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Article

Consensus towards Partially Cooperative Strategies in Self-regulated Evolutionary Games on Networks

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Abstract: Cooperation is widely recognized to be challenging for the well-balanced development of human societies. The emergence of cooperation in populations has been largely studied in the context of the Prisoner's Dilemma game, where temptation to defect and fear to be betrayed by others often activate defective strategies. In this paper we analyze the decision making mechanisms fostering cooperation in the two-strategy Stag-Hunt and Chicken games, which include the mixed strategy Nash equilibrium, describing partially cooperative behavior. We find the conditions for which cooperation is asymptotically stable in both full and partial cases, and we show that the partially cooperative steady state is also globally stable in the simplex. Furthermore, we show that the last can be more rewarding than the first, thus making the mixed strategy effective, although people cooperate at a lower level with respect to the maximum allowed, as it is reasonably expected in real situations. Our findings highlight the importance of Stag-Hunt and Chicken games in understanding the emergence of cooperation in social networks.

Keywords: Evolutionary Games; Cooperation; Consensus; Dynamics on Networks; Stag-Hunt Game; Chicken Game; Mixed Nash Equilibrium; Self-regulation; Stable Equilibrium; Complex Systems

1. Introduction

Cooperation in a population is a key emerging phenomenon, which have fascinated many scientists in several fields, ranging from biology to social and economics science [1–7], and recently also considered in technological applications [8,9]. Although cooperation has been sometimes seen to contrast the Darwinian concept of natural selection, it emerges in many complex systems providing substantial benefits for all members of groups and organizations [10–17]. Generally, this topic is tackled using the tools of evolutionary game theory, which constitute the mathematical foundations for modeling the decisions of a population of players taking part in a replicator/selection competition [18]. These mechanisms are described by means of the well known replicator equation [3,19]. Moreover, the structure of the society plays an important role; studies aimed to embed networks into games have been developed for infinite lattices [20], when the players are assumed to be able to choose between two strategies. Further developments have been proposed in [21–23] where the graph topology is general, and players are allowed to choose a strategy in the continuous set $\Delta = [0, 1]$. It has been shown that the presence of a network of connections acts as a catalyst for the emergence of cooperation [14,24–28]. In [29] it has been shown that cooperation can emerge if the average connectivity level k, accounted by the average degree of the underlying graph, is smaller than the benefits/costs ratio of altruistic behavior. Moreover, to solve the problem of cooperation, almost all works present in the literature add and/or change the rule of the played games [30]. For example, as part of a society, individuals often use punishment mechanisms which limit the detrimental behavior of free riders or they can be awarded if they

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prefer cooperative behaviors [10,17,31–34]. Furthermore, mechanisms based on discount and synergies among players have been also proposed [13]. In summary, all these efforts are aimed to point out the role of exogenous factors in the emergence of cooperation.

Recently the fact that every human player is characterized by endogenous factors, like the awareness of conflicts, which can act as a motivation for cooperation has been emphasised. In other words, the rules of the game are important, but also other elements that distinguish human and animals, must have a crucial effect. In particular, in [35] the Self-Regulated Evolutionary Game on Network equation (SR-EGN) has been introduced, by extending the EGN equation studied in [22,23]. The SR-EGN equation is a set of ordinary differential equations able to model the social pressure on each individual (exogenous factors) and their innate tendency to cooperate with one another even when it goes against their rational self-interest (endogenous factors).

Consensus solutions, where all players agree to converge to a common level of cooperation, have been also found significant for the sustainable development of interacting real societies. Consensus is a puzzling topic since it is often achieved without centralized control [36–38]. Remarkably, when cooperation spreads all over a population, also consensus of all individuals to the same strategy is reached, thus allowing social individuals to wipe out the cost of indecision [39]. In the context of cooperation, consensus is usually studied in the full sense, where all players have the ability to make fully cooperative decisions (pure NE equilibrium). For the purpose of being more realistic, we notice that real players can be partially cooperative, i.e. people can be cooperative with some one and defective with others, moreover their decision may change over time. Thus, a revision of the concept of consensus towards cooperation in a more general sense is required.

The SR-EGN is suitable for this scope, since the modeled individuals are naturally able to exhibit both full and partial level of cooperation, as well as full defection. Indeed, it embeds three different consensus steady states: $\mathbf{x}^*_{AC}$, where all players are fully cooperative, $\mathbf{x}^*_{AM}$ where all players are partially cooperative, and $\mathbf{x}^*_{AD}$, where all players are fully defective. While $\mathbf{x}^*_{AD}$ should be avoided, $\mathbf{x}^*_{AC}$ and $\mathbf{x}^*_{AM}$ are both desirable. In this paper we study and compare the stability conditions and the effectiveness of the last two states in the Stag-Hunt (SH) [13,40–42] and Chicken (CH) [12,13,26,40] games. In [35], the conditions for the onset of the fully cooperative consensus steady state have been found for the Prisoner's Dilemma game (PD). Anyway, for this specific game, the convergence towards a partially cooperative consensus is not allowed.

The main finding of this paper is that in both SH and CH games, while consensus over the full cooperative state is asymptotically stable for feasible values of the self-regulating parameter, the consensus over partial cooperative state is globally asymptotically stable. As a consequence, the stronger cooperative consensus is reached only from a suitable set of initial conditions. On the other hand, the weaker cooperative consensus is reached from any initial condition. Additionally, we show that $\mathbf{x}^*_{AM}$ can produce an higher than $\mathbf{x}^*_{AC}$. This results highlight the importance of studying SH and CH games, which can lead to an deeper understanding of the mechanisms leading towards cooperation in real world situations.

The paper is structured as follows: Section 2 illustrates previous results that will be used along the study. In Section 3 the definition of the main concepts is introduced. Section 4 presents the results on asymptotic stability of $\mathbf{x}^*_{AC}$ and $\mathbf{x}^*_{AD}$, and on asymptotic and global stability of $\mathbf{x}^*_{AM}$. In Section 5 several numerical experiments are developed and the discussion on the main findings is reported. Finally, some conclusions and further developments are presented in Section 6.

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2. Preliminaries

The study developed in this paper is grounded on a recently introduced equation, namely the SR-EGN model [35], which represents a framework for understanding the evolution of cooperation under the effect of self regulation in a structured population.

Following [35], we consider a population of N players, labeled by $v = \{1, \dots, N\}$, arranged on an undirected graph described by the adjacency matrix $\mathbf{A} = \{a_{v,w}\}$ ($a_{v,w} = a_{w,v} = 1$ when v plays against w, 0 otherwise). Individuals play two-person games with their neighbors, will we assume $a_{v,v} = 0 \ \forall v \in \mathcal{V}$. The degree of player v is the number of its connection and it is denoted by $k_v = \sum_{w=1}^{N} a_{v,w}$. The vector $\mathbf{k}$ collects all the degrees k_vs. We denote with $x_v \in \Delta = [0,1]$ the level of cooperation of player v, and with $1 - x_v$ its level of defection. Specifically, a player with $x_v \in \text{int}(\Delta) = (0,1)$ is exhibiting a partial level of cooperation, while full cooperator has $x_v = 1$ and a free rider is characterized by $x_v = 0$. The cooperation level of each individual is collected in the vector $\mathbf{x} = [x_1, \dots, x_N]^\top \in \Delta^N$. When any two connected players, let's say v and w, take part in a game, the outcome of v is defined by the payoff function $\phi : \Delta \times \Delta \to \mathbb{R}$. In the considered interconnected context, the total payoff function $\phi_v : \Delta^N \to \mathbb{R}$ of a generic player v corresponds to the sum of all payoffs gained with neighbors, and it is defined as follows:

$$\phi_v(\mathbf{x}) = \sum_{w=1}^{N} a_{v,w}\phi(x_v, x_w). \tag{1}$$

The player v is able to appraise if a change of his own strategy x_v leads to an improvement of the payoff ϕ_v. That is, if the derivative of ϕ_v with respect to x_v is positive (negative), the player will increase (decrease) his strategy over time. Of course, when this derivative is null, then the player has reached a steady state. In this sense, $\frac{\partial \phi_v}{\partial x_v}$ represents the external feedback perceived by player v from the environment which influences his own strategy dynamics [35].

It is important to notice that the payoff ϕ assumes the following form:

$$\phi(x_v, x_w) = \begin{bmatrix} x_v \\ 1 - x_v \end{bmatrix}^\top \mathbf{B} \begin{bmatrix} x_w \\ 1 - x_w \end{bmatrix}. \tag{2}$$

In the previous formula, $\mathbf{B}$ represents the payoff matrix which entries are the income obtained by player v when playing against w, and it is defined as follows:

$$\mathbf{B} = \begin{bmatrix} 1 & S \\ T & 0 \end{bmatrix}, \tag{3}$$

where 1 represents the reward collected when both players cooperate, T is the temptation to defect when opponent cooperates, S is the sucker's payoff earned by a cooperative player when the opponent is a free rider, and 0 is the punishment for mutual defection.

Using (3), equation (2) can be rewritten as follows:

$$\phi(x_v, x_w) = x_v(1 \cdot x_w + S(1 - x_w)) + (1 - x_v)(Tx_w + 0 \cdot (1 - x_w)). \tag{4}$$

Accordingly, the derivative of (4) with respect to x_v is:

$$\frac{\partial \phi(x_v, x_w)}{\partial x_v} = (1 - T - S)x_w + S. \tag{5}$$

For the ease of notation, we introduce the function:

$$f(z) = (1 - T - S)z + S. \tag{6}$$

Hence, $\frac{\partial \phi(x_v,x_w)}{\partial x_v} = f(x_w)$, and the derivative of the total payoff (1) is:

$$\begin{aligned}\frac{\partial \phi_v}{\partial x_v} &= \frac{\partial}{\partial x_v}\sum_{w=1}^{N} a_{v,w}\phi(x_v,x_w) \\ &= \sum_{w=1}^{N} a_{v,w}\frac{\partial \phi(x_v,x_w)}{\partial x_v} \\ &= \sum_{w=1}^{N} a_{v,w}[(1-T-S)x_w + S] \\ &= k_v[(1-T-S)\overline{x}_v + S] \\ &= k_v f(\overline{x}_v),\end{aligned} \tag{7}$$

where $\overline{x}_v = \frac{1}{k_v}\sum_{w=1}^{N} a_{v,w}x_w$ represents the equivalent player perceived by player v, i.e. the average of the strategies of all his neighbors.

The internal feedback is assumed to have the same form of (6) as a function of x_v itself:

$$f(x_v) = (1-T-S)x_v + S. \tag{8}$$

In equation (8) an individual v has the role of one of his "opponent", thus answering questions like "What kind of reward can I earn if I apply this strategy to myself?". SR-EGN accounts for internal feedback induced by cultural traits, awareness, altruism, learning and so on [1,11]. Indeed, when an individual judges cooperation as a greater good, there must be some inertial mechanisms reducing the (rational) temptation to defect, represented by the function $f(x_v)$, weighted by a parameter β_v. Both external and internal feedback represent the two main ingredients of the SR-EGN equation, which reads as follows:

$$\dot{x}_v = x_v(1-x_v)\left(\frac{\partial \phi_v}{\partial x_v} - \beta_v f(x_v)\right). \tag{9}$$

When $\beta_v = 0$, the individual is somehow "member of the flock", since his strategy changes only according to the outcomes of the game interaction with neighbors. This effect is particularly dramatic in the classical PD context, as cooperation completely disappears from the population. In this direction, β_v can be also interpreted as a resistance of the player to external feedback [38]. In particular, $\beta_v > 0$ represents a negative feedback, $\beta_v < 0$ stands for a positive feedback, while $\beta_v = 0$ refers to situations where the player v does not play a self game.

Since parameter β_v regulates the self game, we introduce the following modified adjacency matrix

$$\mathbf{A}'(\boldsymbol{\beta}) = \mathbf{A} - \mathrm{diag}(\boldsymbol{\beta}), \tag{10}$$

where

$$\boldsymbol{\beta} = [\beta_1, \ldots, \beta_N]^\top.$$

The parameters β_v on the diagonal of $\mathbf{A}'(\boldsymbol{\beta})$ are the weights of the self loops in the network, and thus model the presence of self games.

3. Emergence of cooperation and consensus

Social dilemmas involving the cooperation are described by three main game classes [13,25,26]: the PD game, characterized by $T > 1 > 0 > S$, the SH game with $1 > T > 0 > S$, and the CH game where $T > 1 > S > 0$. All these games are characterized by social tensions [25]: when $T > 1$ (CH and PD games) players prefer unilateral defection than mutual cooperation; when $S < 0$ (SH and PD games) players have a preference for

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mutual defection in spite of unilateral cooperation. For completeness, the tension-free case where cooperation is always preferred is called Harmony game (HA), and it is characterized by $0 < T < 1$ and $0 < S < 1$. The game classification with respect to parameters T and S is briefly reported in Figure 1.

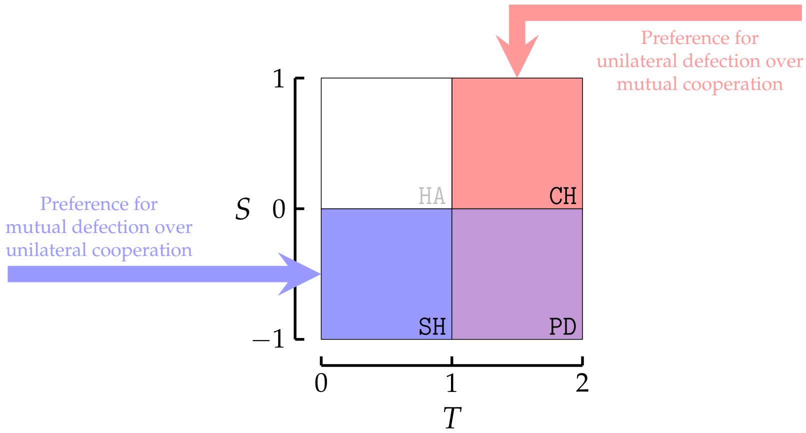

Figure 1. Different game types according to the values of parameters T and S.

Figure 2 shows five possible asymptotic configurations in a simple social network with $N = 5$ individuals. The color of each node of the graph denotes the level of cooperation of the corresponding player - red for full defectors, yellow for full cooperators and orange shadings for intermediate levels. In the first graph of Figure 2, a generic configuration is shown, including defectors (players 1 and 5), one cooperator (player 2), and mixed ones (players 3 and 4). The second graph in Figure 2 shows a population without full defectors (i.e. $x_v > 0\ \forall v \in \mathcal{V}$). Moreover, the third graph shows consensus towards the partially cooperative steady state $x_v^* = m \in \text{int}(\Delta)\ \forall v \in \mathcal{V}$.

In this study, we focus on the following consensus states that are also steady states of (9):

- Full cooperation (pure strategy): $\mathbf{x}_{AC}^* = [1, 1, \ldots, 1]^\top$.
- Full defection (pure strategy): $\mathbf{x}_{AD}^* = [0, 0, \ldots, 0]^\top$.
- Partial cooperation (mixed strategy): $\mathbf{x}_{AM}^* = [m, m, \ldots, m]^\top$, where $m \in \text{int}(\Delta)$.

Their stability will be deeply analyzed later in this paper. For this reason we will refer to them as *consensus steady states*.

Following [25–27,29,43], full cooperation is reached when all the members of a social network turn their strategies to cooperation. This concept can be formally defined as follows:

Definition 1. *In SR-EGN equation* (9) *full cooperation emerges if*

$$\lim_{t \to +\infty} x_v(t) = 1\ \forall v \in \mathcal{V},$$

for any initial condition $\mathbf{x}(0) \in int(\Delta^N)$.

In the previous definition we used the adjective "full", since the concept of cooperation can be weakened dealing with players which can be partial cooperators. Therefore, we introduce the following weaker definition of emergence of cooperation:

Definition 2. *In SR-EGN equation* (9) *consensus on partial cooperation emerges if:*

$$\lim_{t \to +\infty} x_v(t) = m\ \forall v \in \mathcal{V},$$

with $m \in int(\Delta)$ *for any initial condition* $\mathbf{x}(0) \in int(\Delta^N)$.

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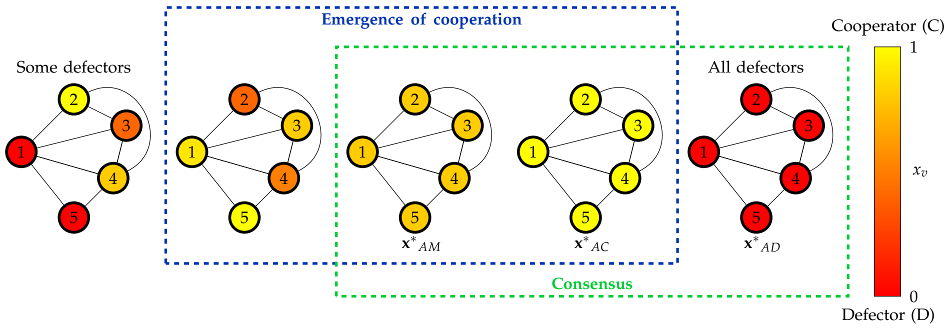

Figure 2. Emergence of cooperation and consensus. Five configurations of a social network of 5 players. The cooperation level of each player is represented by a color ranging from red (full defection, $x_v = 0$) to yellow (full cooperation, $x_v = 1$), as reported on the shaded box on the rightmost part of the figure. The first graph shows a generic configuration with cooperators, defectors and mixed players. The second, third and forth graphs show situations where no full defector is present. These three situations, highlighted by the blue dashed line box, are examples of emergence of cooperation. In particular, the third graph corresponds to the steady state $\mathbf{x}^*{}_{AM}$, while the forth represents the steady state $\mathbf{x}^*{}_{AC}$. The fifth graph shows a population composed of full defectors only, thus SR-EGN converged to the steady state $\mathbf{x}^*{}_{AD}$. The third, forth and fifth graphs represent consensus states, since all players reach the same level of cooperation (green dashed line box).

In [35], sufficient conditions for full cooperation and full defection have been proven for the PD game case (see Main Results 2 and Main Results 3). In the next sections, we will develop results on the of cooperation and consensus when the game played in the population is SH or CH. Specifically, in order to study the stability of steady states $\mathbf{x}^*_{AC}$ and $\mathbf{x}^*_{AM}$, we start by analyzing their linear stability. Alongside this analysis, we also investigate the stability of $\mathbf{x}^*_{AD}$. Finally, appropriate Lyapunov functions will be found to guarantee the emergence of cooperation in partial sense.

4. Results on the emergence of cooperative consensus

4.1. Steady states

A steady state $\mathbf{x}^*$ is a solution of equation (9) satisfying $\dot{x}_v = 0 \; \forall v \in \mathcal{V}$. In order to be feasible, the steady state components must lay in Δ. Formally, the set of feasible steady states is:

$$\Theta = \left\{ \mathbf{x}^* \in \Delta^N : x_v^*(1 - x_v^*) \left(\frac{\partial \phi_v}{\partial x_v}(\mathbf{x}^*) - \beta_v f(x_v^*) \right) = 0 \; \forall v \in \mathcal{V} \right\}.$$

It is clear that all points such that for all v, $x_v^* = 0$ or $x_v^* = 1$ are in the set Θ. They are 2^N and we will refer to them as pure steady states. We denote with $\Theta^P \subseteq \Theta$ their set, which includes $\mathbf{x}^*_{AC}$ and $\mathbf{x}^*_{AD}$ among the others.

Mixed steady states may exist when:

$$\frac{\partial \phi_v}{\partial x_v}(\mathbf{x}^*) - \beta_v f(x_v^*) = 0 \; \forall v \in \mathcal{V}, \tag{11}$$

and $x_v^* \in \text{int}(\Delta)$. We denote the set of mixed steady states with $\Theta^M \subset \Theta$. In [22], it has been shown that, if $1 - T - S \neq 0$, the solution of (11), is $\mathbf{x}^*_{AM} = [m, \dots, m]^\top$, where

$$m = \frac{S}{S + T - 1}. \tag{12}$$

$\mathbf{x}^*_{AM}$ is feasible when $m \in \text{int}(\Delta)$; this is allowed only in SH and CH games. Indeed:

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- for SH games, $S < 0$ and $0 < T < 1$. Then, $S < S + T - 1 < 0$ and hence $m \in \text{int}(\Delta)$;
- for CH games, $S > 0$ and $T > 1$. Then, $0 < S < S + T - 1$ implying that $m \in \text{int}(\Delta)$.

For the sake of completeness, we remark that SR-EGN may also have pure-mixed steady states, which belong to the set $\Theta^{PM} = \Theta \setminus (\Theta^P \cup \Theta^M)$. These are not considered in this work, since they do not represent consensus steady states.

4.2. Linearization of SR-EGN model

The Jacobian matrix of system (9), $\mathbf{J}(\mathbf{x}) = \{j_{v,w}(\mathbf{x})\}$, is defined as follows:

$$j_{v,w}(\mathbf{x}) = \frac{\partial \dot{x}_v}{\partial x_w} = \begin{cases} x_v(1-x_v)(1-T-S), & \text{if } a_{v,w} = 1 \\ -\beta_v x_v(1-x_v)(1-T-S) \\ +(1-2x_v)(k_v f(\overline{x}_v) - \beta_v f(x_v)), & \text{if } w = v \\ 0, & \text{otherwise} \end{cases}. \tag{13}$$

Evaluating the Jacobian matrix on a generic steady state $\mathbf{x}^* \in \Theta$, we have the following cases:

- if $x_v^* \in \{0,1\}$ (player v uses a pure strategy at steady state), then all in (13) non-diagonal entries $j_{v,w}(\mathbf{x})$ with $v \neq w$ are null, while

$$j_{v,v}(\mathbf{x}^*) = \begin{cases} k_v f(\overline{x}_v^*) - \beta_v f(x_v^*), & \text{if } x_v^* = 0 \\ -(k_v f(\overline{x}_v^*) - \beta_v f(x_v^*)), & \text{if } x_v^* = 1 \end{cases}. \tag{14}$$

- if $x_v^* \in (0,1)$ (player v uses a mixed strategy at steady state), then, according to equations (11) and (13), the entries of the v-th row of $\mathbf{J}(\mathbf{x}^*)$ are:

$$j_{v,w}(\mathbf{x}^*) = \begin{cases} x_v^*(1-x_v^*)(1-T-S), & \text{if } a_{v,w} = 1 \\ -\beta_v x_v^*(1-x_v^*)(1-T-S), & \text{if } w = v \\ 0, & \text{otherwise} \end{cases}. \tag{15}$$

4.3. Stability of pure consensus steady states $\mathbf{x}^*_{AC}$ and $\mathbf{x}^*_{AD}$

Recall that the spectrum of $\mathbf{J}(\mathbf{x}^*)$ characterizes the linear stability of a steady state $\mathbf{x}^*$ of the SR-EGN equation (see [44]). Therefore, the role of the eigenvalues of the Jacobian matrix $\mathbf{J}(\mathbf{x}^*)$ is fundamental to tackle the problem of the emergence of cooperation.

According to equation (14), the Jacobian matrix evaluated in the consensus steady states $\mathbf{x}^*_{AC}$ and $\mathbf{x}^*_{AD}$ are both diagonal, and they read as:

$$\mathbf{J}(\mathbf{x}^*_{AC}) = -(1-T) \cdot \text{diag}(\mathbf{k} - \boldsymbol{\beta}), \tag{16}$$

and

$$\mathbf{J}(\mathbf{x}^*_{AD}) = S \cdot \text{diag}(\mathbf{k} - \boldsymbol{\beta}). \tag{17}$$

The following results hold.

Theorem 1.

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- *SH game. If $\beta_v < k_v \; \forall v \in \mathcal{V}$ then $\mathbf{x}^*_{AC}$ is asymptotically stable for equation* (9);
- *CH game. If $\beta_v > k_v \; \forall v \in \mathcal{V}$ then $\mathbf{x}^*_{AC}$ is asymptotically stable for equation* (9).

Proof. The diagonal elements of the Jacobian matrix evaluated in $\mathbf{x}^*_{AC}$ (equation (16)) correspond to its eigenvalues and they are read as:

$$j_{v,v}(\mathbf{x}^*_{AC}) = \lambda_v = -(1-T)(k_v - \beta_v).$$

In SH Game, $T < 1$ and $\beta_v < k_v$, hence, all eigenvalues are negative. In CH Game, $T > 1$ and $\beta_v > k_v$, and again all eigenvalues are negative. Thus, $\mathbf{x}^*_{AC}$ is asymptotically stable in both cases. □

Notice that Theorem 1 is an extension of the *Main result 1* of [35] to SH and CH games. Additionally, Theorem 2 stated in [22] ensures that any asymptotically stable pure steady state is also a Nash equilibrium, and viceversa. Then, under the hypotheses of Theorem 1, $\mathbf{x}^*_{AC}$ is a Nash equilibrium of the networked game.

Theorem 2.

- *SH game. If $\beta_v < k_v \; \forall v \in \mathcal{V}$ then $\mathbf{x}^*_{AD}$ is asymptotically stable for equation* (9);
- *CH game. If $\beta_v > k_v \; \forall v \in \mathcal{V}$ then $\mathbf{x}^*_{AD}$ is asymptotically stable for equation* (9).

Proof. The eigenvalues of the Jacobian matrix relative to the steady state $\mathbf{x}^*_{AD}$ (equation (17)) are:

$$j_{v,v}(\mathbf{x}^*_{AD}) = \lambda_v = S(k_v - \beta_v).$$

In SH game, $S < 0$ and $\beta_v < k_v$, hence all eigenvalues are negative. In CH game, then $S > 0$ and $\beta_v > k_v$, hence all eigenvalues are negative. Thus, $\mathbf{x}^*_{AD}$ is an asymptotically stable steady state in both cases. □

Also Theorem 2 represents an extension of *Main Result 1* in [35] to SH and CH games. Moreover, the result of the previous Theorem, jointly with the Theorem 2 enunciated in [22], imply that $\mathbf{x}^*_{AD}$ is not a Nash equilibrium of the underlying game.

These results are synthesized in Figures 3.a and 3.b. In PD game, defection dominates over cooperation. Then, if the system does not present any feedback mechanisms (i.e. $\beta_v = 0 \; \forall v \in \mathcal{V}$), the whole social network will converge to $\mathbf{x}^*{}_{AD}$, where cooperation vanishes. Anyway, if $\beta_v > k_v$ for all the members of the population, $\mathbf{x}^*{}_{AD}$ is destabilized and $\mathbf{x}^*{}_{AC}$ becomes attractive.

A SH game naturally exhibits bistability, a common property of many social and biological systems [13,40–42]. Indeed, in the natural case where $\boldsymbol{\beta} = \mathbf{0}$ we have a repulsive equilibrium $\mathbf{x}^*_{AM}$ standing between two attractive equilibria $\mathbf{x}^*_{AC}$ and $\mathbf{x}^*_{AD}$. In the SR-EGN, for $\beta_v < k_v \; \forall v \in \mathcal{V}$, $\mathbf{x}^*_{AC}$ and $\mathbf{x}^*_{AD}$ are both attractive, while they are both unstable for $\beta_v > k_v \; \forall v \in \mathcal{V}$.

CH games represent an important class of social dilemmas [12,13,26,40], where cooperators and free riders coexist. In the standard replicator equation, total cooperation and total defection are now repulsive equilibria, while the mixed steady state is attractive. In the base case in the natural case where $\boldsymbol{\beta} = \mathbf{0}$ of the SR-EGN model, the mixed stead state $\mathbf{x}^*{}_{AM}$ is also present, but it is not attractive when no feedback is present, and it stands between the two repulsive steady states $\mathbf{x}^*_{AC}$ and $\mathbf{x}^*_{AD}$. When self-regulation is active, and in particular $\beta_v > k_v \; \forall v \in \mathcal{V}$, then both $\mathbf{x}^*_{AC}$ and $\mathbf{x}^*_{AD}$ become asymptotically stable.

Following Theorems 1 and 2, if in both SH and CH games, $\mathbf{x}^*_{AC}$ and $\mathbf{x}^*_{AD}$ are unstable, we can investigate the presence of other asymptotically stable steady states in $\mathrm{int}(\Delta^N)$. In the next Section 4.4, we find the conditions for which the state $\mathbf{x}^*_{AM}$ is asymptotically

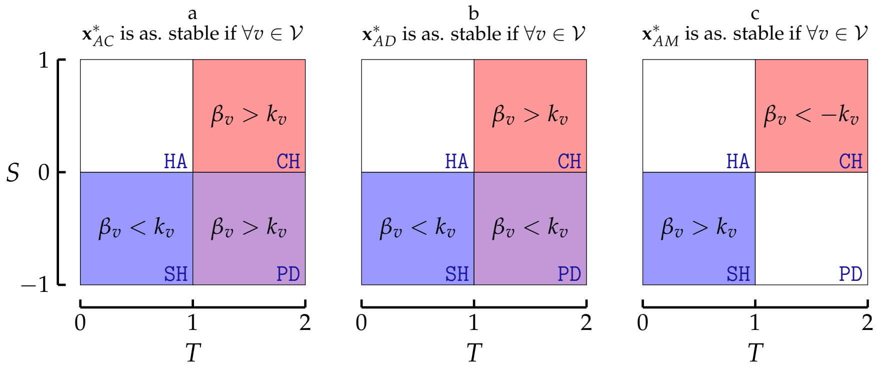

Figure 3. Graphical representation of Theorems 1 (subplot a), 2 (subplot b) and 3 (subplot c).

stable, and prove that it is also globally asymptotically stable. This means that no other attractive states exists in $\mathrm{int}(\Delta^N)$.

4.4. Results on asymptotical and global stability of mixed consensus steady states $\mathbf{x}^*_{AM}$

According to equations (12) and (15) the Jacobian matrix for the mixed steady state $\mathbf{x}^*_{AM}$ is:

$$\mathbf{J}(\mathbf{x}^*_{AM}) = m(1-m)(1-T-S)(\mathbf{A} - \mathrm{diag}(\boldsymbol{\beta})) = \frac{S(T-1)}{1-T-S}\mathbf{A}'(\boldsymbol{\beta}). \tag{18}$$

The following result holds.

Theorem 3.

- *SH game. If $\beta_v > k_v$ $\forall v \in \mathcal{V}$ then $\mathbf{x}^*_{AM}$ is asymptotically stable for equation (9);*
- *CH game. If $\beta_v < -k_v$ $\forall v \in \mathcal{V}$ then $\mathbf{x}^*_{AM}$ is asymptotically stable for equation (9).*

Proof. If $|\beta_v| > k_v$ $\forall v \in \mathcal{V}$, then $\mathbf{A}'(\boldsymbol{\beta})$ is a strictly diagonally dominant matrix. Indeed, k_v corresponds to the sum of all non-diagonal entries of the v-th row of $\mathbf{A}'(\boldsymbol{\beta})$, while $-\beta_v$ are the diagonal entries of $\mathbf{A}'(\boldsymbol{\beta})$.

In the SH game, since $\beta_v > k_v$ $\forall v \in \mathcal{V}$ and $\mathbf{A}$ has null diagonal, then diagonal entries of $\mathbf{A}'(\boldsymbol{\beta})$ are negative. Therefore, from the strict diagonal dominance of $\mathbf{A}'(\boldsymbol{\beta})$ it follows that all the eigenvalues of $\mathbf{A}'(\boldsymbol{\beta})$ are negative. Moreover, since $T < 1$ and $S < 0$, then $1 - T - S > 0$ and $S(T-1) > 0$. Thus, according to equation (18), the eigenvalues of $\mathbf{J}(\mathbf{x}^*_{AM})$ are all negative.

In the CH game, for the same reasons as above, since $\beta_v < -k_v$ $\forall v \in \mathcal{V}$, then all the eigenvalues of $\mathbf{A}'(\boldsymbol{\beta})$ are positive. Moreover, since $T > 1$ and $S > 0$, then $1 - T - S < 0$ and $S(T-1) > 0$. Thus, according to equation (18), the eigenvalues of $\mathbf{J}(\mathbf{x}^*_{AM})$ are all negative.

In both cases, $\mathbf{x}^*_{AM}$ is an asymptotically stable steady state for equation (9). □

This result is graphically shown in Figure 3.c.

The consensus on full cooperation can emerge under the condition of Theorem 1. Anyway, the basin of attraction of $\mathbf{x}^*_{AC}$ does not correspond to the whole set $\mathrm{int}(\Delta^N)$, since $\mathbf{x}^*_{AD}$ is asymptotically stable for the same parameters. Hence, we cannot expect to reach global consensus towards full cooperation. Moreover, we showed in Theorem 3 that $\mathbf{x}^*_{AM}$ is asymptotically stable for suitable conditions. Then, we further check if there

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are conditions for which it is also globally asymptotically stable.

Hereafter we introduce a function $V(\mathbf{x})$ that will be proved to be a Lyapunov function, allowing us to prove that $\mathbf{x}^*_{AM}$ is globally asymptotically stable:

$$V(\mathbf{x}) = \sum_{v=1}^{N} \left(m \log\left(\frac{m}{x_v}\right) + (1-m)\log\left(\frac{1-m}{1-x_v}\right)\right), \tag{19}$$

for $\mathbf{x} \in \text{int}(\Delta^N)$. First of all, notice that $V(\mathbf{x}^*_{\mathbf{AM}}) = 0$. Moreover, the gradient of $V(\mathbf{x})$ is null for $\mathbf{x}^*_{\mathbf{AM}}$. Indeed, the partial derivatives of $V(\mathbf{x})$ with respect to $\mathbf{x}$ are:

$$\frac{\partial V(\mathbf{x})}{\partial x_v} = -\frac{m}{x_v} + \frac{1-m}{1-x_v} = \frac{x_v - m}{x_v(1-x_v)}.$$

It is straightforward to see that the Hessian matrix $\mathbf{H}(\mathbf{x}) = \{h_{v,w}(\mathbf{x})\}$ of $V(\mathbf{x})$ is diagonal:

$$h_{v,v}(\mathbf{x}) = \frac{\partial^2 V(\mathbf{x})}{\partial x_v^2} = \frac{x_v^2 - 2mx_v + m}{x_v^2(1-x_v)^2}. \tag{20}$$

From (20) it follows that $V(\mathbf{x}) \in \mathcal{C}^2$ for all $\mathbf{x}$ and it is definite positive $\forall \mathbf{x} \in \text{int}(\Delta^N)$. Indeed the denominator of $h_{v,v}(\mathbf{x})$ is always positive as its numerator:

$$x_v^2 - 2mx_v + m = x_v^2 - 2mx_v + m^2 + m - m^2 = (x_v - m)^2 + m(1-m) > 0.$$

This proves that $V(\mathbf{x})$ is a strictly convex function in the set $\text{int}(\Delta^N)$. So, $\mathbf{x}^*_{\mathbf{AM}}$ is a global minimum of $V(\mathbf{x})$ in the set $\text{int}(\Delta^N)$.

The time derivative of $V(\mathbf{x})$ is:

$$\begin{aligned} \dot{V}(\mathbf{x}) = \frac{\partial V(\mathbf{x})}{\partial t} &= \sum_{v=1}^{N} \frac{\partial V(\mathbf{x})}{\partial x_v} \dot{x}_v \\ &= \sum_{v=1}^{N} \frac{x_v - m}{x_v(1-x_v)} x_v(1-x_v)[k_v f(\overline{x}_v) - \beta_v f(x_v)] \\ &= \sum_{v=1}^{N} (x_v - m)[k_v f(\overline{x}_v) - \beta_v f(x_v)]. \end{aligned} \tag{21}$$

By substituting (12) in equation (8), we get that:

$$f(z) = (1-T-S)z + S = (1-T-S)\left(z + \frac{S}{1-T-S}\right) = (1-T-S)(z-m),$$

and equation (21) becomes:

$$\dot{V}(\mathbf{x}) = (1-T-S) \sum_{v=1}^{N} (x_v - m)[k_v(\overline{x}_v - m) - \beta_v(x_v - m)].$$

Notice that:

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$$
\begin{aligned}
k_v(\bar{x}_v - m) &= k_v\bar{x}_v - k_v m \\
&= k_v\left(\frac{1}{k_v}\sum_{w=1}^{N} a_{v,w}x_w\right) - \left(\sum_{w=1}^{N} a_{v,w}\right)m \\
&= \sum_{w=1}^{N} a_{v,w}x_w - \sum_{w=1}^{N} a_{v,w}m \\
&= \sum_{w=1}^{N} a_{v,w}(x_w - m).
\end{aligned}
$$

This yields to:

$$
\begin{aligned}
\dot{V}(\mathbf{x}) &= (1-T-S)\sum_{v=1}^{N}(x_v - m)\left[\sum_{w=1}^{N} a_{v,w}(x_w - m) - \beta_v(x_v - m)\right]. \\
&= (1-T-S)\sum_{v=1}^{N}\left(\sum_{w=1}^{N} a_{v,w}(x_w - m)(x_v - m) - \beta_v(x_v - m)^2\right) \\
&= (1-T-S)\left(\sum_{v=1}^{N}\sum_{w=1}^{N} a_{v,w}(x_w - m)(x_v - m) - \sum_{v=1}^{N}\beta_v(x_v - m)^2\right) \\
&= (1-T-S)(\mathbf{x} - \mathbf{x}^*_{\mathbf{AM}})^\top(\mathbf{A} - \mathrm{diag}(\boldsymbol{\beta}))(\mathbf{x} - \mathbf{x}^*_{\mathbf{AM}}) \\
&= (\mathbf{x} - \mathbf{x}^*_{\mathbf{AM}})^\top((1-T-S)\mathbf{A}'(\boldsymbol{\beta}))(\mathbf{x} - \mathbf{x}^*_{\mathbf{AM}}) \\
&= (\mathbf{x} - \mathbf{x}^*_{\mathbf{AM}})^\top\mathbf{M}(\mathbf{x} - \mathbf{x}^*_{\mathbf{AM}}),
\end{aligned}
$$

where $\mathbf{M} = (1-T-S)\mathbf{A}'(\boldsymbol{\beta})$ is a symmetric matrix. Notice that $\dot{V}(\mathbf{x})$ is a quadratic form and $\dot{V}(\mathbf{x}^*_{\mathbf{AM}}) = 0$.

We now prove the main result which states that x^*_{AM} is globally asymptotically stable for SH and CH games.

Theorem 4.

- *SH game. If $\beta_v > k_v\ \forall v \in \mathcal{V}$ then $\mathbf{x}^*_{AM}$ is globally asymptotically stable for equation (9);*
- *CH game. If $\beta_v < -k_v\ \forall v \in \mathcal{V}$ then $\mathbf{x}^*_{AM}$ is globally asymptotically stable for equation (9).*

Proof. According to equation (18), the matrix $\mathbf{M}$ is related to the the Jacobian matrix evaluated in the internal steady state $\mathbf{x}^*_{AM}$ as follows:

$$
\mathbf{M} = \frac{1}{m(1-m)}\mathbf{J}(\mathbf{x}^*_{AM}). \tag{22}
$$

Since $m(1-m) > 0$, then using the same arguments of Theorem (3), we have that all the eigenvalues of $\mathbf{M}$ are negative.

Therefore, the quadratic form $\dot{V}(\mathbf{x})$ is negative definite in the set $\mathrm{int}(\Delta^N)$, and then $V(\mathbf{x})$ (equation (19)) is a Lyapunov function in the set $\mathrm{int}(\Delta^N)$. Thus, $\mathbf{x}^*_{AM}$ is a global attractor in the set $\mathrm{int}(\Delta^N)$.
□

Theorem 4 states that the system (9) shows consensus towards partial cooperation. Notice that under the assumptions of Theorem 4, both Theorems 1 and 2 are not satisfied, and hence both $\mathbf{x}^*{}_{AC}$ and $\mathbf{x}^*{}_{AD}$ are unstable.

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5. Discussion and numerical results

The theoretical results presented in the previous Sections have been numerically tested by means of several simulation experiments.

In the first experiment we show, for each game, the numerical solutions of equation (9) over time for a graph of 20 nodes arranged over scale-free topology and different types of initial conditions (see Figure 4). Subplots 4.a and 4.d have been realized using $\beta_v = -2k_v \; \forall v \in \mathcal{V}$; in subplots 4.b and 4.e we fixed $\beta_v = 0 \; \forall v \in \mathcal{V}$; finally, for subplots 4.c and 4.f, we have set $\beta_v = 2k_v \; \forall v \in \mathcal{V}$. In all cases, the first and third slices show the dynamics obtained for i.c.s of all players close to defection and cooperation, respectively. The central slice is obtained by using random i.c.s. The bistable dynamics of the two games is visible: i.c.s near 0 lead to defective asymptotic states and i.c.s near 1 drive towards cooperation (subplots 4.a, 4.b and 4.f). In these three cases, both Theorems 1 and 2 are satisfied. Regarding subplot 4.e, we observe that, using random initial conditions, the solutions convergence pure-mixed steady states ($\beta_v = 0 \; \forall v \in \mathcal{V}$). Finally, the subplots 4.c and 4.d are realized using the parameters of the hypotheses of Theorems 3 and 4, and hence consensus towards a partial cooperation is reached.

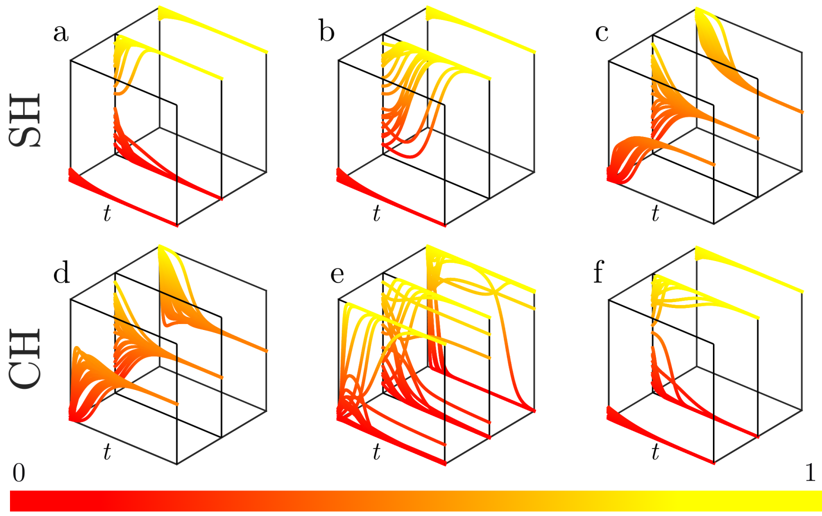

Figure 4. First row: simulations of the SR-EGN equation (9) for the SH game ($T = 0.5$ and $S = -0.5$) with $\beta_v = -2k_v < -k_v \; \forall v \in \mathcal{V}$ (subplot a), $\beta_v = 0 \; \forall v \in \mathcal{V}$ (subplot b) and $\beta_v = 2k_v > k_v \; \forall v \in \mathcal{V}$ (subplot c). The first slice refers to simulations with random initial conditions close to full defection, the middle one refers to simulations with random initial conditions in $\text{int}(\Delta^N)$, while the last one refers to simulations with random initial conditions close to full cooperation. Second row: the same simulations are carried out for CH game ($T = 1.5$ and $S = 0.5$).

In order to evaluate the performance of players decisions, in Figure 5 (first row) we draw the payoffs obtained by each individual in a two-players game, when the stable steady state is $\mathbf{x}^*_{AM}$, as a function of parameters T and S, while the second row shows the value of such steady state in the same conditions. The white line in all subplots indicates the combinations of T and S for which the payoff is equal to 1, which corresponds to the hypothetical value earned if a full cooperative state is considered. It is worth noticing that in the green regions the payoff (subplots 5.a and 5.b) earned by the partially cooperative state $\mathbf{x}^*_{AM}$ (subplots 5.c and 5.d) is larger then the one obtained by full cooperation, although $\mathbf{x}^*_{AM}$ is lower than $\mathbf{x}^*_{AC}$. This fact is particularly relevant since this kind of consensus is more reasonable in any real situation, thus highlighting the importance of studying SH and CH games with respect to the over-exploited PD game.

Further experiments consider a population of $N = 100$ individuals arranged on different types of networks, such as Erdös-Rényi (ER) and Scale-Free (SF), with average degree equal 10. We set the same value of β_v for all members of the population. For each

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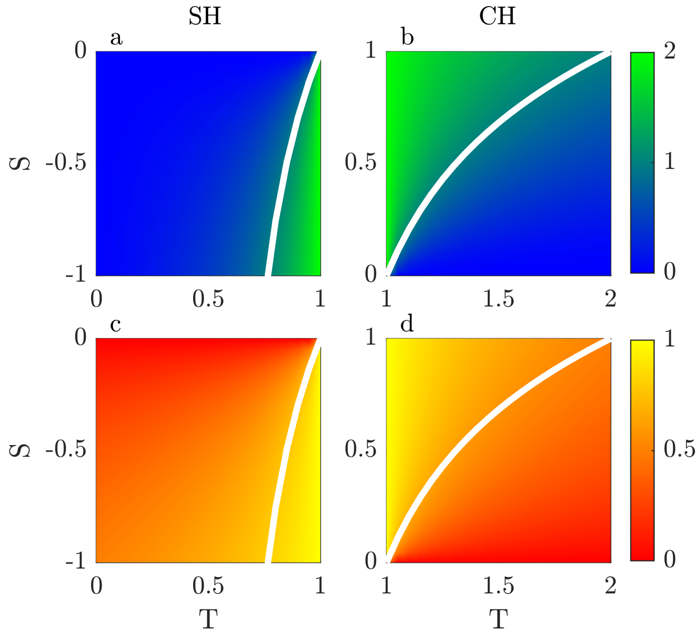

Figure 5. Payoff $\phi(x_v, x_w)$ (equation (4)) and mixed equilibrium $\mathbf{x}^*_{AM}$ (equation (12)). SH game (subplots a and c): $T \in [0,1]$ and $S \in [-1,0]$. CH game (subplots b and d): $T \in [1,2]$ and $S \in [0,1]$. The white lines indicate the locus of values S and T for which the players choose full cooperation (subplots a and b) and the corresponding values of $\mathbf{x}^*_{AM}$ (subplots c and d) .

network type, we simulate 500 instances of the problem starting with initial conditions randomly chosen in the set int(Δ). ER and SF graphs are also randomly generated for each instance. The same experiment has been repeated for the SH and CH game configurations using the values of T and S used in Figure 6. For each player and for each simulation, the value of an indicator which measures the difference between the level of cooperation x_v of player v and the average cooperation level $\overline{x}$ of his neighbors at steady state, is calculated (Figure 6). This indicator, hereafter called c_v, is defined as follows:

$$c_v = x_v(\infty) - \overline{x}_v(\infty).$$

If $c_v > 0$, then player v is an altruist since it cooperates more than the average of his neighbors, while $c_v < 0$ indicates a more selfish behavior. In each subplot, we also depict the degree distributions of the underlying connection networks with superposed gray lines. The subdivision into two subgroups (non-central players and hubs) is observed for all the considered games. In the SH case (subplots 6.a and 6.c), this subdivision is coherent with the players connectivity: low connections present negative c_v (selfish behavior), while high connectivity drives players to adopt an altruistic strategy as indicated by positive c_v. For the CH case (subplots 6.b and 6.d), the relationship between the degree k_v and c_v vanishes: indeed, there are both altruistic and selfish players independently on their connectivity. This phenomenon is more evident in the SF case.

6. Conclusions

In this paper, the emergence of cooperative consensus for SH and CH games have been tackled in the framework of the SR-EGN equation, which describes the behavior of a population of randomly interconnected individuals, driven by game theoretic mechanisms jointly with internal self-regulating factors.

We proved that in both SH and CH games, consensus over the fully cooperative state is asymptotically stable for feasible values of the self-regulating parameter. Unfortunately, the same conditions also ensure the asymptotic stability of the fully defective consensus. Starting from this point, the possibility to observe a global converge of the SR-

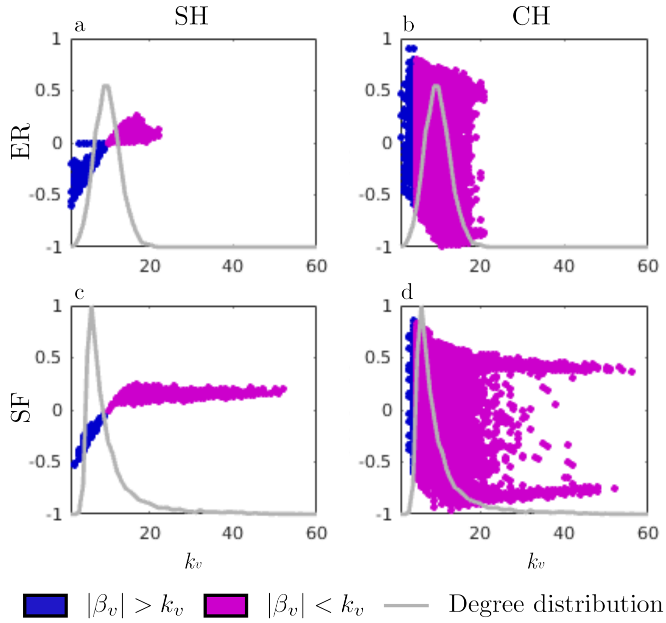

Figure 6. Altruistic and selfish behavior at steady state. The indicator c_v is evaluated for SH and Ch games in both ER and SF networks as a function of the average degree k. β_v has been set equal for all players: values of k_v lower and larger than β_v are reported in blue and pink colors, respectively. In the simulation, the parameters used for the SH game are: $T = -1$, $S = -1$ and $\beta_v = 10\ \forall v \in \mathcal{V}$, while the ones of the CH game are: $T = 2$, $S = 2$ and $\beta = -5\ \forall v \in \mathcal{V}$.

EGN towards different consensus steady states has also been investigated. In particular, we found a Lyapunov function to show that the unique partially cooperative consensus is globally asymptotically stable. An important consequence of our findings is that, the full cooperative consensus is reached only from a suitable set of initial conditions, while for the weaker one the basin of attraction corresponds to the whole feasible set.

According to the evolutionary game theory, partial cooperation means that people can behave cooperatively with some opponent, and defectively against others, which is effective in many real situations. Moreover, we showed that the reward obtained by the population when the partially cooperative consensus is reached, is larger than the case where full cooperation is present.

The global asymptotic stability of the partially cooperative steady state and, on the other hand, the simultaneous stability of full cooperation and full defection, which induce bistable behavior, can be fruitfully exploited for planning good policies taking into account the actual state of the population described by the initial state of the SR-EGN equation. Indeed, in the first case, the cooperation is independent on the initial state, while, in the second case, a more aware population is required in order to observe a full cooperation. All the above results highlight the importance of studying SH and CH games, which can lead to a deeper understanding of the mechanisms leading towards cooperation in social networks.

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