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Time optimal Control for semilinear stochastic functional differential equations with delays

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Abstract: The purpose of this paper is to find the time optimal control to a target set for semilinear stochastic functional differential equations involving time delays or memories under general conditions on a target set and nonlinear terms even though the equations contain unbounded principal operators. Our research approach is construct a fundamental solution for corresponding linear systems and establish variations of constant formula of solutions for given stochastic equations. The existence result of time optimal controls for one point target set governed by the given semilinear stochastic equation is also established.

Keywords: stochastic differential equation; retarded control system; time optimal control; admissible set; analytic semigroup.

1. Introduction

This paper deals with the existence of optimal control to reach the target set governed by semilinear stochastic differential equations:

$$\begin{cases} x'(t) = Ax(t) + \int_{-h}^{0} a_1(s)A_1x(t+s)ds + f(t,x_t)d\omega + Bu(t), \quad t>0, \\ x(0) = \phi^0 \in L^2(\Omega, H), \quad x(s) = \phi^1(s), \quad s \in [-h, 0]. \end{cases} \tag{1}$$

Here, A is an elliptic differential operator of the second order induced by the sesquilinear form, A_1 is a closed linear operator with domain $D(A_1)$ containing the $D(A)$, $h>0$, and the function $a(\cdot)$ is real valued and Hölder continuous. Moreover, $\omega(t)$ represents K-valued Brownian motion or Wiener process with a finite trace nuclear covariance operator Q, and f is a nonlinear mapping satisfying some assumptions. Let U be a Banach space as a control space and the controller B be linear bounded from U to H.

The purpose of this paper is to find the time optimal control governed by (1) to a bounded target set in the shortest time. This kind of stochastic differential equations arises in many practical mathematical models, such as, option pricing, population dynamics, physical, engineering and biological problems, etc. (see [1–4]). Many many literature works have been studied for the theory of stochastic differential equations in a variety of ways in [5,6] and reference therein. The approximate controllability of stochastic equations have been studied by authors [7–10]. Similar considerations of semilinear stochastic systems have been dealt with in many references [11,12].

Some standard optimal control results for general linear systems with delays in reflexive Banach spaces were studied in [13]. Recently, a survey of results on optimal control problems governed by delay differential inclusions is referred in Mordukhovich et al. [14,15], and in [16,17]. Micu et al. [18] discussed the time optimal boundary controls for the heat equation, and for parabolic equations with the Neumann condition was considered by Krakowiak [19]. If the principal operator is unbounded, Jeong and Son [20]

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investigated the time optimal control results to a target set for semilinear control equations involving time delay.

In the case (1), since the mild solution has discontinuities at delay times, we have difficulty inducing the basic properties of solutions of semilinear stochastic control systems. But by interpreting the fundamental solutions for linear functional equations with time delay as is seen in [21], we can overcome the difficulty and obtain some essential results of solutions of stochastic differential equations with delays in Hilbert spaces. Moreover, motivated by the above mentioned works [13,20], we deal with the time optimal control problem to a bounded target set for semilinear stochastic control equations involving time delays or memories even though the equations contain unbounded principal operators and nonlinear terms by using an easy consequence of real interpolation spaces.

We enumerate the contents of this paper. Section 2 introduces some basic results on the general retarded linear equations by constructing the fundamental solution, and deal with a variation of constant formula of L^2-primitive process and properties of the strict solutions of (1). In Section 3, we deals with the existence of optimal control to reach the target set governed by semilinear stochastic differential equations. As the time optimal control theory for the standard results, we refer to the linear case as in [13](or semilinear equation [20]) and extend the results in our semilinear stochastic functional differential equations with delays. Finally, the existence of the optimal control to a singleton target is also derived from the convergence of optimal controls to decreasing target sets containing the singleton.

2. PRELIMINARIES AND LEMMAS

2.1. RETARDED LINEAR EQUATIONS

Let H be a Hilbert space densely and V be continuously embedded in H. The norms of V, H and the dual space V^* with V are denoted by $||\cdot||$, $|\cdot|$ and $||\cdot||_*$, respectively. It can generally be considered

$$||u||_* \le |u| \le ||u||, \quad u \in V.$$

If $b(\cdot,\cdot)$ is a continuous sesquilinear form in $V \times V$ satisfying Gårding's inequality:

$$\text{Re } b(u,u) \ge c_0||u||^2 - c_1|u|^2, \quad c_0 > 0, \quad c_1 \ge 0. \tag{2}$$

We define A as follows:

$$< (c_1 - A)u, v >= -b(u,v), \quad u,\ v \in V,$$

where $< \cdot,\cdot >$ denotes also the duality pairing between V and V^*. By (2), we have

$$\text{Re } < Au, u >\ge c_0||u||^2, \quad u \in V.$$

According to the Lax-Milgram theorem, we know that A is a bounded linear operator from V to V^*. Moreover, as seen in Theorem 3.6.1 of [22], A generates an analytic semigroup $S(t) = e^{tA}$ in both H and V^*. The the restriction of A to domain

$$D(A) = \{u \in V; Au \in H\}$$

with the graph norm is also denoted by A. By identifying the dual of H with H, we may consider the following relation

$$D(A) \subset V \subset H \subset V^* \subset D(A)^*. \tag{3}$$

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Let $(D(A), H)_{1/2,2}$ be the real intermediate space between $D(A)$ and H in the sense of Section 1.3.3 of [23]. In relation to (3), it is well known that

$$(V, V^*)_{1/2,2} = H, \quad (D(A), H)_{1/2,2} = V.$$

If X and Y are two Banach space, $\mathcal{L}(X, Y)$ is the collection of all bounded linear operators from X into Y, and $\mathcal{L}(X, X)$ is simply written as $\mathcal{L}(X)$. For simplicity, we assume that $S(t)$ is uniformly bounded, that is, there exists a constant $C_0 > 0$ such that

$$||S(t)||_{\mathcal{L}(H)} \leq C_0.$$

Lemma 1. *Let $S(t)$ be an analytic semigroup by generated by A. Then we have*

$$||S(t)||_{\mathcal{L}(H)} \leq C_0, \quad ||AS(t)||_{\mathcal{L}(H)} \leq \frac{C_0}{t}, \quad \text{and } ||A^2S(t)||_{\mathcal{L}(H)} \leq \frac{C_0}{t^2}$$

for each $t \geq 0$(e.g. [22]), and for $0 < s < t$ and $0 < \alpha < 1$

$$||S(t) - S(s)|| \leq \frac{C_0}{\alpha}(t-s)^{\alpha} s^{-\alpha} \tag{4}$$

Proof. For $0 < s < t$

$$||S(t) - S(s)|| = \int_0^t AS(\tau)d\tau \leq C_0 \log(\frac{t}{s}). \tag{5}$$

Noting that for $0 < \alpha < 1$

$$\log(\frac{t}{s}) = \int_1^{1+\frac{t-s}{s}} \frac{1}{y} dy \leq \int_1^{1+\frac{t-s}{s}} \frac{1}{y^{1-\alpha}} dy \leq \frac{1}{\alpha}(\frac{t-s}{s})^{\alpha},$$

combining this and (5), we get (4) for $0 < s < t$ and $0 < \alpha < 1$. □

Now, consider the following retarded linear functional differential equation:

$$\begin{cases} x'(t) = Ax(t) + \int_{-h}^0 a(s)A_1 x(t+s)ds + k(t), & t > 0, \\ x(0) = \phi^0, \quad x(s) = \phi^1(s) & -h \leq s \leq 0, \end{cases} \tag{6}$$

where A_1 is a closed linear unbounded operator with $D(A) \subset D(A_1)$, for instance, A_1 is an elliptic differential operator of second order induced by sesquilinear form. As seen in Harakiri [24,25], we introduce the fundamental solution $W(\cdot)$ of the retarded linear equation (6) defined by

$$\begin{cases} W(t) & = S(t) + \int_0^t S(t-s)\{\int_{-h}^0 a(\tau)A_1 W(s+\tau)d\tau\}ds, \quad t > 0. \\ W(0) & = I, \qquad W(s) = 0, \quad -h \leq s < 0, \end{cases}$$

where $a(\cdot)$ is a real valued Hölder continuous function:

$$|a(s)| \leq H_0, \quad |a(s) - a(\tau)| \leq H_0(s-\tau)^{\rho}, \quad -h \leq \tau, s \leq 0 \tag{7}$$

for a constant H_0.

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By the property of $S(\cdot)$ as in lemma 1, $W(t)$ is strongly continuous in both H and V^*. For each $t > 0$, if we define the operator $U_t(\cdot)$ by

$$U_t(s) = \int_{-h}^{s} W(t - s + \sigma)a(\sigma)A_1 d\sigma : V \to V, \quad s \in [-h, 0].$$

Then (4) is represented

$$x(t) = W(t)\phi^0 + \int_{-h}^{0} U_t(s)\phi^1(s)ds + \int_{0}^{t} W(t - s)k(s)ds.$$

By Lemma 1, we obtain basic properties of $W(t)$(see Proposition 4.1 of [26] or Theorem 1 of [21]).

Lemma 2. *The fundamental solution $W(t)$ exists uniquely and there exists a constant $C_0 > 0$ such that*

$$\begin{aligned} &||W(t)||_{\mathcal{L}(H)}(t \geq 0) \leq C_0, \ \ ||W(t)||_{\mathcal{L}(V^*)}(t \geq 0) \leq C_0, \\ &||W(t') - W(t)||_{\mathcal{L}(H)} \leq \frac{C_0}{\alpha}(t' - t)^{\alpha} t^{-\alpha}, \quad t > 0, \\ &||W(t') - W(t)||_{\mathcal{L}(V^*)} \leq C_0(t' - t), \\ &||W(t') - W(t)||_{\mathcal{L}(V^*,V)} \leq C_0(t' - t)^{\kappa}(t - h)^{-\kappa} \end{aligned} \tag{8}$$

for $h < t < t'$, and $\kappa < \rho$, where ρ is the Hölder constant in (7)

2.2. SEMILINEAR STOCHASTIC DIFFERENTIAL EQUATIONS

Let $(H, |\cdot|)$ and $(K, |\cdot|_K)$ be real separable Hilbert spaces. Consider the following retarded semilinear stochastic control system in Hilbert space H:

$$\begin{cases} x'(t) = Ax(t) + \int_{-h}^{0} a_1(s)A_1 x(t+s)ds + f(t, x_t)d\omega + Bu(t), \quad t > 0, \\ x(0) = \phi^0 \in L^2(\Omega, H), \quad x(s) = \phi^1(s), \quad s \in [-h, 0]. \end{cases} \tag{9}$$

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with complete family of right continuous increasing sub σ-algebras $\{\mathcal{F}_t, t \in I\}$ satisfying $\mathcal{F}_t \subset \mathcal{F}$.

The collection of random variables $\mathcal{S} = \{\mathcal{F}-$measurablefunction $x(t, w) : \Omega \to H : t \in [0, T], \ w \in \Omega\}$ is a stochastic process. Generally, we just write $x(t)$ instead of $x(t, w)$ and $x(t) : [0, T] \to H$ in the space of $\mathcal{S}$

Let $\{e_n\}_{n=1}^{\infty}$ be a complete orthonormal basis of K, and let $Q \in \mathcal{L}(K)$ be an operator defined by $Qe_n = \lambda_n e_n$ with finite $\mathrm{Tr}(Q) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} = \lambda < \infty$ (Tr denotes the trace of the operator), where $\lambda_n \geq 0 (n = 1, 2, \cdots)$. Here, $\mathcal{L}(K, H)$ denotes the space of all bounded linear operators from K into H, we denote simply $\mathcal{L}(K)$ if $H = K$.

$\{\omega(t) : t \geq 0\}$ be a cylindrical K-valued Wiener process with a finite trace nuclear covariance operator Q over $(\Omega, \mathcal{F}, P)$, which satisfies that

$$\omega(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} w_i(t) e_n, \quad t \geq 0,$$

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where $\{w_i(t)\}_{i=1}^{\infty}$ be mutually independent one dimensional standard Wiener processes over $(\Omega, \mathcal{F}, P)$. Then the above K-valued stochastic process $\omega(t)$ is called a Q-Wiener process.

We assume that $\mathcal{F}_t = \sigma\{\omega(s) : 0 \le s \le t\}$ is the σ-algebra generated by w and $\mathcal{F}_T = \mathcal{F}$. Let $\psi \in \mathcal{L}(K, H)$ and define

$$|\psi|_Q^2 = \mathrm{Tr}(\psi Q \psi^*) = \sum_{n=1}^{\infty} |\sqrt{\lambda_n}\psi e_n|^2.$$

If $|\psi|_Q^2 < \infty$, then ψ is called a Q-Hilbert-Schmidt operator. $\mathcal{L}_Q(K, H)$ stands for the space of all Q-Hilbert-Schmidt operators. The completion $\mathcal{L}_Q(K, H)$ of $\mathcal{L}(K, H)$ with respect to the topology induced by the norm $|\psi|_Q$, where $|\psi|_Q^2 = (\psi, \psi)$ is a Hilbert space with the above norm topology.

Assume that V is s dense space of H as seen in Section 2.1. For $T > 0$ we define

$$M^2(-h, T; V) = \{x : [-h, T] \to V : E(\int_{-h}^{T} ||x(s)||^2 ds) < \infty\}.$$

The spaces $M^2(-h, 0; V)$, $M^2(0, T; V)$, and $M^2(0, T; V^*)$ are also defined as the same way and the basic theory of M_2-spaces can be founded in [2].

For $h > 0$, we assume that $\phi^1 : [-h, 0) \to V$ is a given initial value satisfying

$$E(\int_{-h}^{0} ||\phi^1(s)||^2 ds) < \infty,$$

that is, $\phi^1 \in M^2(-h, 0; V)$. In this note, a random variable $x(t) : \Omega \to H$ will be called an L^2-primitive process if $x \in M^2(-h, T; V)$.

For every $s \in [0, T]$, we define $x_s : [-h, 0] \to H$ as

$$x_s(r) = x(s + r), \quad -h \le r \le 0.$$

For brevity, we will set

$$\Pi = M^2(-h, 0; V).$$

Definition 1. *A stochastic process $x : [-h, T] \times \Omega \to H$ is called a solution of (9) if*

(i) *$x(t)$ is measurable and $\mathcal{F}_t$-adapted for each $t \ge 0$.*
(ii) *$x(t) \in H$ has cádlág paths on $t \in (0, T)$ such that*

$$x(t) = W(t)\phi^0 + \int_{-h}^{0} U_t(s)\phi^1(s)ds + \int_0^t W(t-s)\{f(s, x_s)d\omega + Bu(s)\}ds,$$

where

$$U_t(s) = \int_{-h}^{s} W(t - s + \sigma)a(\sigma)A_1 d\sigma.$$

(iii) *$x \in M^2(0, T; V)$ i.e., $E(\int_0^T ||x(s)||^2 ds) < \infty$ and $x \in C([0, T]; H)$.*

To get our results, we need the following assumptions on (9).

Assumption (F). Let $f : \mathbb{R} \times \Pi \to \mathcal{L}(K, H)$ be a nonlinear mapping satisfying the following:

(i) For each $x \in \Pi$, the mapping $f(\cdot, x)$ is strongly measurable.

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(ii) There is a function $L_f : \mathbb{R}_+ \to \mathbb{R}$ such that

$$E|f(t,x) - f(t,y)|^2 \le L_f(r)||x-y||^2_{\Pi}, \quad t \in [0,T]$$
$$E|f(t,x)|^2 \le L_f(r)(||x||_{\Pi} + 1)^2$$

for $||x||_{\Pi} \le r$ and $||y||_{\Pi} \le r$.

Let the solution spaces $\mathcal{Z}(T)$ and $\mathcal{Z}_0(T)$ be defined by

$$\mathcal{Z}(T) := M^2(0,T;V) \cap C([0,T];H),$$
$$\mathcal{Z}_0(T) := M^2(0,T;D(A)) \cap C([0,T];V).$$

We can briefly summarize about the solvability of the system (9) from [27].

Proposition 1. *1) Let Assumption (F) be satisfied. Suppose that* $(\phi^0, \phi^1) \in L^2(\Omega, H) \times \Pi$ *and* $k \in M^2(0,T;V^*)$ *for* $T > 0$*. Then, there is a solution x of the system (9) such that*

$$x \in \mathcal{Z}(T).$$

Moreover, there is a positive constant C_1 *independent of the initial data* (ϕ^0, ϕ^1) *and the forcing term k such that*

$$||x||_{\mathcal{Z}(T)} \le C_1(1 + E(|\phi^0|^2) + ||\phi^1||_{\Pi} + ||k||_{M^2(0,T;V^*)}).$$

2) Let Assumption (F) be satisfied. Suppose that the initial data $(\phi^0, \phi^1) \in L^2(\Omega, H) \times \Pi$ *and the forcing term* $k \in M^2(0,T;V^*)$*. Then the solution x of the equation (9) belongs to* $x \in M^2(0,T;V)$ *and the mapping*

$$L^2(\Omega, H) \times \Pi \times M^2(0,T;V^*) \ni (\phi^0, \phi^1, k) \mapsto x \in M^2(0,T;V)$$

is continuous.

3. TIME OPTIMAL CONTROL PROBLEMS

Let U be a Banach space as a control space and the controller $B \in \mathcal{L}(U,H)$. The solution $x(t) = x(t;\phi,F,u)$ of the system (9) is the following form:

$$\begin{cases} x(t;\phi,f,u) = W(t)\phi^0 + \int_{-h}^0 U_t(s)\phi^1(s)ds + \int_0^t W(t-s)\{f(s,x_s)d\omega + Bu(s)ds\}, \\ u \in U_{ad}, \end{cases} \tag{10}$$

where

$$U_t(s) = \int_{-h}^s W(t-s+\sigma)a(\sigma)A_1 d\sigma,$$

A_1 is the operators mentioned in Section 2, and U_{ad} is a class od admissible controls. For $x_t \in \Pi$ and $(t,s) \in [0,\infty) \times [-h,0]$, we set

$$f(t,x_t) = \int_{-h}^0 g(t,s,x_t(s))d\mu.$$

We need the following assumptions on the nonlinear operator f:

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Let $\mathcal{A}$ and $\mathcal{B}$ be the Lebesgue σ-field on $[0,\infty[$ and the Borel σ-field on $[-h,0]$, respectively. Let μ be a Borel measure on $[-h,0]$.

Assumption (G). Let g be a nonlinear mapping from $[0,\infty[\times[-h,0]\times V$ to $\mathcal{L}(K,H)$ satisfying the following:

(i) For any $x\in V$ the mapping $g(\cdot,\cdot,x)$ is strongly $\mathcal{A}\times\mathcal{B}$-measurable and $x\mapsto g(\cdot,\cdot,x)$ is compact;
(ii) $g(t,s,x)$ is Lipschitz continuous in x, uniformly in t and s, i.e., there exist a positive constant L_1 such that

$$E|g(t,s,x)|^2\le L_1(1+||x||)^2,$$
$$E|g(t,s,x)-g(t,s,\hat{x})|^2\le L_1E||x-\hat{x}||^2,$$

for all $(t,s)\in[0,\infty[\times[-h,0]$ and $x,\hat{x}\in V$.

Remark 1. *The nonlinear operator g mentioned above is related with the semilinear case of the nonlinear part of quasilinear equations introduced by Yong and Pan [28].*

Now, we will establish the time optimal control problem governed by (10) in H. Throughout this section, let the admissible set U_{ad} be a weakly compact subset in $L^2(0,T;U)$. We sometimes represent the solution $x(t)$ in (10) by $x_u(t)$ to express the dependence on $u\in U_{ad}$. The function x_u is called the trajectory corresponding to a control u. Let the target set W be bounded in H. Define

$$U_0=\{u\in U_{ad}:x_u(t)\in W\text{ for some }\ t\in[0,T]\}$$

and assume that $U_0\neq\emptyset$. Then, a control $u\in U_0$ is equivalent that there is a $u\in U_{ad}$ such that for some $w\in W$

$$E|w-x_u(t)|=0.$$

The optimal time is defined by low limit t_0 of

$$\{t:x_u(t)\in W\ \ \text{for}\ \ \text{some}\ \ u\in U_0\}.$$

For every $u\in U_0$, we are able to define the shortest time $\tilde{t}(u)$ such that $x_u(\tilde{t})\in W$. Our purpose is to seek a $\bar{u}\in U_0$ satisfying

$$\tilde{t}(\bar{u})\le\tilde{t}(u)\quad\text{for all }u\in U_0$$

constrained to the system (10). Since $x_u\in C([0,T];H)$, the transition time $\tilde{t}(u)$ is well defined for each $u\in U_{ad}$. Finally, the existence of the optimal control to a singleton target is also derived from the convergence of optimal controls to decreasing target sets containing the singleton.

We define the linear operators K_1 from $M^2(0,T;V)$ to H and K_2 from $\mathcal{L}(K,H)$ to H by

$$K_1(t)p=\int_0^t W(t-s)p(s)ds,\quad t\le T, \tag{11}$$
$$K_2(t)q=\int_0^t W(t-s)q(s)d\omega,\quad t\le T, \tag{12}$$

respectively.

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Lemma 3. *Let $x \in M^2(0,T;V)$. Then for $0 \le t \le T$, $x_t \in \Pi$ and $f(\cdot, x_t) \in M^2(0,T;H)$ and*

$$||f(\cdot, x.)||_{M^2(0,T;H)} \le \mu([-h,0])\{L_0\sqrt{T} + L_1||x||_{M^2(0,T;V)} + L_1||x||_{\Pi}\}.$$

Moreover, if $x^1,\ x^2 \in M^2(0,T;V)$, then

$$||f(\cdot, x^1_\cdot) - f(\cdot, x^2_\cdot)||_{M^2(0,T;H)} \le \mu([-h,0])L_1\{||x_1 - x_2||_{M^2(0,T;H)} + ||x_1 - x_2||_{M^2(-h,0;H)}\}.$$

Proof. Noting that

$$||x_t||^2_{\Pi} = E\Big(\int_{-h}^0 ||x(t+\tau)||^2 d\tau\Big) \le E\Big[\int_{-h}^t ||x(\tau)||^2 d\tau\Big] \le ||x||^2_{M^2(-h,t;V)},\ t > 0,$$

from (ii) of Assumption (G), we have

$$\begin{aligned} ||f(\cdot, x^1_\cdot) - f(\cdot, x^2_\cdot)||_{M^2(0,T;H)} &\le \mu([-h,0])L_1||x^1_t - x^2_t||_{\Pi} \\ &\le \mu([-h,0])L_1||x^1 - x^2||_{M^2(-h,T;V)}. \end{aligned}$$

The first paragraph is similar. □

By virtue of Lemma 3, from Proposition 2.1 it follows that a solution x of (10) exists in $\mathcal{Z}(T)$. Now, we find a time optimal control which transfers from the initial data to the target set to the trajectory of the constraint system (10) in the first time as follows.

Theorem 1. *Let Assumption (G) be satisfied and $U_0 \neq \emptyset$. Then there exists a time optimal control.*

Proof. Let $t_n \downarrow t_0$ as $n \to \infty$, $u_n \in U_{ad}$, and suppose that $x_n \in W$. Since U_{ad} is weakly compact, there is an $u_0 \in U_{ad}$, $\hat{x} \in W$ and a subsequences, which is denoted again by $\{u_n\}$ and $\{x_n(t_n)\}$ such that $u_n \to u_0$ weakly in $L^2(0,T;Y)$, i.e., $w - \lim_{n\to\infty} u_n = u_0$ and $x_n(t_n) \to \hat{x}$ weakly in W.

Let $\mathcal{F}$ and $\mathcal{B}$ be the Nijinsky operators corresponding to the maps f and $\mathcal{B}$, which are defined by

$$(\mathcal{F}u)(\cdot) = f(\cdot, x_u), \quad \text{and} \quad \mathcal{B}u(\cdot) = Bu(\cdot),$$

respectively. Then

$$x_n(t_n) = x(t_n;\phi,0) + K_2(t_n)\mathcal{F}u_n + K_1(t_n)\mathcal{B}u_n, \tag{13}$$

where

$$x(t_n;\phi,0) = W(t_n)\phi^0 + \int_{-h}^0 U_{t_n}(s)\phi^1(s)ds,$$

and K_1 and K_2 are the operator defined by (11) and (12), respectively. From the strong continuity of $W(t)$ it follows that

$$x(t_n;\phi,0) \to x(t_0;\phi,0) \quad \text{strongly in } H. \tag{14}$$

By the property of $W(t)$, see [25**?** ,26], we have

$$W(t_n - s) = S(t_n - t_0)W(t_0 - s) + \int_{t_0 - s}^{t_n - s} S(t_n - s - \sigma)\int_{-h}^0 a(\tau)A_2W(s+\tau)d\tau d\sigma.$$

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Hence,

$$\begin{aligned} E|K_2(t_n)\mathcal{F}u_n - K_2(t_0)\mathcal{F}u_0| \leq & E\Big|\int_{t_0}^{t_n} W(t_n-s)(\mathcal{F}u_n)(s)d\omega\Big| \\ & + E\Big|\int_0^{t_0} \big(W(t_n-s)-W(t_0-s)\big)(\mathcal{F}u_n)(s)d\omega\Big| \\ & + E\Big|\int_0^{t_0} W(t_0-s)\big((\mathcal{F}u_n)(s)-(\mathcal{F}u_0)(s)\big)d\omega\Big| \\ = & I + II + III. \end{aligned} \tag{15}$$

By Assumption (G), we get

$$I \leq \sqrt{t_n - t_0} C_0 L_f(r) Tr(Q)(||x_u||_{\Pi} + 1), \tag{16}$$

and by(8)

$$II \leq C_0\alpha^{-1}(1-2\alpha)^{-1/2}(t_n-t_0)^{\alpha} t^{(1-2\alpha)/2} Tr(Q)\mu([-h,0])(L_0\sqrt{t} + L_1||x_u||_{M^2(-h,t;V)}). \tag{17}$$

Hence, $I, II \to 0$ as $n \to \infty$ if $\alpha < 1/2$. Since $\mathcal{F}$ is compact, we have that $\mathcal{F}u_n \to \mathcal{F}u_0$ tends to zero as $t_n \downarrow t_0$, so are III and (13) by (14)-(17).

Now we will show that K_1 is a completely continuous mapping. If so, then from the formula

$$\begin{aligned} E|K_1(t_n)\mathcal{B}u_n - K_1(t_0)\mathcal{B}u_0| = & E\Big|\int_{t_0}^{t_n} W(t_n-s)(\mathcal{B}u_n)(s)ds\Big| \\ & + E\Big|\int_0^{t_0} \big(W(t_n-s)-W(t_0-s)\big)(\mathcal{B}u_n)(s)ds\Big| \\ & + E\Big|\int_0^{t_0} W(t_0-s)\big((\mathcal{B}u_n)(s)-(\mathcal{B}u_0)(s)\big)ds\Big|, \end{aligned}$$

it holds

$$E|K_1(t_n)\mathcal{B}u_n - K_1(t_0)\mathcal{B}u_0| = 0$$

as $t_n \downarrow t_0$ in a similar way to (15). In order to proof of compactness of K_1, we will show that

$$\lim_{n\to\infty} |K_1\mathcal{B}u_n - K_1\mathcal{B}u_0| = 0, \tag{18}$$

which means that the completely continuity of K_1 since H is reflexive. Set

$$(hu)(s) = W(t_0 - s)(\mathcal{B}u)(s).$$

Then by (6), we have

$$|h(s)| \leq C_0||B||_{(Y,H)}||u_n(s) - u_0||_Y \leq \infty$$

For a fixed $s \in [0, t_0]$, let $x_s^*(u) = (hu)(s)$ for every $u \in L^2(0, T_1; Y)$. Then $x_s^* \in L^2(0, t_0; Y^*)$ and we have $\lim_{n\to\infty} x_s^*(u_n) = x_t^*(u_0)$ since $w - \lim_{n\to\infty} u_n = u_0$. Hence,

$$\lim_{n\to\infty} (fu_n)(s) = (fu_0)(s), \quad s \in [0, t_0].$$

Therefore, by Lebesgue's dominated convergence theorem, we obtain (18). From (14), (15) and (18), it follows that $x_n(t_n) \to x_0(t_0) = \hat{x} \in W$ □

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Now we deal with the time optimal control if the target set W is a singleton. Let $W = w_0$ such that $\phi^0 \neq w_0$ and $\phi^1(s) \neq w_0$ for some $s \in [-h, 0[$. Then we can choose a decreasing target set $\{W_n\}$ of convex and weakly compact sets with nonempty interior satisfying

$$w_0 \in \bigcap_{n=1}^{\infty} W_n, \text{ and } \operatorname{dist}(w_0, W) = \sup_{x \in W_n} E|x - w_0| \to 0 (n \to \infty). \tag{19}$$

Define

$$U_0^n = \{u \in U_{ad} : x_u(t) \in W_n \text{ for some } t \in [0, T]\}.$$

Then, we may consider that u_n is the time optimal control with the optimal time t_n to the target set W_n, $n = 1, 2, \ldots$.

Theorem 2. *Let Assumption (G) be satisfied. Suppose that $\{W_n\}$ be a sequence of closed convex and weakly compact sets in H satisfying (19) and $U_0^n \neq \emptyset$. Then there is a time optimal control u_0 with the optimal time $t_0 = \sup_{n \geq 1}\{t_n\}$ to the singleton $\{w_0\}$, which is defined by the weak limit of some subsequence of $\{u_n\}$ in $L^2(0, t_0; U)$.*

Proof. Let $w_n = x_n(t_n) \in W_n$. Because of the weak compactness of U_{ad}, there are a $u_0 \in U_{ad}$ and subsequences(which are denoted again by $\{u_n\}$ and $\{w_n\}$) such that $t_n \uparrow t_0$ as $n \to \infty$, $u_n \to u_0$ weakly in $L^2(0, t_0; U)$, and $w_n = x_n(t_n) \in W_n \to w_0$ strongly in H. Then by virtue of (19), we have

$$E|w_n - w_0| \to 0$$

strongly in H. Thus, in the similar argument as the proof of Theorem 2, we can easily prove that u_0 and t_0 are the time optimal control and the optimal time to the target $\{w_0\}$, respectively. □

Example 1.

Let

$$H = L^2(0, \pi), \ V = H_0^1(0, \pi), \ V^* = H^{-1}(0, \pi),$$

and let U be a Banach space of control variables. Consider the following control system which is described a retarded neutral stochastic differential system on H:

$$\begin{cases} \partial x(t, y) = & [\Delta x(t, y) + \int_{-h}^{0} a(s)\Delta x(t + s, y)ds + Bu(t, y)]\partial t \\ & +f(t, x_t)d\omega(t), \quad (t, y) \in [0, T] \times [0, \pi], \\ x(0, y) = & \phi^0(y) \in L^2(\Omega, H), \ \ x(s, y) = \phi^1(s, y), \ \ (s, y) \in [-h, 0) \times [0, \pi], \end{cases} \tag{20}$$

where $h > 0$, $a(\cdot)$ is Hölder continuous, and $\omega(t)$ stands for a standard cylindrical Winner process in H defined on a stochastic basis $(\Omega, \mathcal{F}, P)$. Define sesquilinear form $b(\cdot, \cdot)$ on $V \times V$ by

$$b(u, v) = \int_0^{\pi} \frac{du(y)}{dy} \overline{\frac{dv(y)}{dy}} dy.$$

satisfying Gårding's inequality (2). Let

$$A = \partial^2/\partial y^2 \quad \text{with} \quad D(A) = \{x \in H^2(0, \pi) : x(0) = x(\pi) = 0\}.$$

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Then the eigenvalue of $A = \Delta$ is $\lambda_n = -n^2$ and its eigenfunction is given by $z_n(y) = (2/\pi)^{1/2}\sin ny$. Since $\{z_n : n \in N\}$ is an orthogonal basis of H, the semigroup $S(t)$ generated by A is represented as

$$S(t)x = \sum_{n=1}^{\infty} e^{n^2 t}(x, z_n)z_n, \quad \forall x \in H, \ \ t > 0,$$

and there is a positive constant M_0 such that $||S(t)||_{\mathcal{L}(H)} \le M_0$. For any $x \in \Pi)$, set

$$f(t, x_t)) = \int_0^t \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\sigma_i(t, \nabla x_t)ds.$$

We assume that there is a constant $L > 0$ such that

$$|\sigma_i(t, \xi) - \sigma_i(t, \widehat{\xi})| \le L|\xi - \widehat{\xi}|, \tag{21}$$

where $|\cdot|$ is the norm of $H = L^2(\Omega)$. For simplicity, we assume that $\sigma_i(t, 0) = 0$. Hence, we have

$$|\sigma_i(t, \xi)| \le L|\xi|.$$

Put

$$g_1(t, x_t) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\sigma_i(t, \nabla x_t).$$

Then $g_1(t, x) \in V^*$. For each $w \in H_0^1(\Omega)$, we satisfy the following that

$$(g_1(t, x_t), w) = -\sum_{i=1}^{n}(\sigma_i(t, \nabla x_t), \frac{\partial}{\partial x_i}w).$$

The nonlinear term is given by

$$f(t, x_t) = \int_0^t g_1(t, x_t)ds.$$

For any $w \in H_0^1(\Omega)$, if u and $\hat{u}$ belong to $H_0^1(\Omega)$, by (21) we obtain

$$E|(g(t, x_t(s)) - g(t, \hat{x}_t(s))), w| \le LTE||x_t(s) - \hat{x}_t(s)||\,||w||.$$

Thus, it is easily seen that Assumption (G) has been satisfied. Define $B \in \mathcal{L}(U, H)$ by

$$Bu(t) = \gamma(y)u, \quad 0 \le y \le \pi, \ \ u \in L^2(0, T; U), \ \ \gamma(y) \in M^2([0, \pi]).$$

Let U be a real Banach space and let the admissible set U_{ad} be a weakly compact subset in $L^2(0, T; U)$. If U_0 is nonempty, then by Theorem 2, there is a control $\bar{u} \in U_0$ such that

$$\tilde{t}(\bar{u}) \le \tilde{t}(u) \quad \text{for all } u \in U_0$$

governed to the constraint (20).

4. Conclusions

The difficulty of finding optimal control to a the target set for stochastic functional differential equations perturbing unbounded nonlinear terms with delays is due to the discontinuity of unbounded principal operators and delay terms. Thanks to configuring and interpreting the fundamental solution

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for linear functional equations with time delays as seen in [21], we can overcome difficulties and achieve some essential results of the solution of stochastic differential equations with delays in Hilbert spaces. By the basic consequence of real interpolation spaces and establishing variations of constant formula of solutions, we investigate the time optimal control problem to a bounded target set for semilinear stochastic control equations involving time delays or memories although the equations contain unbounded principal operators and nonlinear terms. The presence of time optimal controls only for a set of one-point targets governed by the given semilinear stochastic equation is also established. Based on this approach, we intend to study the approaches and applicable methods of various nonlinear stochastic equations in science.

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