 = 0.8, \alpha_3 = 0.7$$

$$y_1(k+1) - y_1(k) = [(-y_2(\mathrm{t}-\tau) - y_3(\mathrm{t}-\tau) + P_1 I_1 D_1 + \mathrm{u}_1)]\Delta$$
$$y_2(k+1) - y_2(k) = [(y_1(\mathrm{t}-\tau) + 0.2\mathrm{y}_2(\mathrm{t}-\tau)) + P_2 I_2 D_2 + \mathrm{u}_2]\Delta$$
$$y_3(k+1) - y_3(k) = [0.2 - 5.7y_3(\mathrm{t}-\tau) + \mathrm{y}_1(\mathrm{t}-\tau)\mathrm{y}_3(\mathrm{t}-\tau) + P_3 I_3 D_3 + \mathrm{u}_3]\Delta$$

Where y_1, y_2, y_3 are the states of the system (11). The phase plane for system (11) with $u_i = 0$, and $P_i I_i D_i = 0\ \ \forall\, i$, where u_1, u_2, u_3 are the active nonlinear controllers to be designed.

The anti-synchronization error $e \in \mathbb{R}^3$ is defined as:

$$e_{as_i} = y_i(\mathrm{t}-\tau) + \mathrm{u_i} + x_i\,, i = 1,2,3$$

The error dynamics equations are obtained as follows:

$$\nabla^{\alpha_i} e_{as_i}(k+1) = f\left(e_{as_i}(k)\right) - e_{as_i}(k),$$
$$\alpha_i = 1, 2,3$$
$$\alpha_1 = 0.9, \alpha_2 = 0.8, \alpha_3 = 0.7$$

$$\begin{gathered} e_{as_1}(k+1) - e_{as_1}(k) = y_1(k) + [(-y_2(\mathrm{t}-\tau) - y_3(\mathrm{t}-\tau) + \mathrm{u}_1 + P_1 I_1 D_1)]\Delta \\ +\{\mathrm{x}_1(k) + [35(x_2 - \mathrm{x}_1)]\Delta\}= \\ [-(y_1(\mathrm{t}-\tau) + x_1) + (y_1(\mathrm{t}-\tau) + x_1) - y_2(\mathrm{t}-\tau) - y_3(\mathrm{t}-\tau) + 35x_2 - 35x_1 + u_1 + P_1 I_1 D_1]\Delta + \\ y_1(k) + x_1(k) = \\ [-e_1 + (y_1(\mathrm{t}-\tau) + x_1) - y_2(\mathrm{t}-\tau) - y_3(\mathrm{t}-\tau) + 35x_2 - 35x_1 + u_1 + P_1 I_1 D_1\,]\Delta + \\ y_1(k) + x_1(k) \end{gathered} \tag{20}$$

$$\begin{gathered} e_{as_2}(k+1) - e_{as_2}(k) = y_2(k) + [y_1(\mathrm{t}-\tau) + 0.2\mathrm{y}_2(\mathrm{t}-\tau) + P_2 I_2 D_2 + \mathrm{u}_2]\Delta \\ +\{x_2(k) + [-7x_1 - \mathrm{x}_1\mathrm{x}_3 + 28x_2]\Delta\} = \\ \{-(y_2(\mathrm{t}-\tau) + x_2) + (y_2(\mathrm{t}-\tau) + x_2) + [y_1(\mathrm{t}-\tau) + 0.2\mathrm{y}_2(\mathrm{t}-\tau) + P_2 I_2 D_2 + u_2 - 7x_1 - \mathrm{x}_1\mathrm{x}_3 + 28x_2]\}\Delta + \\ y_2(k) + x_2(k) = \\ [-e_2 + y_2(\mathrm{t}-\tau) + x_2 + y_1(\mathrm{t}-\tau) + 0.2\mathrm{y}_2(\mathrm{t}-\tau) + +P_2 I_2 D_2 + u_2 - 7x_1 - \mathrm{x}_1\mathrm{x}_3 + 28x_2]\Delta + \\ y_2(k) + x_2(k) \end{gathered}$$

$$e_{as_3}(k+1) - e_{as_3}(k) =$$
$$y_3(k) + [0.2 - 5.7y_3(\mathrm{t}-\tau) + \mathrm{y}_1(\mathrm{t}-\tau)\mathrm{y}_3(\mathrm{t}-\tau) + P_3I_3D_3 + \mathrm{u}_3]\Delta + [\mathrm{x}_3(\mathrm{k}) + (-3x_3 + \mathrm{x}_1\mathrm{x}_2)\Delta] =$$
$$\{-(y_3(\mathrm{t}-\tau) + x_3) + (y_3(\mathrm{t}-\tau) + x_3) +$$
$$0.2 - 5.7y_3(\mathrm{t}-\tau) + \mathrm{y}_1(\mathrm{t}-\tau)\mathrm{y}_3(\mathrm{t}-\tau) + P_3I_3D_3 + \mathrm{u}_3 - 3x_3 + \mathrm{x}_1\mathrm{x}_2\}\Delta + y_3(k) + x_3(k) =$$
$$[-e_3 + (y_3(\mathrm{t}-\tau) + x_3) +$$
$$0.2 - 5.7y_3(\mathrm{t}-\tau) + \mathrm{y}_1(\mathrm{t}-\tau)\mathrm{y}_3(\mathrm{t}-\tau) + P_3I_3D_3 + \mathrm{u}_3 - 3x_3 + \mathrm{x}_1\mathrm{x}_2\}\Delta +$$
$$y_3(k) + x_3(k)$$

We need to find the nonlinear active control law for u_i, $\forall\, i$ in such a manner that the error dynamics of (13) is globally asymptotically stable. Let

$$\begin{aligned} u_1 &= (-y_1(\mathrm{t}-\tau) - x_1) + y_2(\mathrm{t}-\tau) + y_3(\mathrm{t}-\tau) - 35x_2 + 35x_1 + u_1 - P_1I_1D_1 \\ u_2 &= -y_2(\mathrm{t}-\tau) - x_2 - y_1(\mathrm{t}-\tau) - .2\mathrm{y}_2(\mathrm{t}-\tau) - P_2I_2D_2 + 7x_1 + \mathrm{x}_1\mathrm{x}_3 - 28x_2 \\ u_3 &= (-y_3(\mathrm{t}-\tau) - x_3) - 0.2 + 5.7y_3(\mathrm{t}-\tau) - \mathrm{y}_1(\mathrm{t}-\tau)\mathrm{y}_3(\mathrm{t}-\tau) - P_3I_3D_3 + 3x_3 - \mathrm{x}_1\mathrm{x}_2\} \end{aligned} \quad (21)$$

Substituting the controller dynamics (21) in error dynamics (20), we have error dynamics as:

$$\begin{aligned} e_{as_1}(k+1) - e_{as_1}(k) &= [-e_{as_1}]\Delta + y_1(k) + x_1(k) \\ e_{as_2}(k+1) - e_{as_2}(k) &= [-e_{as_2}]\Delta + y_2(k) + x_2(k) \\ e_{as_3}(k+1) - e_{as_3}(k) &= [-e_{as_3}]\Delta + y_3(k) + x_3(k) \end{aligned}$$

$$\begin{aligned} e_{as_1}(k+1) - e_{as_1}(k) &= [-e_{as_1}]\Delta + e_{as_1} \\ e_{as_2}(k+1) - e_{as_2}(k) &= [-e_{as_2}]\Delta + e_{as_2} \\ e_{as_3}(k+1) - e_{as_3}(k) &= [-e_{as_3}]\Delta + e_{as_3} \end{aligned} \quad (22)$$

The synchronization problem is to determine the nonlinear controller U, so that:

$$\lim_{t\to\infty} \|e(t)\| = 0$$

Considering a positive definite Lyapunov function as [14], [15] and [16]:

$$V(e_{as}) = \tfrac{1}{2}(e_{as_1}^2 + e_{as_2}^2 + e_{as_3}^2) + \int_{t-\tau}^{t} g(x(\zeta))d\zeta$$

This Lyapunov-Krasovskii function is defined for systems that are continuous in time, and for discrete systems, which is our case study, the integral is replaced by the following summation, the function thus obtained is called the function of Lyapunov-Krasovskii for discrete systems in time.

$$V_1(e_t) = \sum_{i=t-h}^{t-1} e^T(i)Qe(i)$$

$$V(e_{as}) = \tfrac{1}{2}(e_{as_1}^2 + e_{as_2}^2 + e_{as_3}^2) + \textstyle\sum_{i=t-h}^{t-1} e^T(i)Qe(i) \quad (23)$$

Assuming first order partial derivatives of (23) exists, we obtained, using the procedure in (7)

$$\Delta\left(V\left(e_{as_1}\right)\right) = e_{as_1}\dot{e}_{as_1} + e_{as_2}\dot{e}_{as_2} + e_{as_3}\dot{e}_{as_3} + e^T(t)Qe(t) - e^T(t-h)Qe(t-h) \tag{24}$$

Substituting (22) in (24), we obtain

$$\Delta\left(V(e_{as})\right) = -e_{as_1}^2\Delta + \mathrm{e}_{as_1}^2 - e_{as_2}^2\Delta + \mathrm{e}_{as_2}^2 - e_{as_3}^2\Delta + \mathrm{e}_{as_3}^2 + e^T(t)Qe(t) - e^T(t-h)Qe(t-h)$$

$$\Delta\left(V(e_{as})\right) = -e_{as_1}^2\Delta + \left\|\mathrm{e}_{as_2}\right\| - e_{as_3}^2\Delta + \left\|\mathrm{e}_{as_3}\right\| - e_{as_3}^2\Delta + \left\|\mathrm{e}_{as_3}\right\| + \left\|\mathrm{e}_{as_3}\right\| + \left\|e(t-h)\right\| \tag{25}$$

$$\Delta\left(V(e_{as})\right) = -e_{as_1}^2\Delta - e_{as_2}^2\Delta - e_{as_3}^2\Delta$$

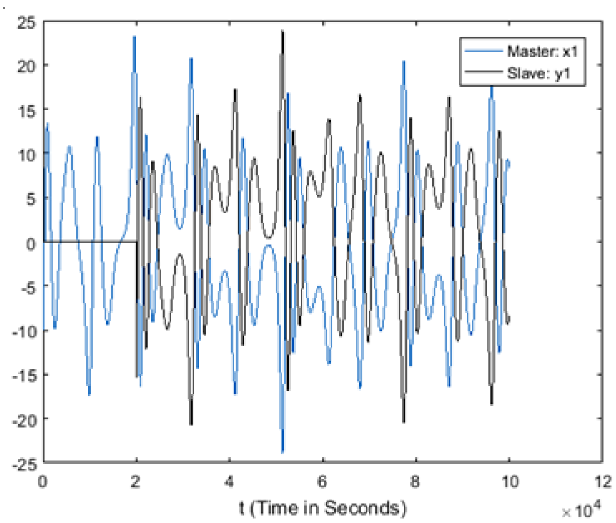
Since, $\Delta\left(V(e)\right)$ is negative definite. For the Lyapunov stability theory, the error dynamics (22) is globally asymptotically stable and the error dynamics will converge to zero as $t \to \infty$ with the control law in (21). The chaotic systems Chen (10) and Rossler (11) are globally asymptotically anti-synchronized for any initial condition, and with this analysis, we have the next theorem:

The analytical results obtained through examples developed via simulation are illustrated below for anti-sinchronization:

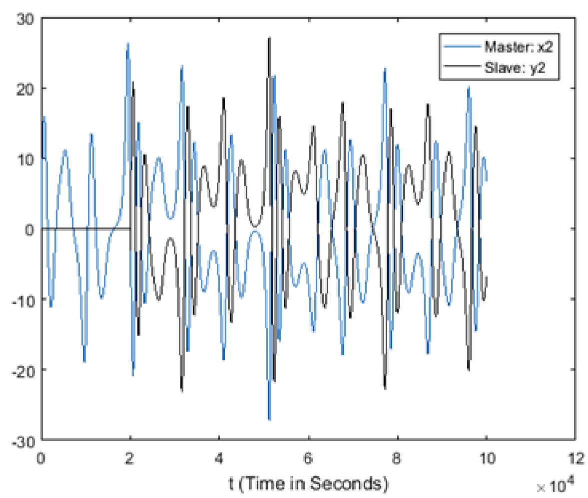
Chen and Rossler systems are simulated in simulink matlab using the control law U (21) for anti-synchronization:

The initial conditions for these systems are $x(0) = [-10, 0, 37]^T$ and $y(0) = [0.1, 0, 0]^T$ Respectively for simulation:

Evolution over time of the states of the Chen and Rossler systems for anti-sinchronization:



a



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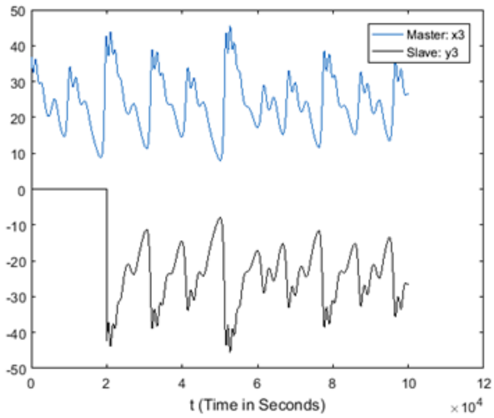

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Figure 7. Time response of synchronized states of master and slave.

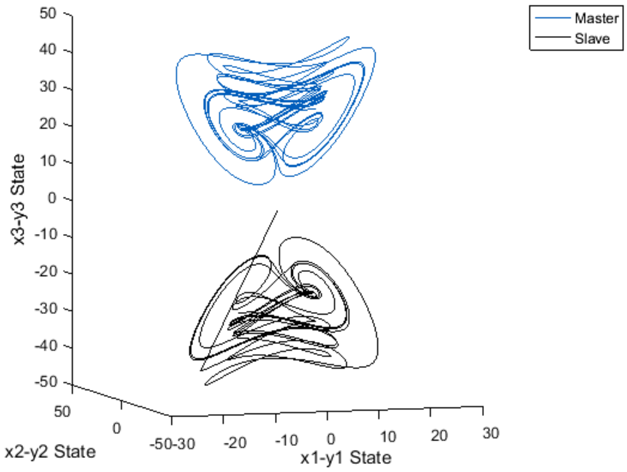

Figure 8. Phase space of the anti-synchronization of the original master slave systems.

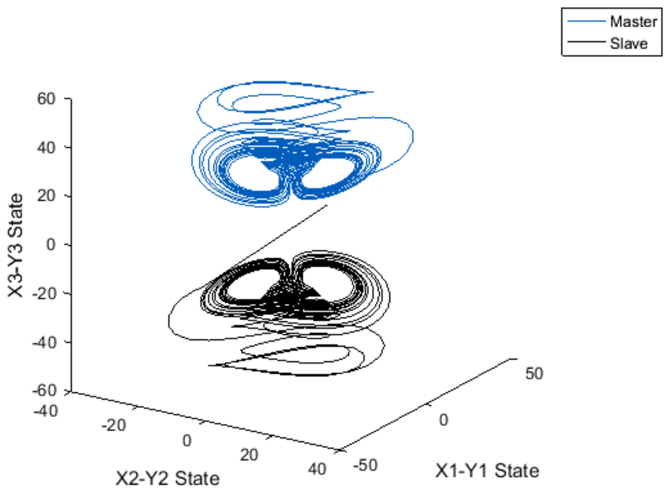

Figure 9. Phase space of the anti-synchronization of the master-slave system whit fractional order given by c = 0.9, c1 = 0.8, c2 = 0.7.

For these simulations we use a $\Delta= 0.001$, and $\tau = 20\ sec$.
Anti-synchronization errors of states are shown in the **figure 10**.

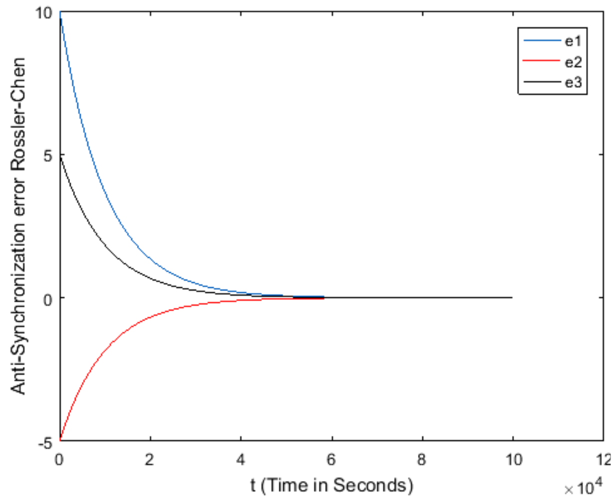

Figure 10. Anti-synchronization errors with time-delay between the states of master and slave systems of variable fractional order derivadtve.

Theorem: The synchronization and anti-sinchronization problem of discrete fractional order chaotic systems in time is solved by means of control laws (14) and (21) which are obtained using the stability analysis by Lyapunov-Krasovskii and PID control laws for fractional order systems , so we ensure that: $\Delta(\mathrm{V(e)}) < 0 \quad \forall \mathrm{e} \neq 0$ and then $\lim_{k\to\infty} e(k) = 0$, $\Delta(\mathrm{V}(e_{as})) < 0$ $\forall\, e_{as} \neq 0$ and then $\lim_{k\to\infty} e_{as}(k) = 0$, therefore the synchronization and anti-sinchronization problem is solved.

6. Conclusions

In this research work, a solution is given to the problem of synchronization and anti-synchronization of chaotic systems described by differential equations of variable order derivative and discrete time with time delay tau, said problem is solved by means of a control law which is deduced by the well-known discrete Lyapunov-Krasovskii stability analysis and discrete PID control laws, as can be seen in the simulations of sections 4. and 5. the analytical results obtained are illustrated by simulations in these sections, as can be seen, the results are satisfactory, these simulations were carried out in the Simulink-MatLab environment.

Remarks:

Although the study that was carried out was for the chaotic systems of Chen and Rossler, of variable fractional order with time delay, the methodology used can be used for other chaotic, hyperchaotic or other systems.

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Conflicts of Interest: "The authors declare no conflict of interest ". The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript, or in the decision to publish the results".

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