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Polyadic braid operators and higher braiding gates

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A new kind of quantum gates, higher braiding gates, as matrix solutions of the polyadic braid equations (different from the generalized Yang-Baxter equations) is introduced. Such gates lead to another special multiqubit entanglement which can speed up key distribution and accelerate algorithms. Ternary braiding gates acting on three qubit states are studied in details. We also consider exotic noninvertible gates which can be related with qubit loss, and define partial identities (which can be orthogonal), partial unitarity, and partially bounded operators (which can be noninvertible). We define two classes of matrices, star and circle ones, such that the magic matrices (connected with the Cartan decomposition) belong to the star class. The general algebraic structure of the introduced classes is described in terms of semigroups, ternary and 5-ary groups and modules. The higher braid group and its representation by the higher braid operators are given. Finally, we show, that for each multiqubit state there exist higher braiding gates which are not entangling, and the concrete conditions to be non-entangling are given for the obtained binary and ternary gates.

1. Introduction

The modern development of the quantum computing technique implies various extensions of its foundational concepts [1–3]. One of the main problems in the physical realization of quantum computers is presence of errors, which implies that it is desirable that quantum computations be provided with error correction, or that ways be found to make the states more stable, which leads to the concept of topological quantum computation (for reviews, see, e.g., [4–6], and references therein). In the Turaev approach [7], link invariants can be obtained from the solutions of the constant Yang-Baxter equation (the braid equation). It was realized that the topological entanglement of knots and links is deeply connected with quantum entanglement [8,9]. Indeed, if the solutions to the constant Yang-Baxter equation [10] (Yang-Baxter operators/maps [11,12]) are interpreted as a special class of quantum gate, namely braiding quantum gates [13,14], then the inclusion of non-entangling gates does not change the relevant topological invariants [15,16]. For further properties and applications of braiding quantum gates, see [17–20].

In this paper we obtain and study the solutions to the higher arity (polyadic) braid equations introduced in [21,22], as a polyadic generalization of the constant Yang-Baxter equation (which is considerably different from the generalized Yang-Baxter equation of [23–26]). We introduce special classes of matrices (star and circle types), to which most of the solutions belong, and find that the so-called magic matrices [18,27,28] belong to the star class. We investigate their general non-trivial group properties and polyadic generalizations. We then consider the invertible and non-invertible matrix solutions to the higher braid equations as the corresponding higher braiding gates acting on multi-qubit states. It is important that multi-qubit entanglement can speed up quantum key distribution [29] and accelerate various algorithms [30]. As an example, we have made detailed computations for the ternary braiding gates as solutions to the ternary braid equations [21,22]. A particular solution to the n-ary braid equation is also presented. It is shown, that for each multi-qubit state there exist higher braiding gates which are not entangling, and the concrete relations for that are obtained, which can allow us to build non-entangling networks.

2. Yang-Baxter operators

Recall here [9,13] the standard construction of the special kind of gates we will consider, the braiding gates, in terms of solutions to the *constant Yang-Baxter equation* [10] (called also *algebraic* Yang-Baxter equation [31]), or the (binary) *braid equation* [21].

2.1. Yang-Baxter maps and braid group

First we consider a general abstract construction of the (binary) braid equation. Let V be a vector space over a field $\mathbb{K}$ and the mapping $\mathrm{C}_{\mathrm{V}^2} : \mathrm{V} \otimes \mathrm{V} \to \mathrm{V} \otimes \mathrm{V}$, where $\otimes = \otimes_{\mathbb{K}}$ is the tensor product over $\mathbb{K}$. A linear operator (*braid operator*) $\mathrm{C}_{\mathrm{V}^2}$ is called a *Yang-Baxter operator* (denoted by R in [13] and by B in [10]) or *Yang-Baxter map* [12] (denoted by F in [11]), if it satisfies the *braid equation* [32–34]

$$(\mathrm{C}_{\mathrm{V}^2} \otimes \mathrm{id}_{\mathrm{V}}) \circ (\mathrm{id}_{\mathrm{V}} \otimes \mathrm{C}_{\mathrm{V}^2}) \circ (\mathrm{C}_{\mathrm{V}^2} \otimes \mathrm{id}_{\mathrm{V}}) = (\mathrm{id}_{\mathrm{V}} \otimes \mathrm{C}_{\mathrm{V}^2}) \circ (\mathrm{C}_{\mathrm{V}^2} \otimes \mathrm{id}_{\mathrm{V}}) \circ (\mathrm{id}_{\mathrm{V}} \otimes \mathrm{C}_{\mathrm{V}^2}), \tag{2.1}$$

where $\mathrm{id}_{\mathrm{V}} : \mathrm{V} \to \mathrm{V}$, is the identity operator in V. The connection of $\mathrm{C}_{\mathrm{V}^2}$ with the R-matrix R is given by $\mathrm{C}_{\mathrm{V}^2} = \tau \circ \mathrm{R}$, where τ is the flip operation [10,11,32].

Let us introduce the operators $\mathrm{A}_{1,2} : \mathrm{V} \otimes \mathrm{V} \otimes \mathrm{V} \to \mathrm{V} \otimes \mathrm{V} \otimes \mathrm{V}$ by

$$\mathrm{A}_1 = \mathrm{C}_{\mathrm{V}^2} \otimes \mathrm{id}_{\mathrm{V}}, \quad \mathrm{A}_2 = \mathrm{id}_{\mathrm{V}} \otimes \mathrm{C}_{\mathrm{V}^2}, \tag{2.2}$$

It follows from (2.1) that

$$\mathrm{A}_1 \circ \mathrm{A}_2 \circ \mathrm{A}_1 = \mathrm{A}_2 \circ \mathrm{A}_1 \circ \mathrm{A}_2. \tag{2.3}$$

If $\mathrm{C}_{\mathrm{V}^2}$ is invertible, then $\mathrm{C}_{\mathrm{V}^2}^{-1}$ is also the Yang-Baxter map with A_1^{-1} and A_2^{-1}. Therefore, the operators A_i represent the braid group $\mathcal{B}_3 = \{e, \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2\}$ by the mapping π_3 as

$$\mathcal{B}_3 \xrightarrow{\pi_3} \mathrm{End}(\mathrm{V} \otimes \mathrm{V} \otimes \mathrm{V}), \;\; \sigma_1 \overset{\pi_3}{\mapsto} \mathrm{A}_1, \;\; \sigma_2 \overset{\pi_3}{\mapsto} \mathrm{A}_2, \;\; e \overset{\pi_3}{\mapsto} \mathrm{id}_{\mathrm{V}}. \tag{2.4}$$

The representation π_m of the braid group with m strands

$$\mathcal{B}_m = \left\{e, \sigma_1, \ldots, \sigma_{m-1} \,\middle|\, \begin{array}{c} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \;\; i = 1, \ldots, m-1, \\ \sigma_i\sigma_j = \sigma_j\sigma_i, \quad |i-j| \geq 2, \end{array} \right\} \tag{2.5}$$

can be obtained using operators $\mathrm{A}_i(m) : \mathrm{V}^{\otimes m} \to \mathrm{V}^{\otimes m}$ analogous to (2.2)

$$\mathrm{A}_i(m) = \overbrace{\mathrm{id}_{\mathrm{V}} \otimes \ldots \otimes \mathrm{id}_{\mathrm{V}}}^{i-1} \otimes \mathrm{C}_{\mathrm{V}^2} \otimes \overbrace{\mathrm{id}_{\mathrm{V}} \otimes \ldots \mathrm{id}_{\mathrm{V}}}^{m-i-1}, \quad \mathrm{A}_0(m) = (\mathrm{id}_{\mathrm{V}})^{\otimes m}, \;\; i = 1, \ldots, m-1, \tag{2.6}$$

by the mapping $\pi_m : \mathcal{B}_m \to \mathrm{End}\mathrm{V}^{\otimes m}$ in the following way

$$\pi_m(\sigma_i) = \mathrm{A}_i(m), \qquad \pi_m(e) = \mathrm{A}_0(m). \tag{2.7}$$

In this notation (2.2) is $\mathrm{A}_i = \mathrm{A}_i(2)$, $i = 1, 2$, and so (2.3) represents $\mathcal{B}_3$ by (2.4).

2.2. Constant matrix solutions to the Yang-Baxter equation

Consider next a concrete version of the vector space V which is used in the quantum computation, a d-dimensional euclidean vector space V_d over complex numbers $\mathbb{C}$ with a basis $\{e_i\}$, $i = 1, \ldots, d$. A linear operator $V_d \to V_d$ is given by a complex $d \times d$ matrix, the identity operator id_{V} becomes the identity $d \times d$ matrix I_d, and the Yang-Baxter map $\mathrm{C}_{\mathrm{V}^2}$

is a $d^2 \times d^2$ matrix C_{d^2} (denoted by R in [31]) satisfying the matrix algebraic Yang-Baxter equation

$$(C_{d^2} \otimes I_d)(I_d \otimes C_{d^2})(C_{d^2} \otimes I_d) = (I_d \otimes C_{d^2})(C_{d^2} \otimes I_d)(I_d \otimes C_{d^2}), \tag{2.8}$$

being an equality between two matrices of size $d^3 \times d^3$. We use the unified notations which can be straightforwardly generalized for higher braid operators. In components

$$C_{d^2} \circ (e_{i_1} \otimes e_{i_2}) = \sum_{j'_1,j'_2=1}^{d} c_{i_1 i_2}^{j'_1 j'_2} \cdot e_{j'_1} \otimes e_{j'_2}, \tag{2.9}$$

the Yang-Baxter equation (2.8) has the shape (where summing is by primed indices)

$$\sum_{j'_1,j'_2,j'_3=1}^{d} c_{i_1 i_2}^{j'_1 j'_2} \cdot c_{j'_2 i_3}^{j'_3 k_3} \cdot c_{j'_1 j'_3}^{k_1 k_2} = \sum_{l'_1,l'_2,l'_3=1}^{d} c_{i_2 i_3}^{l'_2 l'_3} \cdot c_{i_1 l'_2}^{k_1 l'_1} \cdot c_{l'_1 l'_3}^{k_2 k_3} \equiv q_{i_1 i_2 i_3}^{k_1 k_2 k_3}. \tag{2.10}$$

The system (2.10) is highly overdetermined, because the matrix C_{d^2} contains d^4 unknown entries, while there are d^6 cubic polynomial equations for them. So for $d = 2$ we have 64 equations for 16 unknowns, while for $d = 3$ there are 729 equations for the 81 unknown entries of C_{d^2}. The unitarity of C_{d^2} imposes a further d^2 quadratic equations, and so for $d = 2$ we have in total 68 equations for 16 unknowns. This makes the direct discovery of solutions for the matrix Yang-Baxter equation (2.10) very cumbersome. Nevertheless, using a conjugation classes method, the unitary solutions and their classification for $d = 2$ were presented in [31].

In the standard matrix form (2.9) can be presented by introducing the 4-dimensional vector space $\tilde{V}_4 = V \otimes V$ with the natural basis $\tilde{e}_{\tilde{k}} = \{e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2\}$, where $\tilde{k} = 1, \ldots, 8$ is a cumulative index. The linear operator $\tilde{C}_4 : \tilde{V}_4 \to \tilde{V}_4$ corresponding to (2.9) is given by 4×4 matrix $\tilde{c}_{\tilde{i}\tilde{j}}$ as $\tilde{C}_4 \circ \tilde{e}_{\tilde{i}} = \sum_{\tilde{j}=1}^{4} \tilde{c}_{\tilde{i}\tilde{j}} \cdot \tilde{e}_{\tilde{j}}$. The operators (2.2) become two 8×8 matrices $\tilde{A}_{1,2}$ as

$$\tilde{A}_1 = \tilde{c} \otimes_K I_2, \quad \tilde{A}_2 = I_2 \otimes_K \tilde{c}, \tag{2.11}$$

where $\otimes_K$ is the Kronecker product of matrices and I_2 is the 2×2 identity matrix. In this notation (which is universal and also used for higher braid equations) the operator binary braid equations (3.7) become a single matrix equation

$$\tilde{A}_1 \tilde{A}_2 \tilde{A}_1 = \tilde{A}_2 \tilde{A}_1 \tilde{A}_2, \tag{2.12}$$

which we call the *matrix binary braid equation* (and also the constant Yang-Baxter equation [31]). In component form (2.12) is a highly overdetermined system of 64 cubic equations for 16 unknowns, the entries of $\tilde{c}$.

The matrix equation (2.12) has the following "gauge invariance", which allows a classification of Yang-Baxter maps [35]. Introduce an invertible operator $Q : V \to V$ in the two-dimensional vector space $V \equiv V_{d=2}$. In the basis $\{e_1, e_2\}$ its 2×2 matrix q is given by $Q \circ e_i = \sum_{j=1}^{2} q_{ij} \cdot e_j$. In the natural 4-dimensional basis $\tilde{e}_{\tilde{k}}$ the tensor product of operators $Q \otimes Q$ is presented by the Kronecker product of matrices $\tilde{q}_4 = q \otimes_K q$. If the 4×4 matrix $\tilde{c}$ is a fixed solution to the Yang-Baxter equation (2.12), then the family of solutions $\tilde{c}(q)$ corresponding to the invertible 2×2 matrix q is the conjugation of $\tilde{c}$ by $\tilde{q}_4$ such that

$$\tilde{c}(q) = \tilde{q}_4 \tilde{c} \tilde{q}_4^{-1} = (q \otimes_K q) \tilde{c} \left(q^{-1} \otimes_K q^{-1}\right), \tag{2.13}$$

which follows from conjugating (2.12) by $q \otimes_K q$ and using (2.11). If we include the obvious invariance of (2.12) with respect to an overall factor $t \in \mathbb{C}$, the general family of solutions becomes (cf. the Yang-Baxter equation [35])

$$\tilde{c}(q,t) = t\tilde{q}_4\tilde{c}\tilde{q}_4^{-1} = t(q \otimes_K q)\tilde{c}\left(q^{-1} \otimes_K q^{-1}\right). \tag{2.14}$$

It follows from (2.13) that the matrix $q \in \mathrm{GL}(2,\mathbb{C})$ is defined up to a complex non-zero factor. In this case we can put

$$q = \begin{pmatrix} a & 1 \\ c & d \end{pmatrix}, \tag{2.15}$$

and the manifest form of $\tilde{q}_4$ is

$$\tilde{q}_4 = \begin{pmatrix} a^2 & a & a & 1 \\ ac & ad & c & d \\ ac & c & ad & d \\ c^2 & cd & cd & d^2 \end{pmatrix}. \tag{2.16}$$

The matrix $\tilde{q}_4^{\star}\tilde{q}_4$ (where $\star$ represents Hermitian conjugation) is diagonal (this case is important in a further classification similar to the binary one [31]), when the condition

$$c = -a/d^* \tag{2.17}$$

holds, and so the matrix q takes the special form (depending on 2 complex parameters)

$$q = \begin{pmatrix} a & 1 \\ -a/d^* & d \end{pmatrix}. \tag{2.18}$$

We call two solutions $\tilde{c}_1$ and $\tilde{c}_2$ of the constant Yang-Baxter equation (2.12) *q-conjugated*, if

$$\tilde{c}_1\tilde{q}_4 = \tilde{q}_4\tilde{c}_2, \tag{2.19}$$

and we will not distinguish between them. The q-conjugation in the form (2.19) does not require the invertibility of the matrix q, and therefore the solutions of different ranks (or invertible and not invertible) can be q-conjugated (for the invertible case, see [35–37]).

The matrix equation (2.12) does not imply the invertibility of solutions, i.e. matrices $\tilde{c}$ being of full rank (in the binary Yang-Baxter case of rank 4 and $d = 2$). Therefore, below we introduce in a unified way invertible and non-invertible solutions to the matrix Yang-Baxter equation (2.10) for any rank of the corresponding matrices.

2.3. Partial identity and unitarity

To be as close as possible to the invertible case, we introduce "non-invertible analogs" of identity and unitarity. Let M be a diagonal $n \times n$ matrix of rank $r \leq n$, and therefore with $n - r$ zeroes on the diagonal. If the other diagonal elements are units, such a diagonal M can be reduced by row operations to a block matrix, being a direct sum of the identity matrix $I_{r\times r}$ and the zero matrix $Z_{(n-r)\times(n-r)}$. We call such a diagonal matrix a *block r-partial identity* $I_n^{(block)}(r) = \mathrm{diag}\left\{\overbrace{1,\ldots 1}^{r},\overbrace{0,\ldots,0}^{n-r}\right\}$, and without the block reduction—a *shuffle r-partial identity* $I_n^{(shuffle)}(r)$ (these are connected by conjugation). We will use the term partial identity and $I_n(r)$ to denote any matrix of this form. Obviously, with the full rank $r = n$ we have $I_n(n) \equiv I_n$, where I_n is the identity $n \times n$ matrix. As with the invertible case and identities, the partial identities (of the corresponding form) are *trivial solutions* of the Yang-Baxter equation.

If a matrix $M = M(r)$ of size $n \times n$ and rank r satisfies the following *r-partial unitarity* condition

$$M(r)^{\star}M(r) = I_n^{(1)}(r), \tag{2.20}$$
$$M(r)M(r)^{\star} = I_n^{(2)}(r), \tag{2.21}$$

where $M(r)^{\star}$ is the conjugate-transposed matrix and $I_n^{(1)}(r)$, $I_n^{(2)}(r)$ are partial identities (of any kind, they can be different), then $M(r)$ is called a *r-partial unitary matrix*. In the case, when $I_n^{(1)}(r) = I_n^{(2)}(r)$, the matrix $M(r)$ is called *normal*. If $M(r)^{\star} = M(r)$, then it is called *r-partial self-adjoint*. In the case of full rank $r = n$, the conditions (2.20)–(2.21) become ordinary unitarity, and $M(n)$ becomes an unitary (and normal) matrix, while a r-partial self-adjoint matrix becomes a self-adjoint matrix or Hermitian matrix.

As an example, we consider a 4×4 matrix of rank 3

$$M(3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & e^{i\beta} & 0 & 0 \\ 0 & 0 & 0 & e^{i\gamma} \\ e^{i\alpha} & 0 & 0 & 0 \end{pmatrix}, \quad \alpha, \beta, \gamma \in \mathbb{R}, \tag{2.22}$$

which satisfies the 3-partial unitarity conditions (2.20)–(2.21) with two different 3-partial identities on the r.h.s.

$$M(3)^{\star}M(3) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = I_4^{(1)}(3) \neq I_4^{(2)}(3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = M(3)M(3)^{\star}. \tag{2.23}$$

For a non-invertible matrix $M(r)$ one can define a *pseudoinverse* $M(r)^{+}$ (or the *Moore-Penrose inverse*) [38] by

$$M(r)M(r)^{+}M(r) = M(r), \;\; M(r)^{+}M(r)M(r)^{+} = M(r)^{+}, \tag{2.24}$$

and $M(r)M(r)^{+}$, $M(r)^{+}M(r)$ are Hermitian. In the case of (2.22) the partial unitary matrix $M(3)$ coincides with its pseudoinverse

$$M(3)^{\star} = M(3)^{+}, \tag{2.25}$$

which is similar to the standard unitarity $M_{inv}^{\star} = M_{inv}^{-1}$ for an invertible matrix M_{inv}. It is important that (2.22) is a solution of the matrix Yang-Baxter equation (2.12), and so is an example of a non-invertible Yang-Baxter map.

If only the first (second) of the conditions (2.20)–(2.21) holds, we call such $M(r)$ a *left (right) r-partial unitary matrix*. An example of such a non-invertible Yang-Baxter map of rank 2 is the left 2-partial unitary matrix

$$M(2) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & e^{i\alpha} \\ 0 & e^{i\beta} & 0 & 0 \\ 0 & e^{i\beta} & 0 & 0 \\ 0 & 0 & 0 & e^{i\beta} \end{pmatrix}, \quad \alpha, \beta \in \mathbb{R}, \tag{2.26}$$

which satisfies (2.20), but not (2.21), and so $M(2)$ is not normal

$$M(2)^{\star}M(2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 & 0 & e^{i(\alpha-\beta)} \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ e^{i(\beta-\alpha)} & 0 & 0 & 1 \end{pmatrix} = M(2)M(2)^{\star}. \tag{2.27}$$

Nevertheless, the property (2.25) still holds and $M(2)^{\star} = M(2)^{+}$.

2.4. Permutation and parameter-permutation 4-vertex Yang-Baxter maps

The system (2.12) with respect to all 16 variables is too cumbersome for direct solution. The classification of all solutions can only be accomplished in special cases, e.g. for matrices over finite fields [35] or for fewer than 16 vertices. Here we will start from 4-vertex permutation and parameter-permutation matrix solutions and investigate their group structure. It was shown [13,31] that the special 8-vertex solutions to the Yang-Baxter equation are most important for further applications including braiding gates. We will therefore study the 8-vertex solutions in the most general way: over $\mathbb{C}$ and in various configurations, invertible and not invertible, and also consider their group structure.

First, we introduce the *permutation Yang-Baxter maps* which are presented by the permutation matrices (binary matrices with a single 1 in each row and column), i.e. 4-vertex solutions. In total, there are 64 permutation matrices of size 4×4, while only 4 of them have the full rank 4 and simultaneously satisfy the Yang-Baxter equation (2.12). These are the following

$$\tilde{c}^{perm}_{bisymm} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \tag{2.28}$$

$$\begin{gathered} \operatorname{tr}\tilde{c} = 2, \;\; \det\tilde{c} = -1, \\ \text{eigenvalues: } \{1\}^{[2]}, \{-1\}^{[2]}, \end{gathered} \tag{2.29}$$

$$\tilde{c}^{perm}_{90symm} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tag{2.30}$$

$$\begin{gathered} \operatorname{tr}\tilde{c} = 0, \;\; \det\tilde{c} = -1, \\ \text{eigenvalues: } 1, i, -1, -i. \end{gathered} \tag{2.31}$$

Here and next we list eigenvalues to understand which matrices are conjugated, and after that, if and only if the conjugation matrix is of the form (2.16), then such solutions to the Yang-Baxter equation (2.12) coincide. The traces are important in the construction of corresponding link invariants [7] and local invariants [39,40], and determinants are connected with the concurrence [41,42]. Note that the first matrix in (2.28) is the SWAP quantum gate [1].

To understand symmetry properties of (2.28)–(2.30), we introduce the so called *reverse matrix* $J \equiv J_n$ of size $n \times n$ by $(J_n)_{ij} = \delta_{i,n+1-i}$. For $n = 4$ it is

$$J_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{2.32}$$

For any $n \times n$ matrix $M \equiv M_n$ the matrix JM is the matrix M reflected vertically, and the product MJ is M reflected horizontally. In addition to the standard *symmetric matrix* satisfying $M = M^T$ (T is the transposition), one can introduce

$$M \text{ is } \textit{persymmetric}, \text{ if } JM = (JM)^T, \tag{2.33}$$
$$M \text{ is } \textit{90°-symmetric}, \text{ if } M^T = JM. \tag{2.34}$$

Thus, a persymmetric matrix is symmetric with respect to the minor diagonal, while a 90°-symmetric matrix is symmetric under 90°-rotations. A *bisymmetric matrix* is symmetric and persymmetric simultaneously. In this notation, the first family of the permutation solutions (2.28) are bisymmetric, but not 90°-symmetric, while the second family of the solutions (2.30) are, oppositely, 90°-symmetric, but not symmetric and not persymmetric (which explains their notation).

In the next step, we define the corresponding *parameter-permutation solutions* replacing the units in (2.28) by parameters. We found the following four 4-vertex solutions to the Yang-Baxter equation (2.12) over $\mathbb{C}$

$$\tilde{c}^{perm,star}_{rank=4}(x,y,z,t) = \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & z & 0 & 0 \\ 0 & 0 & 0 & t \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & y \\ 0 & x & 0 & 0 \\ 0 & 0 & t & 0 \\ z & 0 & 0 & 0 \end{pmatrix}, \tag{2.35}$$

$$\begin{gathered} \operatorname{tr} \tilde{c} = x + t, \\ \det \tilde{c} = -xyzt, \quad x,y,z,t \neq 0, \\ \text{eigenvalues: } x, t, \sqrt{yz}, -\sqrt{yz}, \end{gathered} \tag{2.36}$$

$$\tilde{c}^{perm,circ}_{rank=4}(x,y) = \begin{pmatrix} 0 & 0 & x & 0 \\ y & 0 & 0 & 0 \\ 0 & 0 & 0 & x \\ 0 & y & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x & 0 & 0 \\ 0 & 0 & 0 & x \\ y & 0 & 0 & 0 \\ 0 & 0 & y & 0 \end{pmatrix}, \tag{2.37}$$

$$\begin{gathered} \operatorname{tr} \tilde{c} = 0, \\ \det \tilde{c} = -x^2 y^2, \quad x,y \neq 0, \\ \text{eigenvalues: } \sqrt{xy}, -\sqrt{xy}, i\sqrt{xy}, -i\sqrt{xy}. \end{gathered} \tag{2.38}$$

The first pair of solutions (2.35) correspond to the bi-symmetric permutation matrices (2.28), and we call them *star-like solutions*, while the second two solutions (2.37) correspond to the 90°-symmetric matrices (2.28) which are called *circle-like solutions*.

The first (second) star-like solution in (2.35) with $y = z$ ($x = t$) becomes symmetric (persymmetric), while on the other hand with $x = t$ ($y = z$) it becomes persymmetric (symmetric). They become bisymmetric parameter-permutation solutions if all the parameters are equal $x = y = z = t$. The circle-like solutions (2.37) are 90°-symmetric when $x = y$.

Using q-conjugation (2.14) one can next get families of solutions depending from the entries of q, the additional complex parameters in (2.15).

2.5. Group structure of 4-vertex and 8-vertex matrices

Let us analyze the group structure of 4-vertex matrices (2.35)–(2.37) with respect to matrix multiplication, i.e. which kinds of subgroups in $\mathrm{GL}(4,\mathbb{C})$ they can form. For this we introduce four 4-vertex 4×4 matrices over $\mathbb{C}$: two star-like matrices

$$N_{star1} = \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & z & 0 & 0 \\ 0 & 0 & 0 & t \end{pmatrix}, N_{star2} = \begin{pmatrix} 0 & 0 & 0 & y \\ 0 & x & 0 & 0 \\ 0 & 0 & t & 0 \\ z & 0 & 0 & 0 \end{pmatrix}, \quad \begin{gathered} \operatorname{tr} N = x + t, \\ \det N = -xyzt, \quad x,y,z,t \neq 0, \\ \text{eigenvalues: } x, t, \sqrt{yz}, -\sqrt{yz}, \end{gathered} \tag{2.39}$$

and two circle-like matrices

$$N_{circ1} = \begin{pmatrix} 0 & 0 & x & 0 \\ y & 0 & 0 & 0 \\ 0 & 0 & 0 & z \\ 0 & t & 0 & 0 \end{pmatrix}, \quad N_{circ2} = \begin{pmatrix} 0 & x & 0 & 0 \\ 0 & 0 & 0 & y \\ z & 0 & 0 & 0 \\ 0 & 0 & t & 0 \end{pmatrix}, \tag{2.40}$$

$$\begin{array}{c} \operatorname{tr} N = 0, \\ \det N = -xyzt, \quad x, y, z, t \neq 0, \\ \text{eigenvalues: } \sqrt[4]{xyzt}, -\sqrt[4]{xyzt}, i\sqrt[4]{xyzt}, -\sqrt[4]{xyzt}, \end{array} . \tag{2.41}$$

Denoting the corresponding sets by $\mathsf{N}_{star1,2} = \{N_{star1,2}\}$ and $\mathsf{N}_{circ1,2} = \{N_{circ1,2}\}$, these do not intersect and are closed with respect to the following multiplications

$$\mathsf{N}_{star1}\mathsf{N}_{star1}\mathsf{N}_{star1} = \mathsf{N}_{star1}, \tag{2.42}$$

$$\mathsf{N}_{star2}\mathsf{N}_{star2}\mathsf{N}_{star2} = \mathsf{N}_{star2}, \tag{2.43}$$

$$\mathsf{N}_{circ1}\mathsf{N}_{circ1}\mathsf{N}_{circ1}\mathsf{N}_{circ1}\mathsf{N}_{circ1} = \mathsf{N}_{circ1}, \tag{2.44}$$

$$\mathsf{N}_{circ2}\mathsf{N}_{circ2}\mathsf{N}_{circ2}\mathsf{N}_{circ2}\mathsf{N}_{circ2} = \mathsf{N}_{circ2}. \tag{2.45}$$

Note that there are no closed binary multiplications among the sets of 4-vertex matrices (2.39)–(2.40).

To give a proper group interpretation of (2.42)–(2.45), we introduce a *k-ary (polyadic) general linear semigroup* $\mathrm{GLS}^{[k]}(n, \mathbb{C}) = \left\{ \mathsf{M}_{full} \mid \mu^{[k]} \right\}$, where $\mathsf{M}_{full} = \{M_{n\times n}\}$ is the set of $n \times n$ matrices over $\mathbb{C}$ and $\mu^{[k]}$ is an ordinary product of k matrices. The full semigroup $\mathrm{GLS}^{[k]}(n, \mathbb{C})$ is *derived* in the sense that its product can be obtained by repeating the binary products which are (binary) closed at each step. However, $n \times n$ matrices of special shape can form k-ary subsemigroups of $\mathrm{GLS}^{[k]}(n, \mathbb{C})$ which can be closed with respect to the product of at minimum k matrices, but not of 2 matrices, and we call such semigroups *k-non-derived*. Moreover, we have for the sets $\mathsf{N}_{star1,2}$ and $\mathsf{N}_{circ1,2}$

$$\mathsf{M}_{full} = \mathsf{N}_{star1} \cup \mathsf{N}_{star2} \cup \mathsf{N}_{circ1} \cup \mathsf{N}_{circ2}, \quad \mathsf{N}_{star1} \cap \mathsf{N}_{star2} \cap \mathsf{N}_{circ1} \cap \mathsf{N}_{circ2} = \varnothing. \tag{2.46}$$

A simple example of a 3-nonderived subsemigroup of the full semigroup $\mathrm{GLS}^{[k]}(n, \mathbb{C})$ is the set of antidiagonal matrices $\mathsf{M}_{\mathrm{adiag}} = \left\{ M_{\mathrm{adiag}} \right\}$ (having nonzero elements on the minor diagonal only): the product $\mu^{[3]}$ of 3 matrices from $\mathsf{M}_{\mathrm{adiag}}$ is closed, and therefore $\mathsf{M}_{\mathrm{adiag}}$ is a subsemigroup $\mathcal{S}^{[3]}_{\mathrm{adiag}} = \left\{ \mathsf{M}_{\mathrm{adiag}} \mid \mu^{[3]} \right\}$ of the full ternary general linear semigroup $\mathrm{GLS}^{[3]}(n, \mathbb{C})$ with the multiplication $\mu^{[3]}$ as the ordinary triple matrix product.

In the theory of polyadic groups [43] an analog of the binary inverse M^{-1} is given by the *querelement*, which is denoted by $\bar{M}$ and in the matrix k-ary case is defined by

$$\overbrace{M \ldots M}^{k-1} \bar{M} = M, \tag{2.47}$$

where $\bar{M}$ can be on any place. If each element of the k-ary semigroup $\mathrm{GLS}^{[k]}(n, \mathbb{C})$ (or its subsemigroup) has its querelement $\bar{M}$, then this semigroup is a *k-ary general linear group* $\mathrm{GL}^{[k]}(n, \mathbb{C})$.

In the set of $n \times n$ matrices the binary (ordinary) product is defined (even it is not closed), and for invertible matrices we formally determine the standard inverse M^{-1}, but for arity $k \geq 4$ it does not coincide with the querelement $\bar{M}$, because, as follows from (2.47) and cancellativity in $\mathbb{C}$ that

$$\bar{M} = M^{2-k}. \tag{2.48}$$

The *k-ary (polyadic) identity* $I_n^{[k]}$ in $\mathrm{GLS}^{[k]}(n,\mathbb{C})$ is defined by

$$\overbrace{I_n^{[k]} \ldots I_n^{[k]}}^{k-1} M = M, \tag{2.49}$$

which holds when M in the l.h.s. is on any place. If M is only on one or another side (but not in the middle places) in (2.49), $I_n^{[k]}$ is called *left (right) polyadic identity*. For instance, in the subsemigroup (in $\mathrm{GLS}^{[k]}(n,\mathbb{C})$) of antidiagonal matrices $\mathcal{S}_{\mathrm{adiag}}^{[3]}$ the ternary identity $I_n^{[3]}$ can be chosen as the $n\times n$ reverse matrix (2.32) having units on the minor diagonal, while the ordinary $n\times n$ unit matrix I_n is not in $\mathcal{S}_{\mathrm{adiag}}^{[3]}$. It follows from (2.49), that for matrices over $\mathbb{C}$ the (left, right) polyadic identity $I_n^{[k]}$ is

$$\left(I_n^{[k]}\right)^{k-1} = I_n, \tag{2.50}$$

which means that for the ordinary matrix product $I_n^{[k]}$ is a $(k-1)$-root of I_n (or $I_n^{[k]}$ is a reflection of $(k-1)$ degree), while both sides cannot belong to a subsemigroup $\mathcal{S}^{[k]}$ of $\mathrm{GLS}^{[k]}(n,\mathbb{C})$ under consideration (as in $\mathcal{S}_{\mathrm{adiag}}^{[3]}$). As the solutions of (2.50) are not unique, there can be many k-ary identities in a k-ary matrix semigroup. We denote the set of k-ary identities by $\mathsf{I}_n^{[k]} = \left\{I_n^{[k]}\right\}$. In the case of $\mathcal{S}_{\mathrm{adiag}}^{[3]}$ the ternary identity $I_n^{[3]}$ can be chosen as any of the $n\times n$ reverse matrices (2.32) with unit complex numbers $e^{i\alpha_j}$, $j=1,\ldots,n$ on the minor diagonal, where α_j satisfy additional conditions depending on the semigroup. In the concrete case of $\mathcal{S}_{\mathrm{adiag}}^{[3]}$ the conditions, giving (2.50), are $(k-1)\alpha_j = 1+2\pi r_j$, $r_j\in\mathbb{Z}$, $j=1,\ldots,n$.

In the framework of the above definitions, we can interpret the closed products (2.42)–(2.43) as the multiplications $\mu^{[3]}$ of the *ternary semigroups* $\mathcal{S}_{star1,2}^{[3]}(4,\mathbb{C}) = \left\{\mathsf{N}_{star1,2} \mid \mu^{[3]}\right\}$. The corresponding querelements are given by

$$\bar{N}_{star1} = N_{star1}^{-1} = \begin{pmatrix} \frac{1}{x} & 0 & 0 & 0\\ 0 & 0 & \frac{1}{z} & 0\\ 0 & \frac{1}{y} & 0 & 0\\ 0 & 0 & 0 & \frac{1}{t}\end{pmatrix}, \bar{N}_{star2} = N_{star2}^{-1} = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{z}\\ 0 & \frac{1}{x} & 0 & 0\\ 0 & 0 & \frac{1}{t} & 0\\ \frac{1}{y} & 0 & 0 & 0\end{pmatrix}, x,y,z,t\neq 0. \tag{2.51}$$

The ternary semigroups having querelements for each element (i.e. the additional operation $\overline{(\,)}$ defined by (2.51)) are the *ternary groups* $\mathcal{G}_{star1,2}^{[3]}(4,\mathbb{C}) = \left\{\mathsf{N}_{star1,2} \mid \mu^{[3]}, \overline{(\,)}\right\}$ which are two (non-intersecting because $\mathsf{N}_{star1}\cap\mathsf{N}_{star2}=\varnothing$) subgroups of the ternary general linear group $\mathrm{GL}^{[3]}(4,\mathbb{C})$. The ternary identities in $\mathcal{G}_{star1,2}^{[3]}(4,\mathbb{C})$ are the following different continuous sets $\mathsf{I}_{star1,2}^{[3]} = \left\{I_{star1,2}^{[3]}\right\}$, where

$$I_{star1}^{[3]} = \begin{pmatrix} e^{i\alpha_1} & 0 & 0 & 0\\ 0 & 0 & e^{i\alpha_2} & 0\\ 0 & e^{i\alpha_3} & 0 & 0\\ 0 & 0 & 0 & e^{i\alpha_4}\end{pmatrix}, \quad e^{2i\alpha_1} = e^{2i\alpha_4} = e^{i(\alpha_2+\alpha_3)} = 1, \quad \alpha_j\in\mathbb{R}, \tag{2.52}$$

$$I_{star2}^{[3]} = \begin{pmatrix} 0 & 0 & 0 & e^{i\alpha_1}\\ 0 & e^{i\alpha_2} & 0 & 0\\ 0 & 0 & e^{i\alpha_3} & 0\\ e^{i\alpha_4} & 0 & 0 & 0\end{pmatrix}, \quad e^{2i\alpha_2} = e^{2i\alpha_3} = e^{i(\alpha_1+\alpha_4)} = 1, \quad \alpha_j\in\mathbb{R}. \tag{2.53}$$

In the particular case $\alpha_j = 0$, $j=1,2,3,4$, the ternary identities (2.52)–(2.53) coincide with the bisymmetric permutation matrices (2.28).

Next we treat the closed set products (2.44)–(2.45) as the multiplications $\mu^{[5]}$ of the 5-*ary semigroups* $\mathcal{S}^{[5]}_{circ1,2}(4,\mathbb{C}) = \left\{ \mathsf{N}_{circ1,2} \mid \mu^{[5]} \right\}$. The querelements are

$$\bar{N}_{circ1} = N^{-3}_{circ1} = \begin{pmatrix} 0 & 0 & \frac{1}{yzt} & 0 \\ \frac{1}{xzt} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{xyt} \\ 0 & \frac{1}{xyz} & 0 & 0 \end{pmatrix}, \bar{N}_{circ2} = N^{-3}_{circ2} = \begin{pmatrix} 0 & \frac{1}{yzt} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{xzt} \\ \frac{1}{xyt} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{xyz} & 0 \end{pmatrix}, \tag{2.54}$$

$x,y,z,t \neq 0$, and the corresponding 5-*ary groups* $\mathcal{G}^{[5]}_{circ1,2}(4,\mathbb{C}) = \left\{ \mathsf{N}_{circ1,2} \mid \mu^{[5]}, \overline{(\,)} \right\}$ which are two (non-intersecting because $\mathsf{N}_{circ1} \cap \mathsf{N}_{circ2} = \varnothing$) subgroups of the 5-ary general linear group $\mathrm{GL}^{[5]}(n,\mathbb{C})$. We have the following continuous sets of 5-ary identities $\mathsf{I}^{[3]}_{circ1,2} = \left\{ I^{[3]}_{circ1,2} \right\}$ in $\mathcal{G}^{[5]}_{circ1,2}(4,\mathbb{C})$ satisfying

$$I^{[5]}_{circ1} = \begin{pmatrix} 0 & 0 & e^{i\alpha_1} & 0 \\ e^{i\alpha_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{i\alpha_3} \\ 0 & e^{i\alpha_4} & 0 & 0 \end{pmatrix}, \quad e^{i(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} = 1, \ \ \alpha_j \in \mathbb{R}, \tag{2.55}$$

$$I^{[5]}_{circ2} = \begin{pmatrix} 0 & e^{i\alpha_1} & 0 & 0 \\ 0 & 0 & 0 & e^{i\alpha_2} \\ e^{i\alpha_3} & 0 & 0 & 0 \\ 0 & 0 & e^{i\alpha_4} & 0 \end{pmatrix}, \quad e^{i(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} = 1, \ \ \alpha_j \in \mathbb{R}. \tag{2.56}$$

In the case $\alpha_j = 0$, $j = 1,2,3,4$, the 5-ary identities (2.55)–(2.56) coincide with the 90°-symmetric permutation matrices (2.30).

Thus, it follows from (2.51)–(2.56) that the 4-vertex star-like (2.39) and circle-like (2.40) matrices form subgroups of the k-ary general linear group $\mathrm{GL}^{[k]}(4,\mathbb{C})$ with significantly different properties: they have different querelements and (sets of) polyadic identities, and even the arities of the subgroups $\mathcal{G}^{[3]}_{star1,2}(4,\mathbb{C})$ and $\mathcal{G}^{[5]}_{circ1,2}(4,\mathbb{C})$ do not coincide (2.42)–(2.45). If we take into account that 4-vertex star-like (2.39) and circle-like (2.40) matrices are (binary) additive and distributive, then they form (with respect to the binary matrix addition $(+)$ and the multiplications $\mu^{[3]}$ and $\mu^{[5]}$) the $(2,3)$-ring $\mathcal{R}^{[3]}_{star1,2}(4,\mathbb{C}) = \left\{ \mathsf{N}_{star1,2} \mid +, \mu^{[3]} \right\}$ and $(2,5)$-ring $\mathcal{R}^{[5]}_{circ1,2}(4,\mathbb{C}) = \left\{ \mathsf{N}_{star1,2} \mid +, \mu^{[5]} \right\}$.

Next we consider the "interaction" of the 4-vertex star-like (2.39) and circle-like (2.40) matrix sets, i.e. their exotic module structure. For this, let us recall the ternary (polyadic) module [44] and s-place action [45] definitions, which are suitable for our case. An abelian group $\mathcal{M}$ is a ternary left (middle, right) $\mathcal{R}$-module (or a module over $\mathcal{R}$), if there exists a ternary operation $\mathcal{R} \times \mathcal{R} \times \mathcal{M} \to \mathcal{M}$ ($\mathcal{R} \times \mathcal{M} \times \mathcal{R} \to \mathcal{M}$, $\mathcal{M} \times \mathcal{R} \times \mathcal{R} \to \mathcal{M}$) which satisfies some compatibility conditions (associativity and distributivity) which hold in the matrix case under consideration (and where the module operation is the triple ordinary matrix product) [44]. A 5-ary left (right) module $\mathcal{M}$ over $\mathcal{R}$ is a 5-ary operation $\mathcal{R} \times \mathcal{R} \times \mathcal{R} \times \mathcal{R} \times \mathcal{M} \to \mathcal{M}$ ($\mathcal{M} \times \mathcal{R} \times \mathcal{R} \times \mathcal{R} \times \mathcal{R} \to \mathcal{M}$) with analogous conditions (and where the module operation is the pentuple matrix product) [45].

First, we have the triple relations "inside" star and circle matrices

$$\mathsf{N}_{star1}(\mathsf{N}_{star2})\mathsf{N}_{star1} = (\mathsf{N}_{star2}), \quad \mathsf{N}_{circ1}\mathsf{N}_{circ2}\mathsf{N}_{circ1} = \mathsf{N}_{circ1}, \tag{2.57}$$
$$\mathsf{N}_{star1}\mathsf{N}_{star1}(\mathsf{N}_{star2}) = (\mathsf{N}_{star2}), \quad \mathsf{N}_{circ1}\mathsf{N}_{circ1}\mathsf{N}_{circ2} = \mathsf{N}_{circ1}, \tag{2.58}$$
$$(\mathsf{N}_{star2})\mathsf{N}_{star1}\mathsf{N}_{star1} = (\mathsf{N}_{star2}), \quad \mathsf{N}_{circ2}\mathsf{N}_{circ1}\mathsf{N}_{circ1} = \mathsf{N}_{circ1}, \tag{2.59}$$
$$\mathsf{N}_{star2}\mathsf{N}_{star2}(\mathsf{N}_{star1}) = (\mathsf{N}_{star1}), \quad \mathsf{N}_{circ2}\mathsf{N}_{circ2}\mathsf{N}_{circ1} = \mathsf{N}_{circ2}, \tag{2.60}$$
$$\mathsf{N}_{star2}(\mathsf{N}_{star1})\mathsf{N}_{star2} = (\mathsf{N}_{star1}), \quad \mathsf{N}_{circ2}\mathsf{N}_{circ1}\mathsf{N}_{circ2} = \mathsf{N}_{circ2}, \tag{2.61}$$
$$(\mathsf{N}_{star1})\mathsf{N}_{star2}\mathsf{N}_{star2} = (\mathsf{N}_{star1}), \quad \mathsf{N}_{circ1}\mathsf{N}_{circ2}\mathsf{N}_{circ2} = \mathsf{N}_{circ2}. \tag{2.62}$$

We observe the following module structures on the left column above (elements of the corresponding module are in brackets, and we informally denote modules by their sets): 1) from (2.57)–(2.59), the set N_{star2} is a middle, right and left module over N_{star1}; 2) from (2.60)–(2.62), the set N_{star1} is a middle, right and left module over N_{star2};

$$\mathsf{N}_{star1}\mathsf{N}_{circ1}\mathsf{N}_{star1} = \mathsf{N}_{circ2}, \quad \mathsf{N}_{star1}\mathsf{N}_{circ2}\mathsf{N}_{star1} = \mathsf{N}_{circ1}, \tag{2.63}$$
$$\mathsf{N}_{star2}\mathsf{N}_{circ1}\mathsf{N}_{star2} = \mathsf{N}_{circ2}, \quad \mathsf{N}_{star2}\mathsf{N}_{circ2}\mathsf{N}_{star2} = \mathsf{N}_{circ1}, \tag{2.64}$$
$$\mathsf{N}_{star1}\mathsf{N}_{star1}(\mathsf{N}_{circ1}) = (\mathsf{N}_{circ1}), \quad (\mathsf{N}_{circ1})\mathsf{N}_{star1}\mathsf{N}_{star1} = (\mathsf{N}_{circ1}), \tag{2.65}$$
$$\mathsf{N}_{star1}\mathsf{N}_{star1}(\mathsf{N}_{circ2}) = (\mathsf{N}_{circ2}), \quad (\mathsf{N}_{circ2})\mathsf{N}_{star1}\mathsf{N}_{star1} = (\mathsf{N}_{circ2}), \tag{2.66}$$
$$\mathsf{N}_{star2}\mathsf{N}_{star2}(\mathsf{N}_{circ1}) = (\mathsf{N}_{circ1}), \quad (\mathsf{N}_{circ1})\mathsf{N}_{star2}\mathsf{N}_{star2} = (\mathsf{N}_{circ1}), \tag{2.67}$$
$$\mathsf{N}_{star2}\mathsf{N}_{star2}(\mathsf{N}_{circ2}) = (\mathsf{N}_{circ2}), \quad (\mathsf{N}_{circ2})\mathsf{N}_{star2}\mathsf{N}_{star2} = (\mathsf{N}_{circ2}), \tag{2.68}$$

3) from (2.65)–(2.68), the sets $\mathsf{N}_{circ1,2}$ are a right and left module over $\mathsf{N}_{star1,2}$;

$$\mathsf{N}_{circ1}(\mathsf{N}_{star1})\mathsf{N}_{circ1} = (\mathsf{N}_{star1}), \quad \mathsf{N}_{circ1}(\mathsf{N}_{star2})\mathsf{N}_{circ1} = (\mathsf{N}_{star2}), \tag{2.69}$$
$$\mathsf{N}_{circ2}(\mathsf{N}_{star1})\mathsf{N}_{circ2} = (\mathsf{N}_{star1}), \quad \mathsf{N}_{circ2}(\mathsf{N}_{star2})\mathsf{N}_{circ2} = (\mathsf{N}_{star2}), \tag{2.70}$$
$$\mathsf{N}_{circ1}\mathsf{N}_{circ1}\mathsf{N}_{star1} = \mathsf{N}_{star2}, \quad \mathsf{N}_{star1}\mathsf{N}_{circ1}\mathsf{N}_{circ1} = \mathsf{N}_{star2}, \tag{2.71}$$
$$\mathsf{N}_{circ1}\mathsf{N}_{circ1}\mathsf{N}_{star2} = \mathsf{N}_{star1}, \quad \mathsf{N}_{star2}\mathsf{N}_{circ1}\mathsf{N}_{circ1} = \mathsf{N}_{star1}, \tag{2.72}$$
$$\mathsf{N}_{circ2}\mathsf{N}_{circ2}\mathsf{N}_{star1} = \mathsf{N}_{star2}, \quad \mathsf{N}_{star1}\mathsf{N}_{circ2}\mathsf{N}_{circ2} = \mathsf{N}_{star2}, \tag{2.73}$$
$$\mathsf{N}_{circ2}\mathsf{N}_{circ2}\mathsf{N}_{star2} = \mathsf{N}_{star1}, \quad \mathsf{N}_{star2}\mathsf{N}_{circ2}\mathsf{N}_{circ2} = \mathsf{N}_{star1}, \tag{2.74}$$

4) from (2.69)–(2.70), the sets $\mathsf{N}_{star1,2}$ are a middle ternary module over $\mathsf{N}_{circ1,2}$;

$$\mathsf{N}_{circ1}\mathsf{N}_{circ1}\mathsf{N}_{circ1}\mathsf{N}_{circ1}(\mathsf{N}_{star1}) = (\mathsf{N}_{star1}), \ \mathsf{N}_{circ1}\mathsf{N}_{circ1}\mathsf{N}_{circ1}\mathsf{N}_{circ1}(\mathsf{N}_{star2}) = (\mathsf{N}_{star2}), \tag{2.75}$$
$$(\mathsf{N}_{star1})\mathsf{N}_{circ1}\mathsf{N}_{circ1}\mathsf{N}_{circ1}\mathsf{N}_{circ1} = (\mathsf{N}_{star1}), \ (\mathsf{N}_{star2})\mathsf{N}_{circ1}\mathsf{N}_{circ1}\mathsf{N}_{circ1}\mathsf{N}_{circ1} = (\mathsf{N}_{star2}), \tag{2.76}$$
$$\mathsf{N}_{circ2}\mathsf{N}_{circ2}\mathsf{N}_{circ2}\mathsf{N}_{circ2}(\mathsf{N}_{star1}) = (\mathsf{N}_{star1}), \ \mathsf{N}_{circ2}\mathsf{N}_{circ2}\mathsf{N}_{circ2}\mathsf{N}_{circ2}(\mathsf{N}_{star2}) = (\mathsf{N}_{star2}), \tag{2.77}$$
$$(\mathsf{N}_{star1})\mathsf{N}_{circ2}\mathsf{N}_{circ2}\mathsf{N}_{circ2}\mathsf{N}_{circ2} = (\mathsf{N}_{star1}), \ (\mathsf{N}_{star2})\mathsf{N}_{circ2}\mathsf{N}_{circ2}\mathsf{N}_{circ2}\mathsf{N}_{circ2} = (\mathsf{N}_{star2}), \tag{2.78}$$
$$\mathsf{N}_{circ1}\mathsf{N}_{circ1}\mathsf{N}_{circ1}\mathsf{N}_{circ1}(\mathsf{N}_{circ2}) = (\mathsf{N}_{circ2}), \ (\mathsf{N}_{circ2})\mathsf{N}_{circ1}\mathsf{N}_{circ1}\mathsf{N}_{circ1}\mathsf{N}_{circ1} = (\mathsf{N}_{circ2}), \tag{2.79}$$
$$\mathsf{N}_{circ2}\mathsf{N}_{circ2}\mathsf{N}_{circ2}\mathsf{N}_{circ2}(\mathsf{N}_{circ1}) = (\mathsf{N}_{circ1}), \ (\mathsf{N}_{circ1})\mathsf{N}_{circ2}\mathsf{N}_{circ2}\mathsf{N}_{circ2}\mathsf{N}_{circ2} = (\mathsf{N}_{circ1}). \tag{2.80}$$

5) from (2.75)–(2.80), the sets $\mathsf{N}_{circ1,2}$ are right and left 5-ary modules over $\mathsf{N}_{circ2,1}$ and $\mathsf{N}_{star1,2}$.

Note that the sum of 4-vertex star solutions of the Yang-Baxter equations (2.35) (with different parameters) gives the shape of 8-vertex matrices, and the same with the

4-vertex circle solutions (2.37). Let us introduce two kind of 8-vertex 4×4 matrices over $\mathbb{C}$: an 8-vertex *star matrix* M_{star} and an 8-vertex *circle matrix* M_{circ} as

$$M_{star} = \begin{pmatrix} x & 0 & 0 & y \\ 0 & z & s & 0 \\ 0 & t & u & 0 \\ v & 0 & 0 & w \end{pmatrix}, \quad \det M_{star} = (xw - yv)(st - uz), \quad \operatorname{tr} M_{star} = x + z + u + w, \tag{2.81}$$

$$M_{circ} = \begin{pmatrix} 0 & x & y & 0 \\ z & 0 & 0 & s \\ t & 0 & 0 & u \\ 0 & v & w & 0 \end{pmatrix}, \quad \det M_{circ} = (xw - yv)(st - uz), \quad \operatorname{tr} M_{circ} = 0. \tag{2.82}$$

If M_{star} and M_{circ} are invertible (the determinants in (2.81)–(2.82) are non-vanishing), then

$$M_{star}^{-1} = \begin{pmatrix} \frac{w}{xw-yv} & 0 & 0 & -\frac{v}{xw-yv} \\ 0 & -\frac{u}{st-uz} & \frac{t}{st-uz} & 0 \\ 0 & \frac{s}{st-uz} & -\frac{z}{st-uz} & 0 \\ -\frac{y}{xw-yv} & 0 & 0 & \frac{x}{xw-yv} \end{pmatrix}, \tag{2.83}$$

$$M_{circ}^{-1} = \begin{pmatrix} 0 & \frac{w}{xw-yv} & -\frac{v}{xw-yv} & 0 \\ -\frac{u}{st-uz} & 0 & 0 & \frac{t}{st-uz} \\ \frac{s}{st-uz} & 0 & 0 & -\frac{z}{st-uz} \\ 0 & -\frac{y}{xw-yv} & \frac{x}{xw-yv} & 0 \end{pmatrix}, \tag{2.84}$$

and therefore the parameter conditions for invertibility are the same in both M_{star} and M_{circ}

$$xw - yv \neq 0, \qquad st - uz \neq 0. \tag{2.85}$$

The corresponding sets $\mathsf{M}_{star} = \{M_{star}\}$ and $\mathsf{M}_{circ} = \{M_{circ}\}$ are closed under the following multiplications

$$\mathsf{M}_{star}\mathsf{M}_{star} = \mathsf{M}_{star}, \tag{2.86}$$

$$\mathsf{M}_{star}\mathsf{M}_{circ} = \mathsf{M}_{circ}, \qquad \mathsf{M}_{circ}\mathsf{M}_{star} = \mathsf{M}_{circ}, \tag{2.87}$$

$$\mathsf{M}_{circ}\mathsf{M}_{circ} = \mathsf{M}_{star}, \tag{2.88}$$

and in terms of sets we can write $\mathsf{M}_{star} = \mathsf{N}_{star1} \cup \mathsf{N}_{star2}$ and $\mathsf{M}_{circ} = \mathsf{N}_{circ1} \cup \mathsf{N}_{circ2}$, while $\mathsf{N}_{star1} \cap \mathsf{N}_{star2} = \varnothing$ and $\mathsf{N}_{circ1} \cap \mathsf{N}_{circ2} = \varnothing$ (see (2.46)). Note that, if M_{star} and M_{circ} are treated as elements of an algebra, then (2.86)–(2.88) are reminiscent of the Cartan decomposition (see, e.g., [46]), but we will consider them from a more general viewpoint, which will treat such structures as semigroups, ternary groups and modules.

Thus the set $\mathsf{M}_{8vertex} = \mathsf{M}_{star} \cup \mathsf{M}_{circ}$ is closed, and because of the associativity of matrix multiplication, $\mathsf{M}_{8vertex}$ forms a non-commutative semigroup which we call a *8-vertex matrix semigroup* $\mathcal{S}_{8vertex}(4, \mathbb{C})$, which contains the zero matrix $Z \in \mathcal{S}_{8vertex}(4, \mathbb{C})$ and is a subsemigroup of the (binary) general linear semigroup $\mathrm{GLS}(4, \mathbb{C})$. It follows from (2.86), that M_{star} is its subsemigroup $\mathcal{S}_{8vertex}^{star}(4, \mathbb{C})$. Moreover, the invertible elements of $\mathcal{S}_{8vertex}(4, \mathbb{C})$ form a *8-vertex matrix group* $\mathcal{G}_{8vertex}(4, \mathbb{C})$, because its identity is a unit 4×4 matrix $I_4 \in \mathsf{M}_{star}$, and so M_{star} is a subgroup $\mathcal{G}_{8vertex}^{star}(4, \mathbb{C})$ of $\mathcal{G}_{8vertex}(4, \mathbb{C})$ and a subgroup of the (binary) general linear group $\mathrm{GL}(4, \mathbb{C})$. The structure of $\mathcal{S}_{8vertex}(4, \mathbb{C})$ (2.86) is similar to that of block-diagonal and block-antidiagonal matrices (of the necessary sizes). So the 8-vertex (binary) matrix semigroup $\mathcal{S}_{8vertex}(4, \mathbb{C})$ in which the parameters satisfy (2.85) is a 8-vertex (binary) matrix group $\mathcal{G}_{8vertex}(4, \mathbb{C})$, having a subgroup $\mathcal{G}_{8vertex}^{star}(4, \mathbb{C}) = \langle \mathsf{M}_{star} \mid \cdot, I_4 \rangle$, where $(\cdot)$ is an ordinary matrix product, and I_4 is its identity.

The group structure of the circle matrices M_{circ} (2.82) follows from

$$\mathsf{M}_{circ}\mathsf{M}_{circ}\mathsf{M}_{circ} = \mathsf{M}_{circ}, \tag{2.89}$$

which means that M_{circ} is closed with respect to the product of three matrices (the product of two matrices from M_{circ} is outside the set (2.88)). We define a ternary multiplication $\nu^{[3]}$ as the ordinary triple product of matrices, then $\mathcal{S}^{circ[3]}_{8vertex}(4,\mathbb{C}) = \left\langle \mathsf{M}_{circ} \mid \nu^{[3]} \right\rangle$ is a ternary (3-nonderived) semigroup with the zero $Z \in \mathsf{M}_{circ}$ which is a subsemigroup of the ternary (derived) general linear semigroup $\mathrm{GLS}^{[3]}(4,\mathbb{C})$. Instead of the inverse, for each invertible element $M_{circ} \in \mathsf{M}_{circ} \setminus Z$ we introduce the unique querelement $\bar{M}_{circ}$ [43] by (2.47), and since the ternary product is the triple ordinary product, we have $\bar{M}_{circ} = M^{-1}_{circ}$ from (2.48). Thus, if the conditions of invertibility (2.85) hold valid, then the ternary semigroup $\mathcal{S}^{circ(3)}_{8vertex}(4,\mathbb{C})$ becomes the ternary group $\mathcal{G}^{circ(3)}_{8vertex}(4,\mathbb{C}) = \left\langle \mathsf{M}_{circ} \mid \nu^{[3]}, \overline{()} \right\rangle$ which does not contain the ordinary (binary) identity, since $I_4 \notin \mathsf{M}_{circ}$. Nevertheless, the ternary group of circle matrices $\mathcal{G}^{circ[3]}_{8vertex}(4,\mathbb{C})$ has the following set $\mathsf{I}^{[3]}_{circ} = \left\{ I^{[3]}_{circ} \right\}$ of left-right 6-vertex and 8-vertex ternary identities (see (2.49)–(2.50))

$$I^{[3]}_{circ} = \begin{pmatrix} 0 & \frac{1}{a} & b & 0 \\ a & 0 & 0 & -\frac{ab}{c} \\ 0 & 0 & 0 & \frac{1}{c} \\ 0 & 0 & c & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\frac{ab}{c} & \frac{1}{c} & 0 \\ 0 & 0 & 0 & \frac{1}{b} \\ c & 0 & 0 & a \\ 0 & b & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\frac{ab}{c} & \frac{1-ad}{c} & 0 \\ -\frac{cd}{b} & 0 & 0 & \frac{1-ad}{b} \\ c & 0 & 0 & a \\ 0 & b & d & 0 \end{pmatrix}, \tag{2.90}$$

which (without additional conditions) depend upon the free parameters $a, b, c, d \in \mathbb{C}$, $b, c \neq 0$, and $\left(I^{[3]}_{circ}\right)^2 = I_4$, $I^{[3]}_{circ} \in \mathsf{M}_{circ}$. In the binary sense, the matrices from (2.90) are mutually similar, but as ternary identities they are different.

If we consider the second operation for matrices (as elements of a general matrix ring), the binary matrix addition $(+)$, the structure of $\mathsf{M}_{8vertex} = \mathsf{M}_{star} \cup \mathsf{M}_{circ}$ becomes more exotic: the set M_{star} is a $(2,2)$-ring $\mathcal{R}^{star[2,2]}_{8vertex} = \langle \mathsf{M}_{star} \mid +, \cdot \rangle$ with the binary addition $(+)$ and binary multiplication $(\cdot)$ from the semigroup $\mathcal{S}^{star}_{8vertex}$, while M_{circ} is a $(2,3)$-ring $\mathcal{R}^{circ[2,3]}_{8vertex} = \left\langle \mathsf{M}_{circ} \mid +, \nu^{[3]} \right\rangle$ with the binary matrix addition $(+)$, the ternary matrix multiplication $\nu^{[3]}$ and the zero Z.

Moreover, because of the distributivity and associativity of binary matrix multiplication, the relations (2.87) mean that the set M_{circ} (being an abelian group under binary addition) can be treated as a left and right binary module $\mathcal{M}^{circ}_{8vertex}$ over the ring $\mathcal{R}^{star(2,2)}_{8vertex}$ with an operation $(*)$: the module action $M_{circ} * M_{star} = M_{circ}$, $M_{star} * M_{circ} = M_{circ}$ (coinciding with the ordinary matrix product (2.87)). The left and right modules are compatible, since the associativity of ordinary matrix multiplication gives the compatibility condition $(M_{circ}M_{star})M'_{circ} = M_{circ}(M_{star}M'_{circ})$, $M_{star} \in \mathcal{R}^{star(2,2)}_{8vertex}$, $M_{circ}, M'_{circ} \in \mathcal{R}^{circ(2,3)}_{8vertex}$, and therefore M_{circ} (as an abelian group under the binary addition $(+)$ and the module action $(*)$) is a $\mathcal{R}^{star(2,2)}_{8vertex}$-bimodule $\mathcal{M}^{circ}_{8vertex}$. The last relation (2.88) shows another interpretation of M_{circ} as a formal "square root" of M_{star} (as sets).

2.6. Star 8-vertex and circle 8-vertex Yang-Baxter maps

Let us consider the star 8-vertex solutions $\tilde{c}$ to the Yang-Baxter equation (2.12), having the shape (2.81), in the most general setting, over $\mathbb{C}$ and for different ranks (i.e. including noninvertible ones). In components they are determined by

$$\begin{aligned}
&vy(u-z)=0, \quad y\left(t^2-wz-x^2+xz\right)=0,\\
&y(s(x-z)+t(u-x))=0, \quad y\left(u(w-x)+x^2-s^2\right)=0,\\
&svy-tuz=0, tvy-suz=0, vwy+xz(x-z)-stz=0,\\
&uz(z-u)=0, \quad suz-tvy=0, y\left(w^2-wz+xz-s^2\right)=0,\\
&y(s(w-u)+t(z-w))=0, \quad v\left(s^2-wz-x^2+xz\right)=0,\\
&tuz-svy=0, \quad stz-vxy+wz(z-w)=0,\\
&stu+u^2x-ux^2-vwy=0, \quad v(s(z-w)+t(w-u))=0,\\
&uz(u-z)=0, \quad y\left(t^2+u(w-x)-w^2\right)=0,\\
&v(s(u-x)+t(x-z))=0, \quad v\left(s^2+u(w-x)-w^2\right)=0,\\
&vy(z-u)=0, \quad v\left(u(w-x)+x^2-t^2\right)=0,\\
&uw^2+vxy-stu-u^2w=0, \quad v\left(w^2-t^2-wz+xz\right)=0.
\end{aligned} \tag{2.91}$$

Solutions from, e.g. [31,35], etc., should satisfy this overdetermined system of 24 cubic equations for 8 variables.

We search for the 8-vertex constant solutions to the Yang-Baxter equation over $\mathbb{C}$ without additional conditions, unitarity, etc. (which will be considered in the next sections). We also will need the matrix functions tr and det which are related to link invariants, as well as eigenvalues which help to find similar matrices and q-conjugated solutions to braid equations. Take into account that the Yang-Baxter maps are determined up to a general complex factor $t \in \mathbb{C}$ (2.14). For eigenvalues (which are determined up to the same factor t) we use the notation: $\{\text{eigenvalue}\}^{[\text{algebraic multiplicity}]}$.

We found the following 8-vertex solutions, classified by rank and number of parameters.

• Rank $=4$ (invertible star Yang-Baxter maps) are

1) quadratic in two parameters

$$\tilde{c}^{\,par=2}_{rank=4}(x,y)=\begin{pmatrix} xy & 0 & 0 & y^2\\ 0 & xy & \pm xy & 0\\ 0 & \mp xy & xy & 0\\ -x^2 & 0 & 0 & xy \end{pmatrix}, \tag{2.92}$$

$$\begin{gathered}
\operatorname{tr}\tilde{c}=4xy,\\
\det\tilde{c}=4x^4y^4, \quad x\neq 0, \ y\neq 0,\\
\text{eigenvalues: } \{(1+i)xy\}^{[2]}, \{(1-i)xy\}^{[2]},
\end{gathered} \tag{2.93}$$

2) quadratic in three parameters

$$\tilde{c}^{\,par=3}_{rank=4,1}(x,y,z)=\begin{pmatrix} xy & 0 & 0 & y^2\\ 0 & zy & \pm xy & 0\\ 0 & \pm xy & zy & 0\\ z^2 & 0 & 0 & xy \end{pmatrix}, \tag{2.94}$$

$$\begin{gathered}
\operatorname{tr}\tilde{c}=2y(x+z),\\
\det\tilde{c}=y^4\left(z^2-x^2\right)^2, \quad z\neq\pm x, \ y\neq 0,\\
\text{eigenvalues: } y(x-z), -y(x-z), \{y(x+z)\}^{[2]},
\end{gathered} \tag{2.95}$$

3) irrational in three parameters

$$\tilde{c}^{par=3}_{rank=4}(x,y,z) = \begin{pmatrix} xy & 0 & 0 & y^2 \\ 0 & \frac{x+z}{2}y & \pm y\sqrt{\frac{x^2+z^2}{2}} & 0 \\ 0 & \pm y\sqrt{\frac{x^2+z^2}{2}} & \frac{x+z}{2}y & 0 \\ \frac{(x+z)^2}{4} & 0 & 0 & yz \end{pmatrix}, \tag{2.96}$$

$$\operatorname{tr}\tilde{c} = 2y(x+z), \quad \det\tilde{c} = \tfrac{1}{16}y^4(x-z)^4, \quad y\neq 0, z\neq x, \tag{2.97}$$

$$\text{eigenvalues: } \left\{\frac{1}{2}y\left(x+z-\sqrt{2}\sqrt{x^2+z^2}\right)\right\}^{[2]}, \left\{\frac{1}{2}y\left(x+z+\sqrt{2}\sqrt{x^2+z^2}\right)\right\}^{[2]}.$$

Note that only the first and the last cases are genuine 8-vertex Yang-Baxter maps, because the three-parameter matrices (2.94) are q-conjugated with the 4-vertex parameter-permutation solutions (2.35). Indeed,

$$\begin{pmatrix} xy & 0 & 0 & y^2 \\ 0 & zy & xy & 0 \\ 0 & xy & zy & 0 \\ z^2 & 0 & 0 & xy \end{pmatrix} = (q\otimes_K q)\begin{pmatrix} y(x+z) & 0 & 0 & 0 \\ 0 & 0 & y(x-z) & 0 \\ 0 & y(x-z) & 0 & 0 \\ 0 & 0 & 0 & y(x+z) \end{pmatrix}\left(q^{-1}\otimes_K q^{-1}\right), \tag{2.98}$$

$$q = \begin{pmatrix} \pm\sqrt{\frac{y}{z}} & b \\ 1 & \mp b\sqrt{\frac{z}{y}} \end{pmatrix}, \tag{2.99}$$

where $b\in\mathbb{C}$ is a free parameter. If $b=\frac{y}{z}$ two matrices q in (2.99) are similar, and we have the unique q-conjugation (2.98). Another solution in (2.94) is q-conjugated to the second 4-vertex parameter-permutation solutions (2.35) such that

$$\begin{pmatrix} xy & 0 & 0 & y^2 \\ 0 & zy & -xy & 0 \\ 0 & -xy & zy & 0 \\ z^2 & 0 & 0 & xy \end{pmatrix} = (q\otimes_K q)\begin{pmatrix} 0 & 0 & 0 & y(x-z) \\ 0 & y(x+z) & 0 & 0 \\ 0 & 0 & y(x+z) & 0 \\ y(x-z) & 0 & 0 & 0 \end{pmatrix}\left(q^{-1}\otimes_K q^{-1}\right), \tag{2.100}$$

$$q = \begin{pmatrix} i\sqrt{\frac{y}{z}} & \pm i\sqrt{\frac{y}{z}} \\ \pm 1 & 1 \end{pmatrix}, \begin{pmatrix} -i\sqrt{\frac{y}{z}} & \pm i\sqrt{\frac{y}{z}} \\ \pm 1 & 1 \end{pmatrix}, \tag{2.101}$$

where q's are pairwise similar in (2.101), and therefore we have 2 different q-conjugations.

• Rank $=2$ (noninvertible star Yang-Baxter maps) are quadratic in parameters

$$\tilde{c}^{par=2}_{rank=2}(x,y) = \begin{pmatrix} xy & 0 & 0 & y^2 \\ 0 & xy & \pm xy & 0 \\ 0 & \pm xy & xy & 0 \\ x^2 & 0 & 0 & xy \end{pmatrix}, \quad \operatorname{tr}\tilde{c} = 4xy, \quad \text{eigenvalues: } \{2xy\}^{[2]}, \{0\}^{[2]}. \tag{2.102}$$

There are no star 8-vertex solutions of rank 3. The above two solutions for $\tilde{c}^{par=2}_{rank=4}$ with different signs are q-conjugated (2.19) with the matrix q being one of the following

$$q = \begin{pmatrix} 0 & 1 \\ \pm i\frac{x}{y} & 0 \end{pmatrix}. \tag{2.103}$$

Further families of solutions can be obtained from (2.92)–(2.102) by applying the general q-conjugation (2.14).

Particular cases of the star solutions are called also X-type operators [37] or magic matrices [18] connected with the Cartan decomposition of $SU(4)$ [27,28,47,48].

The circle 8-vertex solutions $\tilde{c}$ to the Yang-Baxter equation (2.12) of the shape (2.82) are determined by the following system of 32 cubic equations for 8 unknowns over $\mathbb{C}$

$$\begin{aligned}
&x(ty+z(u-y)-vx)=0, tx^2+y^2(v-z)-wx^2=0,\\
&y(-st+tx+wy-xz)=0, su(x-y)-sxy+uxy=0,\\
&z(t(y-x)-sz+wx)=0, v\left(sy+x^2\right)-z\left(s^2+ux\right)=0,\\
&swy-s^2v+xy(v-z)=0, swx-s^2w+yz(u-y)=0,\\
&st^2-t^2x+z^2(y-u)=0, su(v-z)+x(xz-tu)=0,\\
&su(w-v)+xy(z-t)=0, s(tu-uw+yz)-ty^2=0,\\
&s\left(tv+z^2\right)-x\left(v^2+wz\right)=0, svw-vwx+z(xz-wy)=0,\\
&sw(w-t)+yz(z-v)=0, s(sz+u(v-w)-vy)=0,\\
&t(tu-vy+z(y-x))=0, tx(x-s)+u^2v-uvy=0,\\
&xy(t-w)+u^2w-uvx=0, t\left(sy+u^2\right)-w\left(ux+y^2\right)=0,\\
&tz(s-x)-sv^2+tuv=0, tz(x-y)-svw+uvw=0,\\
&u\left(w^2-tz\right)-swz+tyz=0, s^2(t-w)+u^2(v-z)=0,\\
&tx(w-t)+uv(z-v)=0, tvx-t^2y-uvw+vwy=0,\\
&tvy-t^2u+w(wy-uz)=0, u(s(v-w)-tu+wx)=0,\\
&twz-tv(w+z)+vwz=0, v(s(w-t)-uw+vx)=0,\\
&sw^2-uv^2+v^2y-w^2x=0, w(sv+u(z-v)-wy)=0.
\end{aligned} \tag{2.104}$$

We found the 8-vertex solutions, classified by rank and number of parameters.

• Rank $=4$ (invertible circle Yang-Baxter map) are quadratic in parameters

$$\tilde{c}^{par=3}_{rank=4}(x,y,z)=\begin{pmatrix} 0 & xy & yz & 0\\ z^2 & 0 & 0 & xy\\ xz & 0 & 0 & yz\\ 0 & z^2 & xz & 0 \end{pmatrix}, \tag{2.105}$$

$$\begin{gathered}
\operatorname{tr}\tilde{c}=0,\\
\det\tilde{c}=y^2z^2\left(z^2-x^2\right), y\neq 0,\ \ z\neq 0,\ z\neq\pm x,\\
\text{eigenvalues: } \sqrt{-yz}(x-z), -\sqrt{-yz}(x-z), \sqrt{yz}(x+z), -\sqrt{yz}(x+z).
\end{gathered} \tag{2.106}$$

• Rank $=2$ (noninvertible circle Yang-Baxter map) are linear in parameters

$$\tilde{c}^{par=2}_{rank=2}(x,y)=\begin{pmatrix} 0 & -y & -y & 0\\ -x & 0 & 0 & y\\ -x & 0 & 0 & y\\ 0 & x & x & 0 \end{pmatrix},\ \text{eigenvalues: } 2\sqrt{xy}, -2\sqrt{xy}, \{0\}^{[2]}. \tag{2.107}$$

There are no circle 8-vertex solutions of rank 3. The corresponding families of solutions can be derived from the above by using the q-conjugation (2.14).

A particular case of the 8-vertex circle solution (2.105) was considered in [49].

2.7. Triangle invertible 9- and 10-vertex solutions

There are some higher vertex solutions to the Yang-Baxter equations which are not in the above star/circle classification. They are determined by the following system of 15 cubic equations for 9 unknowns over $\mathbb{C}$

$$y(-py - x(u+w-y) + v(y+z)) + s(v-x)(v+x) = 0,$$
$$x(-ty + vz + x(y-z)) = 0, \tag{2.108}$$
$$sx(t-v) + ty(w-z) + vz(y-u) = 0,$$
$$z(pz - t(y+z) + x(u+w-z)) + s\left(x^2 - t^2\right) = 0, \tag{2.109}$$
$$ps(-u+w-y+z) + s(-t(u+z) + x(u-w+y-z)$$
$$+ v(w+y)) + uwy - uwz - uyz + wyz = 0,$$
$$t(y(p-t) + u(x-v)) = 0,\ t(pz - t(u+z) + ux) = 0,$$
$$p^2 s + pu(-u+y+z) - t(st + u(u+w)) + u^2 x = 0,\ t(z(p-v) - tw + wx) = 0,$$
$$ps(t-v) + tw(y-u) + uv(w-z) = 0, v(-py + v(w+y) - wx) = 0,$$
$$v(z(v-p) + tw - wx) = 0, v(y(t-p) + u(v-x)) = 0,$$
$$p^2(-s) + pw(w-y-z) + sv^2 + w(v(u+w) - wx) = 0,\ p(p(w-u) - tw + uv) = 0, \tag{2.110}$$

We found the following 9-vertex Yang-Baxter maps

$$\tilde{c}^{9-vert,1}_{rank=4} = \begin{pmatrix} x & y & z & s \\ 0 & 0 & x & y \\ 0 & x & 0 & z \\ 0 & 0 & 0 & x \end{pmatrix}, \begin{pmatrix} x & y & y & z \\ 0 & 0 & -x & -y \\ 0 & -x & 0 & -y \\ 0 & 0 & 0 & x \end{pmatrix}, \begin{pmatrix} x & y & -y & z \\ 0 & 0 & x & -\frac{zx}{y} \\ 0 & x & 0 & \frac{zx}{y} \\ 0 & 0 & 0 & x \end{pmatrix}, \tag{2.111}$$

$$\operatorname{tr}\tilde{c} = 2x, \quad \det\tilde{c} = -x^4, \quad x \neq 0, \quad \text{eigenvalues: } \{x\}^{[3]}, -x. \tag{2.112}$$

The third matrix in (2.111) is conjugated with the 4-vertex parameter-permutation solutions (2.35) of the form (which has the same the same eigenvalues (2.112))

$$\tilde{c}^{4-vert}_{rank=4}(x) = \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & 0 & x \end{pmatrix} \tag{2.113}$$

by the conjugated matrix

$$U^{9to4} = \begin{pmatrix} 1 & -\frac{y}{2x} & \frac{y}{2x} & 0 \\ 0 & 1 & 0 & -\frac{z}{y} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.114}$$

The matrix (2.114) cannot be presented as the Kronecker product $q \otimes_K q$ (2.16), and so the third matrix in (2.111) and (2.113) are different solutions of the Yang-Baxter equation (2.12). Despite the first two matrices in (2.111) have the same eigenvalues (2.112), they are not similar, because they have different from (2.113) middle Jordan blocks.

Then we have another 3-parameter solutions with fractions

$$\tilde{c}^{9-vert,2}_{rank=4}(x,y,z) = \begin{pmatrix} x & y & y & z \\ 0 & 0 & -x & y - \frac{2xz}{y} \\ 0 & -x & 0 & y - \frac{2xz}{y} \\ 0 & 0 & 0 & x\left(\frac{4xz}{y^2} - 3\right) \end{pmatrix}, \tag{2.115}$$

$$\begin{gathered} \operatorname{tr}\tilde{c} = 2x\frac{2xz-y^2}{y^2}, \\ \det\tilde{c} = x^4\left(3 - \frac{4xz}{y^2}\right), \quad x \neq 0, \;\; y \neq 0, \;\; z \neq \frac{3y^2}{4x}, \\ \text{eigenvalues: } \{x\}^{[2]}, -x, x\left(\frac{4xz}{y^2} - 3\right), \end{gathered} \tag{2.116}$$

and

$$\tilde{c}^{9-vert,3}_{rank=4}(x,y,z) = \begin{pmatrix} x & y & -y & z \\ 0 & 0 & -x & \frac{2zx}{y} + y \\ 0 & 3x & 0 & \frac{2zx}{y} - y \\ 0 & 0 & 0 & \frac{4zx^2}{y^2} + x \end{pmatrix}, \tag{2.117}$$

$$\begin{gathered} \operatorname{tr}\tilde{c} = 2x\left(1 + 2\frac{xz}{y^2}\right) \\ \det\tilde{c} = 3x^4\left(\frac{4zx}{y^2} + 1\right), \quad x \neq 0, \;\; y \neq 0, \;\; z \neq \frac{y^2}{4x} \\ \text{eigenvalues: } x, i\sqrt{3}x, -i\sqrt{3}x, x\left(1 + \frac{4zx}{y^2}\right) \end{gathered} \tag{2.118}$$

The 4-parameter 9-vertex solution is

$$\tilde{c}^{9-vert,par=4}_{rank=4}(x,y,z,s) = \begin{pmatrix} x & y & z & s \\ 0 & 0 & -x & y - \frac{2sx}{z} \\ 0 & x - \frac{2xy}{z} & 0 & z - \frac{2sx}{z} \\ 0 & 0 & 0 & \frac{x(4sx - z(2y+z))}{z^2} \end{pmatrix}, \tag{2.119}$$

$$\begin{gathered} \operatorname{tr}\tilde{c} = 2x\frac{2sx - yz}{z^2} \\ \det\tilde{c} = \frac{x^4(2y-z)(z(2y+z) - 4sx)}{z^3}, x \neq 0, y \neq \frac{z}{2}, z \neq 0, \end{gathered} \tag{2.120}$$

$$\text{eigenvalues: } x, x\sqrt{\frac{2y}{z} - 1}, -x\sqrt{\frac{2y}{z} - 1}, \frac{x(4sx - z(2y+z))}{z^2}. \tag{2.121}$$

We also found 5-parameter, 9-vertex solution of the form

$$\tilde{c}^{9-vert,par=5}_{rank=4}(x,y,z,s,t) = \begin{pmatrix} x & y & z & s \\ 0 & 0 & t & \frac{s(t-x)}{z} + y \\ 0 & \frac{y(t-x)}{z} + x & 0 & \frac{s(t-x)}{z} + z \\ 0 & 0 & 0 & \frac{s(t-x)^2 + tz(y+z) - xyz}{z^2} \end{pmatrix}, \tag{2.122}$$

$$\operatorname{tr}\tilde{c} = \frac{st^2 + sx^2 + tz^2 + xz^2 - 2stx + tyz - xyz}{z^2}, \tag{2.123}$$

$$\det\tilde{c} = \frac{xt(x(y-z) - ty)\left(s(t-x)^2 + tz(y+z) - xyz\right)}{z^3}, \tag{2.124}$$

$$\begin{gathered} \text{eigenvalues: } x, \sqrt{\frac{t}{z}(ty - xy + xz)}, -\sqrt{\frac{t}{z}(ty - xy + xz)}, \\ \frac{st^2 - 2stx + tz^2 + ytz + sx^2 - yxz}{z^2}, \quad x \neq 0, \; z \neq 0, \; t \neq 0. \end{gathered}$$

Finally, we found the following 3-parameter 10-vertex solution

$$\tilde{c}^{10-vert}_{rank=4}(x,y,z) = \begin{pmatrix} x & y & y & \frac{y^2}{x} \\ 0 & 0 & -x & -y \\ 0 & -x & 0 & -y \\ z & 0 & 0 & x \end{pmatrix}, \tag{2.125}$$

$$\begin{aligned} \operatorname{tr} \tilde{c} &= 2x, \\ \det \tilde{c} &= -x\left(x^3 + zy^2\right), \;\; x \neq 0, \\ \text{eigenvalues: } &\{x\}^{[2]}, \sqrt{x^2 + \frac{zy^2}{x}}, -\sqrt{x^2 + \frac{zy^2}{x}}. \end{aligned} \tag{2.126}$$

This 10-vertex solution is conjugated with the 4-vertex parameter-permutation solutions (2.35) of the form (which has the same the same eigenvalues as (2.125))

$$\tilde{c}^{4-vert}_{rank=4}(x,y,z) = \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & 0 & x + \frac{y^2 z}{x^2} & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & 0 & x \end{pmatrix} \tag{2.127}$$

by the conjugated matrix

$$U^{10to4} = \begin{pmatrix} 0 & \frac{x}{z} & -\frac{x}{z} & 0 \\ -1 & -\frac{x^2}{yz} & \frac{x^2}{yz} & -\frac{y}{x} \\ 1 & -\frac{x^2}{yz} & \frac{x^2}{yz} & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}. \tag{2.128}$$

Because the matrix (2.128) cannot be presented as the Kronecker product $q \otimes_K q$ (2.16), therefore (2.125) and (2.127) are different solutions of the Yang-Baxter equation (2.12).

Further families of the higher vertex solutions to the constant Yang-Baxter equation (2.12) can be obtained from the above ones by using the q-conjugation (2.14).

3. Polyadic braid operators and higher braid equations

The polyadic version of the braid equation (2.1) was introduced in [21]. Here we define higher analog of the Yang-Baxter operator and develop its connection with higher braid groups and quantum computations.

Let us consider a vector space V over a field $\mathbb{K}$. A *polyadic (n-ary) braid operator* $\mathrm{C}_{\mathrm{V}^n}$ is defined as the mapping [21]

$$\mathrm{C}_{\mathrm{V}^n} : \overbrace{\mathrm{V} \otimes \ldots \otimes \mathrm{V}}^{n} \to \overbrace{\mathrm{V} \otimes \ldots \otimes \mathrm{V}}^{n}. \tag{3.1}$$

The polyadic analog of the braid equation (2.1) was introduced in [21] using the associative quiver technique [45].

Let us introduce n operators

$$\mathrm{A}_p : \overbrace{\mathrm{V} \otimes \ldots \otimes \mathrm{V}}^{2n-1} \to \overbrace{\mathrm{V} \otimes \ldots \otimes \mathrm{V}}^{2n-1}, \tag{3.2}$$

$$\mathrm{A}_p = \mathrm{id}_{\mathrm{V}}^{\otimes(p-1)} \otimes \mathrm{C}_{\mathrm{V}^n} \otimes \mathrm{id}_{\mathrm{V}}^{\otimes(n-p)}, \quad p = 1, \ldots, n, \tag{3.3}$$

i.e. p is a place of $\mathrm{C}_{\mathrm{V}^n}$ instead of one id_{V} in $\mathrm{id}_{\mathrm{V}}^{\otimes n}$. A system of $(n-1)$ *polyadic (n-ary) braid equations* is defined by

$$\mathrm{A}_1 \circ \mathrm{A}_2 \circ \mathrm{A}_3 \circ \mathrm{A}_4 \circ \ldots \circ \mathrm{A}_{n-2} \circ \mathrm{A}_{n-1} \circ \mathrm{A}_n \circ \mathrm{A}_1 \tag{3.4}$$
$$= \mathrm{A}_2 \circ \mathrm{A}_3 \circ \mathrm{A}_4 \circ \mathrm{A}_5 \circ \ldots \circ \mathrm{A}_{n-1} \circ \mathrm{A}_n \circ \mathrm{A}_1 \circ \mathrm{A}_2 \tag{3.5}$$
$$\vdots$$
$$= \mathrm{A}_n \circ \mathrm{A}_1 \circ \mathrm{A}_2 \circ \mathrm{A}_3 \circ \ldots \circ \mathrm{A}_{n-3} \circ \mathrm{A}_{n-2} \circ \mathrm{A}_{n-1} \circ \mathrm{A}_n. \tag{3.6}$$

In the lowest non-binary case $n = 3$, we have the ternary braid operator $\mathrm{C}_{\mathrm{V}^3} : \mathrm{V} \otimes \mathrm{V} \otimes \mathrm{V} \to \mathrm{V} \otimes \mathrm{V} \otimes \mathrm{V}$ and two ternary braid equations on $\mathrm{V}^{\otimes 5}$

$$\begin{aligned}
&(\mathrm{C}_{\mathrm{V}^3} \otimes \mathrm{id}_{\mathrm{V}} \otimes \mathrm{id}_{\mathrm{V}}) \circ (\mathrm{id}_{\mathrm{V}} \otimes \mathrm{C}_{\mathrm{V}^3} \otimes \mathrm{id}_{\mathrm{V}}) \circ (\mathrm{id}_{\mathrm{V}} \otimes \mathrm{id}_{\mathrm{V}} \otimes \mathrm{C}_{\mathrm{V}^3}) \circ (\mathrm{C}_{\mathrm{V}^3} \otimes \mathrm{id}_{\mathrm{V}} \otimes \mathrm{id}_{\mathrm{V}}) \\
&= (\mathrm{id}_{\mathrm{V}} \otimes \mathrm{C}_{\mathrm{V}^3} \otimes \mathrm{id}_{\mathrm{V}}) \circ (\mathrm{id}_{\mathrm{V}} \otimes \mathrm{id}_{\mathrm{V}} \otimes \mathrm{C}_{\mathrm{V}^3}) \circ (\mathrm{C}_{\mathrm{V}^3} \otimes \mathrm{id}_{\mathrm{V}} \otimes \mathrm{id}_{\mathrm{V}}) \circ (\mathrm{id}_{\mathrm{V}} \otimes \mathrm{C}_{\mathrm{V}^3} \otimes \mathrm{id}_{\mathrm{V}}) \\
&= (\mathrm{id}_{\mathrm{V}} \otimes \mathrm{id}_{\mathrm{V}} \otimes \mathrm{C}_{\mathrm{V}^3}) \circ (\mathrm{C}_{\mathrm{V}^3} \otimes \mathrm{id}_{\mathrm{V}} \otimes \mathrm{id}_{\mathrm{V}}) \circ (\mathrm{id}_{\mathrm{V}} \otimes \mathrm{C}_{\mathrm{V}^3} \otimes \mathrm{id}_{\mathrm{V}}) \circ (\mathrm{id}_{\mathrm{V}} \otimes \mathrm{id}_{\mathrm{V}} \otimes \mathrm{C}_{\mathrm{V}^3}).
\end{aligned} \tag{3.7}$$

Note that the higher braid equations presented above differ from the generalized Yang-Baxter equations of [23,24,50].

The higher braid operators (3.1) satisfying the higher braid equations (3.4)–(3.6) can represent the higher braid group [22] using (2.6) and (3.3). By analogy with (2.6) we introduce m operators by

$$\mathrm{B}_i(m) : \overbrace{\mathrm{V} \otimes \ldots \otimes \mathrm{V}}^{m+n-2} \to \overbrace{\mathrm{V} \otimes \ldots \otimes \mathrm{V}}^{m+n-2}, \quad \mathrm{B}_0(m) = (\mathrm{id}_{\mathrm{V}})^{\otimes(m+n-2)}, \tag{3.8}$$
$$\mathrm{B}_i(m) = \mathrm{id}_{\mathrm{V}}^{\otimes(i-1)} \otimes \mathrm{C}_{\mathrm{V}^n} \otimes \mathrm{id}_{\mathrm{V}}^{\otimes(m-i-1)}, \quad i = 1, \ldots, m-1. \tag{3.9}$$

The representation $\pi_m^{[n]}$ of the higher braid group $\mathcal{B}_m^{[n+1]}$ (of $(n+1)$-degree in the notation of [22]) (having $m-1$ generators σ_i and identity $\boldsymbol{e}$) is given by

$$\pi_m^{[n]} : \mathcal{B}_m^{[n+1]} \longrightarrow \mathrm{End}\, \mathrm{V}^{\otimes(m+n-2)}, \tag{3.10}$$
$$\pi_m^{[n]}(\sigma_i) = \mathrm{B}_i(m), \quad i = 1, \ldots, m-1. \tag{3.11}$$

In this way, the generators σ_i of the higher braid group $\mathcal{B}_m^{[n+1]}$ satisfy the relations

• *n higher braid relations*

$$\overbrace{\sigma_i \sigma_{i+1} \ldots \sigma_{i+n-2} \sigma_{i+n-1} \sigma_i}^{n+1} \tag{3.12}$$
$$= \sigma_{i+1} \sigma_{i+2} \ldots \sigma_{i+n-1} \sigma_i \sigma_{i+1} \tag{3.13}$$
$$\vdots$$
$$= \sigma_{i+n-1} \sigma_i \sigma_{i+1} \sigma_{i+2} \ldots \sigma_i \sigma_{i+1} \sigma_{i+n-1}, \tag{3.14}$$
$$i = 1, \ldots, m-n, \tag{3.15}$$

• *n-ary far commutativity*

$$\overbrace{\sigma_{i_1} \sigma_{i_2} \ldots \sigma_{i_{n-2}} \sigma_{i_{n-1}} \sigma_{i_n}}^{n} \tag{3.16}$$
$$\vdots$$
$$= \sigma_{\tau(i_1)} \sigma_{\tau(i_2)} \ldots \sigma_{\tau(i_{n-2})} \sigma_{\tau(i_{n-1})} \sigma_{\tau(i_n)}, \tag{3.17}$$
$$\text{if all } \left| i_p - i_s \right| \geq n, \quad p, s = 1, \ldots, n, \tag{3.18}$$

where τ is an element of the permutation symmetry group $\tau \in S_n$. The relations (3.12)–(3.17) coincide with those from [22], obtained by another method, that is via the polyadic-binary correspondence.

In the case $m = 4$ and $n = 3$ the higher braid group $\mathcal{B}_4^{[4]}$ is represented by (3.7) and generated by 3 generators σ_1, σ_2, σ_3, which satisfy 2 braid relations only (without far commutativity)

$$\sigma_1\sigma_2\sigma_3\sigma_1 = \sigma_2\sigma_3\sigma_1\sigma_2 = \sigma_3\sigma_1\sigma_2\sigma_3. \tag{3.19}$$

According to (3.16)–(3.17), the far commutativity relations appear when the number of elements of the higher braid groups satisfy

$$m \geq m_{\min} = n(n-1) + 2, \tag{3.20}$$

such that all conditions (3.18) should hold. Thus, to have the far commutativity relations in the ordinary (binary) braid group (2.5) we need 3 generators and $\mathcal{B}_4$, while for $n = 3$ we need at least 7 generators σ_i and $\mathcal{B}_8^{[4]}$ (see *Example* **7.12** in [22]).

In the concrete realization of V as a d-dimensional euclidean vector space V_d over the complex numbers $\mathbb{C}$ and basis $\{e_i\}$, $i = 1, \ldots, d$, the polyadic (n-ary) braid operator $\mathrm{C}_{\mathrm{V}^n}$ becomes a matrix C_{d^n} of size $d^n \times d^n$ which satisfies $n-1$ higher braid equations (3.4)–(3.6) in matrix form. In the components, the matrix braid operator is

$$C_{d^n} \circ \left(e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_n}\right) = \sum_{j'_1, j'_2 \ldots j'_n = 1}^{d} c_{i_1 i_2 \ldots i_n}^{j'_1 j'_2 \ldots j'_n} \cdot e_{j'_1} \otimes e_{j'_2} \otimes \ldots \otimes e_{j'_n}. \tag{3.21}$$

THUS we have d^{2n} entries (unknowns) in C_{d^n} satisfying $(n-1)d^{4n-2}$ equations (3.4)–(3.6) in components of polynomial power $n+1$. In the minimal non-binary case $n = 3$, we have $2d^{10}$ equations of power 4 for d^6 unknowns, e.g. even for $d = 2$ we have 2048 for 64 components, and for $d = 3$ there are 118 098 equations for 729 components. Thus solving the matrix higher braid equations directly is cumbersome, and only particular cases are possible to investigate, for instance by using permutation matrices (2.28), or the star and circle matrices (2.81)–(2.82).

4. Solutions to the ternary braid equations

Here we consider some special solutions to the minimal ternary version ($n = 3$) of the polyadic braid equation (3.4)–(3.6), the ternary braid equation (3.7).

4.1. Constant matrix solutions

Let us consider the following two-dimensional vector space $V \equiv V_{d=2}$ (which is important for quantum computations) and the component matrix realization (3.21) of the ternary braiding operator $C_8 : V \otimes V \otimes V \to V \otimes V \otimes V$ as

$$C_8 \circ \left(e_{i_1} \otimes e_{i_2} \otimes e_{i_3}\right) = \sum_{j'_1, j'_2, j'_3 = 1}^{2} c_{i_1 i_2 i_3}^{j'_1 j'_2 j'_3} \cdot e_{j'_1} \otimes e_{j'_2} \otimes e_{j'_3}, \quad i_{1,2,3}, j'_{1,2,3} = 1, 2. \tag{4.1}$$

We now turn (4.1) to the standard matrix form (just to fix notations) by introducing the 8-dimensional vector space $\tilde{V}_8 = V \otimes V \otimes V$ with the natural basis $\tilde{e}_{\tilde{k}} = \{e_1 \otimes e_1 \otimes e_1, e_1 \otimes e_1 \otimes e_2, \ldots, e_2 \otimes e_2 \otimes e_2\}$, where $\tilde{k} = 1, \ldots, 8$ is a cumulative index. The linear operator $\tilde{C}_8 : \tilde{V}_8 \to \tilde{V}_8$ corresponding to (4.1) is given by the 8×8 matrix $\tilde{c}_{\tilde{i}\tilde{j}}$ as $\tilde{C}_8 \circ \tilde{e}_{\tilde{i}} = \sum_{\tilde{j}=1}^{8} \tilde{c}_{\tilde{i}\tilde{j}} \cdot \tilde{e}_{\tilde{j}}$. The operators (3.2)–(3.3) become three 32×32 matrices $\tilde{A}_{1,2,3}$ as

$$\tilde{A}_1 = \tilde{c} \otimes_K I_2 \otimes_K I_2, \quad \tilde{A}_2 = I_2 \otimes_K \tilde{c} \otimes_K I_2, \quad \tilde{A}_3 = I_2 \otimes_K I_2 \otimes_K \tilde{c}, \tag{4.2}$$

where $\otimes_K$ is the Kronecker product of matrices and I_2 is the 2×2 identity matrix. In this notation the operator ternary braid equations (3.7) become the matrix equations (cf. (3.4)–(3.6)) with $n = 3$)

$$\tilde{A}_1\tilde{A}_2\tilde{A}_3\tilde{A}_1 = \tilde{A}_2\tilde{A}_3\tilde{A}_1\tilde{A}_2 = \tilde{A}_3\tilde{A}_1\tilde{A}_2\tilde{A}_3, \tag{4.3}$$

which we call the *total matrix ternary braid equations*. Some weaker versions of ternary braiding are described by the *partial braid equations*

$$\begin{aligned} &\text{partial 12-braid equation} \quad \tilde{A}_1\tilde{A}_2\tilde{A}_3\tilde{A}_1 = \tilde{A}_2\tilde{A}_3\tilde{A}_1\tilde{A}_2, && (4.4)\\ &\text{partial 13-braid equation} \quad \tilde{A}_1\tilde{A}_2\tilde{A}_3\tilde{A}_1 = \tilde{A}_3\tilde{A}_1\tilde{A}_2\tilde{A}_3, && (4.5)\\ &\text{partial 23-braid equation} \quad \tilde{A}_2\tilde{A}_3\tilde{A}_1\tilde{A}_2 = \tilde{A}_3\tilde{A}_1\tilde{A}_2\tilde{A}_3, && (4.6) \end{aligned}$$

where, obviously, two of them are independent. It follows from (3.4)–(3.6) that the weaker versions of braiding are possible for $n \geq 3$, only, so for higher than binary braiding (the Yang-Baxter equation (2.8)).

Thus, comparing (4.3) and (3.19) we conclude that (for each invertible matrix $\tilde{c}$ in (4.2) satisfying (4.3)) the isomorphism $\tilde{\pi}_4^{[4]} : \sigma_i \mapsto \tilde{A}_i$, $i = 1, 2, 3$ gives a representation of the braid group $\mathcal{B}_4^{[4]}$ by 32×32 matrices over $\mathbb{C}$.

Now we can generate families of solutions corresponding to (4.2)–(4.3) in the following way. Consider an invertible operator $Q : V \to V$ in the two-dimensional vector space $V \equiv V_{d=2}$. In the basis $\{e_1, e_2\}$ its 2×2 matrix q is given by $Q \circ e_i = \sum_{j=1}^{2} q_{ij} \cdot e_j$. In the natural 8-dimensional basis $\tilde{e}_{\tilde{k}}$ the tensor product of operators $Q \otimes Q \otimes Q$ is presented by the Kronecker product of matrices $\tilde{q}_8 = q \otimes_K q \otimes_K q$. Let the 8×8 matrix $\tilde{c}$ be a fixed solution to the ternary braid matrix equations (4.3). Then the family of solutions $\tilde{c}(q)$ corresponding to the invertible 2×2 matrix q is the conjugation of $\tilde{c}$ by $\tilde{q}_8$ so that

$$\tilde{c}(q) = \tilde{q}_8\tilde{c}\tilde{q}_8^{-1} = (q \otimes_K q \otimes_K q)\tilde{c}\left(q^{-1} \otimes_K q^{-1} \otimes_K q^{-1}\right). \tag{4.7}$$

This also follows directly from the conjugation of the braid equations (4.3)–(4.6) by $q \otimes_K q \otimes_K q \otimes_K q \otimes_K q$ and (4.2). If we include the obvious invariance of the braid equations with the respect of an overall factor $t \in \mathbb{C}$, the general family of solutions becomes (cf. the Yang-Baxter equation [35])

$$\tilde{c}(q, t) = t\tilde{q}_8\tilde{c}\tilde{q}_8^{-1} = t(q \otimes_K q \otimes_K q)\tilde{c}\left(q^{-1} \otimes_K q^{-1} \otimes_K q^{-1}\right). \tag{4.8}$$

Let

$$q = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbb{C}), \tag{4.9}$$

and then the manifest form of $\tilde{q}_8$ is

$$\tilde{q}_8 = \begin{pmatrix} a^3 & a^2b & a^2b & ab^2 & a^2b & ab^2 & ab^2 & b^3 \\ a^2c & a^2d & abc & abd & abc & abd & b^2c & b^2d \\ a^2c & abc & a^2d & abd & abc & b^2c & abd & b^2d \\ ac^2 & acd & acd & ad^2 & bc^2 & bcd & bcd & bd^2 \\ a^2c & abc & abc & b^2c & a^2d & abd & abd & b^2d \\ ac^2 & acd & bc^2 & bcd & acd & ad^2 & bcd & bd^2 \\ ac^2 & bc^2 & acd & bcd & acd & bcd & ad^2 & bd^2 \\ c^3 & c^2d & c^2d & cd^2 & c^2d & cd^2 & cd^2 & d^3 \end{pmatrix}. \tag{4.10}$$

It is important that not every conjugation matrix has this very special form (4.10), and that therefore, in general, conjugated matrices are different solutions of the ternary braid equations (4.3). The matrix $\tilde{q}_8^\star\tilde{q}_8$ ($\star$ being the Hermitian conjugation) is diagonal (this case is important for further classification similar to the binary one [31]), when the conditions

$$ab^* + cd^* = 0 \tag{4.11}$$

hold, and so the matrix q has the special form (depending of 3 complex parameters, for $d \neq 0$)

$$q = \begin{pmatrix} a & b \\ -a\frac{b^*}{d^*} & d \end{pmatrix}. \tag{4.12}$$

We can present the families (4.7) for different ranks, because the conjugation by an invertible matrix does not change rank. To avoid demanding (4.11), due to the cumbersome calculations involved, we restrict ourselves to a triangle matrix for q (4.9).

In general, there are $8 \times 8 = 64$ unknowns (elements of the matrix $\tilde{c}$), and each partial braid equation (4.4)–(4.6) gives $32 \times 32 = 1024$ conditions (of power 4) for the elements of $\tilde{c}$, while the total braid equations (4.3) give twice as many conditions $1024 \times 2 = 2048$ (cf. the binary case: 64 cubic equations for 16 unknowns (2.8)). This means that even in the ternary case the higher braid system of equations is hugely overdetermined, and finding even the simplest solutions is a non-trivial task.

4.2. *Permutation and parameter-permutation* 8*-vertex solutions*

First we consider the case when $\tilde{c}$ is a binary (or logical) matrix consisting of $\{0,1\}$ only, and, moreover, it is a permutation matrix (see Subsection 2.4). In the latter case $\tilde{c}$ can be considered as a matrix over the field $\mathbb{F}_2$ (Galois field $GF(2)$). In total, there are $8! = 40\,320$ permutation matrices of the size 8×8. All of them are invertible of full rank 8, because they are obtained from the identity matrix by permutation of rows and columns.

We have found the following four invertible 8-vertex permutation matrix solutions to the ternary braid equations (4.3)

$$\tilde{c}^{bisymm1}_{rank=8} = \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix}, \tilde{c}^{bisymm2}_{rank=8} = \begin{pmatrix} 0&0&0&0&0&0&0&1\\ 0&1&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&1&0\\ 1&0&0&0&0&0&0&0 \end{pmatrix}, \tag{4.13}$$

$$\begin{aligned} \operatorname{tr}\tilde{c} &= 4, \\ \det\tilde{c} &= 1, \\ \text{eigenvalues:}&\{1\}^{[4]}, \{-1\}^{[4]}, \end{aligned} \tag{4.14}$$

and

$$\tilde{c}^{symm1}_{rank=8} = \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&1&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&1&0&0&0&0&0 \end{pmatrix}, \tilde{c}^{symm2}_{rank=8} = \begin{pmatrix} 0&0&0&0&0&1&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&1&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&1 \end{pmatrix}, \tag{4.15}$$

$$\begin{aligned} \operatorname{tr}\tilde{c} &= 4, \\ \det\tilde{c} &= 1, \\ \text{eigenvalues:}&\{1\}^{[4]}, \{-1\}^{[4]}. \end{aligned} \tag{4.16}$$

The first two solutions (4.13) are given by bisymmetric permutation matrices (see (2.33)), and we call them 8-vertex bisymm1 and bisymm2 respectively. The second two solutions (4.15) are symmetric matrices only (we call them 8-vertex symm1 and symm2), but one matrix is a reflection of the other with respect to the minor diagonal (making them mutually persymmetric). No 90°-symmetric (see (2.34)) solution for the ternary braid equations (4.3) was found. The bisymmetric and symmetric matrices have the same eigenvalues, and are therefore pairwise conjugate, but not q-conjugate, because the conjugation matrices do not have the form (4.10). Thus they are 4 different permutation solutions to the ternary braid equations (4.3). Note that the bisymm1 solution (4.13) coincides with the three-qubit swap operator introduced in [18].

All the permutation solutions are reflections (or involutions) $\tilde{c}^2 = I_8$ having $\det \tilde{c} = +1$, eigenvalues $\{1, -1\}$, and are semi-magic squares (the sums in rows and columns are 1, but not the sums in both diagonals). The 8-vertex permutation matrix solutions do not form a binary or ternary group, because they are not closed with respect to multiplication.

By analogy with (2.35)–(2.37), we obtain the 8-vertex parameter-permutation solutions from (4.13)–(4.15) by replacing units with parameters and then solving the ternary braid equations (4.3). Each type of the permutation solutions bisymm1, 2 and symm1, 2 from (4.13)–(4.15) will give a corresponding series of parameter-permutation solutions over $\mathbb{C}$. The ternary braid maps are determined up to a general complex factor (see (2.14) for the Yang-Baxter maps and (4.8)), and therefore we can present all the parameter-permutation solutions in polynomial form.

• The bisymm1 series consists of 2 two-parameter matrices with and 2 two-parameter matrices

$$\tilde{c}^{bisymm1,1}_{rank=8}(x,y) = \begin{pmatrix} xy & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \pm y^2 & 0 \\ 0 & 0 & xy & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \pm x^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \pm y^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & xy & 0 & 0 \\ 0 & \pm x^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & xy \end{pmatrix}, \tag{4.17}$$

$$\begin{gathered} \operatorname{tr} \tilde{c} = 4xy, \\ \det \tilde{c} = x^8 y^8, \quad x, y \neq 0, \\ \text{eigenvalues: } \{xy\}^{[6]}, \{-xy\}^{[2]}, \end{gathered} \tag{4.18}$$

and

$$\tilde{c}^{bisymm1,2}_{rank=8}(x,y) = \begin{pmatrix} xy & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \pm y^2 & 0 \\ 0 & 0 & xy & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \pm x^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mp y^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & xy & 0 & 0 \\ 0 & \mp x^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & xy \end{pmatrix}, \tag{4.19}$$

$$\begin{gathered} \operatorname{tr} \tilde{c} = 4xy, \\ \det \tilde{c} = x^8 y^8, \quad x, y \neq 0, \\ \text{eigenvalues: } \{xy\}^{[4]}, \{ixy\}^{[2]}, \{-ixy\}^{[2]}. \end{gathered} \tag{4.20}$$

• The bisymm2 series consists of 4 two-parameter matrices

$$\tilde{c}^{bisymm2,1}_{rank=8}(x,y) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & x^6 \\ 0 & \pm x^3y^3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x^4y^2 & 0 & 0 \\ 0 & 0 & 0 & \pm x^3y^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \pm x^3y^3 & 0 & 0 & 0 \\ 0 & 0 & x^2y^4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \pm x^3y^3 & 0 \\ y^6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (4.21)$$

$$\begin{aligned} \operatorname{tr}\tilde{c} &= \pm 4x^3y^3, \\ \det \tilde{c}^{bisymm2}_{rank=8}(x,y) &= x^{24}y^{24}, \; x,y \neq 0, \\ \text{eigenvalues: } &\left\{x^3y^3\right\}^{[2]}, \left\{-x^3y^3\right\}^{[2]}, \left\{\pm x^3y^3\right\}^{[4]}, \end{aligned} \quad (4.22)$$

and

$$\tilde{c}^{bisymm2,2}_{rank=8}(x,y) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & x^6 \\ 0 & \pm x^3y^3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x^4y^2 & 0 & 0 \\ 0 & 0 & 0 & \pm x^3y^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \pm x^3y^3 & 0 & 0 & 0 \\ 0 & 0 & -x^2y^4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \pm x^3y^3 & 0 \\ -y^6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (4.23)$$

$$\begin{aligned} \operatorname{tr}\tilde{c} &= \pm 4x^3y^3 \\ \det \tilde{c}^{bisymm2}_{rank=8}(x,y) &= x^{24}y^{24}, x,y \neq 0, \\ \text{eigenvalues: } &\left\{ix^3y^3\right\}^{[2]}, \left\{-ix^3y^3\right\}^{[2]}, \left\{\pm x^3y^3\right\}^{[4]}. \end{aligned} \quad (4.24)$$

• The symm1 series consists of 4 two-parameter matrices

$$\tilde{c}^{symm1,1}_{rank=8}(x,y) = \begin{pmatrix} xy & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \pm xy & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & y^2 \\ 0 & 0 & 0 & xy & 0 & 0 & 0 & 0 \\ 0 & \pm xy & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & xy & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & xy & 0 \\ 0 & 0 & x^2 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (4.25)$$

$$\begin{aligned} \operatorname{tr}\tilde{c} &= 4xy, \\ \det \tilde{c} &= x^8y^8, \; x,y \neq 0, \\ \text{eigenvalues: } &\{xy\}^{[6]}, \{-xy\}^{[2]}, \end{aligned} \quad (4.26)$$

and

$$\tilde{c}^{symm1,2}_{rank=8}(x,y) = \begin{pmatrix} xy & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \pm xy & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & y^2 \\ 0 & 0 & 0 & xy & 0 & 0 & 0 & 0 \\ 0 & \mp xy & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & xy & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & xy & 0 \\ 0 & 0 & -x^2 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4.27}$$

$$\begin{gathered} \operatorname{tr}\tilde{c} = 4xy, \\ \det\tilde{c} = x^8y^8, \quad x,y \neq 0, \\ \text{eigenvalues: } \{xy\}^{[6]}, \{-xy\}^{[2]}, \end{gathered} \tag{4.28}$$

- The symm2 series consists of 4 two-parameter matrices

$$\tilde{c}^{symm2,1}_{rank=8}(x,y) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & y^2 & 0 & 0 \\ 0 & xy & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & xy & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \pm xy & 0 \\ 0 & 0 & 0 & 0 & xy & 0 & 0 & 0 \\ x^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \pm xy & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & xy \end{pmatrix}, \tag{4.29}$$

$$\begin{gathered} \operatorname{tr}\tilde{c} = 4xy, \\ \det\tilde{c} = x^8y^8, \quad x,y \neq 0, \\ \text{eigenvalues: } \{xy\}^{[4]}, \{ixy\}^{[2]}, \{-ixy\}^{[2]}. \end{gathered} \tag{4.30}$$

and

$$\tilde{c}^{symm2,2}_{rank=8}(x,y) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & y^2 & 0 & 0 \\ 0 & xy & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & xy & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \pm xy & 0 \\ 0 & 0 & 0 & 0 & xy & 0 & 0 & 0 \\ -x^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mp xy & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & xy \end{pmatrix}, \tag{4.31}$$

$$\begin{gathered} \operatorname{tr}\tilde{c} = 4xy, \\ \det\tilde{c} = x^8y^8, \quad x,y \neq 0, \\ \text{eigenvalues: } \{xy\}^{[4]}, \{ixy\}^{[2]}, \{-ixy\}^{[2]}. \end{gathered} \tag{4.32}$$

The above matrices with the same eigenvalues are similar, but their conjugation matrices do not have the form of the triple Kronecker product (4.10), and therefore all of them together are 16 different two-parameter invertible solutions to the ternary braid equations (4.3). Further families of solutions can be obtained using ternary q-conjugation (4.8).

4.3. *Group structure of the star and circle 8-vertex matrices*

Here we investigate the group structure of 8×8 matrices by analogy with the star-like (2.39) and circle-like (2.40) 4×4 matrices which are connected with our 8-vertex constant solutions (4.17)–(4.31) to the ternary braid equations (4.3).

Let us introduce the *star-like* 8×8 *matrices* (see their 4×4 analog (2.39)) which correspond to the bisymm series (4.17)–(4.23)

$$N'_{star1} = \begin{pmatrix} x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & y & 0 \\ 0 & 0 & z & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & s & 0 & 0 & 0 \\ 0 & 0 & 0 & t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & u & 0 & 0 \\ 0 & v & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & w \end{pmatrix}, \quad N'_{star2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & y \\ 0 & x & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & s & 0 & 0 \\ 0 & 0 & 0 & z & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & u & 0 & 0 & 0 \\ 0 & 0 & t & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & w & 0 \\ v & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4.33}$$

$$\begin{gathered} \operatorname{tr} N' = x + z + u + w, \quad \det N' = stuvwxyz, \quad s,t,u,v,w,x,y,z \neq 0, \\ \text{eigenvalues: } x, z, u, w, -\sqrt{yv}, \sqrt{yv}, -\sqrt{st}, \sqrt{st}, \end{gathered}$$

and the *circle-like* 8×8 *matrices* (see their 4×4 analog (2.40)) which correspond to the symm series (4.25)–(4.31)

$$N'_{circ1} = \begin{pmatrix} x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & z \\ 0 & 0 & 0 & s & 0 & 0 & 0 & 0 \\ 0 & t & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & u & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & v & 0 \\ 0 & 0 & w & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad N'_{circ2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & y & 0 & 0 \\ 0 & x & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & s & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & z & 0 \\ 0 & 0 & 0 & 0 & u & 0 & 0 & 0 \\ t & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & w & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & v \end{pmatrix}, \tag{4.34}$$

$$\begin{gathered} \operatorname{tr} N' = x + s + u + v, \quad \det N' = stuvwxyz, \quad s,t,u,v,w,x,y,z \neq 0, \\ \text{eigenvalues: } x, s, u, v, -\sqrt{ty}, \sqrt{ty}, -\sqrt{wz}, \sqrt{wz}. \end{gathered} \tag{4.35}$$

Denote the corresponding sets by $\mathsf{N}'_{star1,2} = \left\{ N'_{star1,2} \right\}$ and $\mathsf{N}'_{circ1,2} = \left\{ N'_{circ1,2} \right\}$, then we have for them (which differs from 4×4 matrix sets (2.46))

$$\mathsf{M}'_{full} = \mathsf{N}'_{star1} \cup \mathsf{N}'_{star2} \cup \mathsf{N}'_{circ1} \cup \mathsf{N}'_{circ2}, \quad \mathsf{N}'_{star1} \cap \mathsf{N}'_{star2} \cap \mathsf{N}'_{circ1} \cap \mathsf{N}'_{circ2} = \mathsf{D}, \tag{4.36}$$

where D is the set of diagonal 8×8 matrices. As for 4×4 star-like and circle-like matrices, there are no closed binary multiplications for the sets of 8-vertex matrices (4.33)–(4.34). Nevertheless, we have the following triple set products

$$\mathsf{N}'_{star1}\mathsf{N}'_{star1}\mathsf{N}'_{star1} = \mathsf{N}'_{star1}, \tag{4.37}$$
$$\mathsf{N}'_{star2}\mathsf{N}'_{star2}\mathsf{N}'_{star2} = \mathsf{N}'_{star2}, \tag{4.38}$$
$$\mathsf{N}'_{circ1}\mathsf{N}'_{circ1}\mathsf{N}'_{circ1} = \mathsf{N}'_{circ1}, \tag{4.39}$$
$$\mathsf{N}'_{circ2}\mathsf{N}'_{circ2}\mathsf{N}'_{circ2} = \mathsf{N}'_{circ2}, \tag{4.40}$$

which should be compared with the analogous 4×4 matrices (2.44)–(2.45): note that now we do not have pentuple products.

Using the definitions (2.47)–(2.50), we interpret the closed products (4.37)–(4.38) and (4.39)–(4.40) as the multiplications $\mu^{[3]}$ (being the ordinary triple matrix product) of the *ternary semigroups* $\mathcal{S}^{[3]}_{star1,2}(8,\mathbb{C}) = \left\{\mathsf{N}'_{star1,2} \mid \mu^{[3]}\right\}$ and $\mathcal{S}^{[3]}_{circ1,2}(8,\mathbb{C}) = \left\{\mathsf{N}'_{circ1,2} \mid \mu^{[3]}\right\}$, respectively. The corresponding querelements (2.47) are given by

$$\bar{N}'_{star1} = N'^{-1}_{star1} = \begin{pmatrix} \frac{1}{x} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{v} & 0 \\ 0 & 0 & \frac{1}{z} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{t} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{s} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{u} & 0 & 0 \\ 0 & \frac{1}{y} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{w} \end{pmatrix}, \tag{4.41}$$

$$\bar{N}'_{star2} = N'^{-1}_{star2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{v} \\ 0 & \frac{1}{x} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{t} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{z} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{u} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{s} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{w} & 0 \\ \frac{1}{y} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, s,t,u,v,w,x,y,z \neq 0, \tag{4.42}$$

and

$$\bar{N}'_{circ1} = N'^{-1}_{circ1} = \begin{pmatrix} \frac{1}{x} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{t} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{w} \\ 0 & 0 & 0 & \frac{1}{s} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{y} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{u} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{v} & 0 \\ 0 & 0 & \frac{1}{z} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4.43}$$

$$\bar{N}'_{circ2} = N'^{-1}_{circ2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{t} & 0 & 0 \\ 0 & \frac{1}{x} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{s} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{w} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{u} & 0 & 0 & 0 \\ \frac{1}{y} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{z} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{v} \end{pmatrix}, s,t,u,v,w,x,y,z \neq 0. \tag{4.44}$$

The ternary semigroups $\mathcal{S}^{[3]}_{star1,2}(8,\mathbb{C}) = \left\{\mathsf{N}'_{star1,2} \mid \mu^{[3]}\right\}$ and $\mathcal{S}^{[3]}_{circ1,2}(8,\mathbb{C}) = \left\{\mathsf{N}'_{circ1,2} \mid \mu^{[3]}\right\}$ in which every element has its querelement given by (4.41)–(4.43) become the *ternary groups* $\mathcal{G}^{[3]}_{star1,2}(8,\mathbb{C}) = \left\{\mathsf{N}'_{star1,2} \mid \mu^{[3]}, \overline{(\,)}\right\}$ and $\mathcal{G}^{[3]}_{circ1,2}(8,\mathbb{C}) = \left\{\mathsf{N}'_{circ1,2} \mid \mu^{[3]}, \overline{(\,)}\right\}$, which are four different (3-nonderived) ternary subgroups of the derived ternary general linear group $\mathrm{GL}^{[3]}(8,\mathbb{C})$. The ternary identities in $\mathcal{G}^{[3]}_{star1,2}(8,\mathbb{C})$

and $\mathcal{G}^{[3]}_{circ1,2}(8,\mathbb{C})$ are the following different continuous sets $\mathrm{I}'^{[3]}_{star1,2} = \left\{I'^{[3]}_{star1,2}\right\}$ and $\mathrm{I}'^{[3]}_{circ1,2} = \left\{I'^{[3]}_{circ1,2}\right\}$, where

$$I'^{[3]}_{star1} = \begin{pmatrix} e^{i\alpha_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{i\alpha_2} & 0 \\ 0 & 0 & e^{i\alpha_3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{i\alpha_4} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{i\alpha_5} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{i\alpha_6} & 0 & 0 \\ 0 & e^{i\alpha_7} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{i\alpha_8} \end{pmatrix},$$

$$I'^{[3]}_{star2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{i\alpha_2} \\ 0 & e^{i\alpha_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{i\alpha_4} & 0 & 0 \\ 0 & 0 & 0 & e^{i\alpha_3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{i\alpha_6} & 0 & 0 & 0 \\ 0 & 0 & e^{i\alpha_5} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{i\alpha_8} & 0 \\ e^{i\alpha_7} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$e^{2i\alpha_1} = e^{2i\alpha_3} = e^{2i\alpha_6} = e^{2i\alpha_8} = e^{i(\alpha_2+\alpha_7)} = e^{i(\alpha_4+\alpha_5)} = 1, \;\; \alpha_1,\ldots,\alpha_8 \in \mathbb{R}, \tag{4.45}$$

and

$$I'^{[3]}_{circ1} = \begin{pmatrix} e^{i\alpha_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{i\alpha_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{i\alpha_3} \\ 0 & 0 & 0 & e^{i\alpha_4} & 0 & 0 & 0 & 0 \\ 0 & e^{i\alpha_5} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{i\alpha_6} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{i\alpha_7} & 0 \\ 0 & 0 & e^{i\alpha_8} & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$I'^{[3]}_{circ2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & e^{i\alpha_2} & 0 & 0 \\ 0 & e^{i\alpha_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{i\alpha_4} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{i\alpha_3} & 0 \\ 0 & 0 & 0 & 0 & e^{i\alpha_6} & 0 & 0 & 0 \\ e^{i\alpha_5} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{i\alpha_8} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{i\alpha_7} \end{pmatrix},$$

$$e^{2i\alpha_1} = e^{2i\alpha_4} = e^{2i\alpha_6} = e^{2i\alpha_7} = e^{i(\alpha_3+\alpha_8)} = e^{i(\alpha_2+\alpha_5)} = 1, \;\; \alpha_1,\ldots,\alpha_8 \in \mathbb{R}, \tag{4.46}$$

such that all the identities are the 8×8 matrix reflections $\left(I'^{[3]}\right)^2 = I_8$ (see (2.50)). If $\alpha_j = 0$, $j = 1,\ldots,8$, the ternary identities (4.45)–(4.46) coincide with the 8×8 permutation matrices (4.13)–(4.15), which are solutions to the ternary braid equations (4.3).

The module structure of the 8-vertex star-like (4.33) and circle-like (4.34) 8×8 matrix sets differs from the 4×4 matrix sets (2.57)–(2.80). Firstly, because of the absence of pentuple matrix products (2.75)–(2.80), and secondly through some differences in the ternary closed products of sets.

We have the following triple relations between star and circle matrices separately (the sets corresponding to modules are in brackets, and we informally denote modules by their sets)

$$\mathsf{N}'_{star1}\left(\mathsf{N}'_{star2}\right)\mathsf{N}'_{star1} = \left(\mathsf{N}'_{star2}\right), \quad \mathsf{N}'_{circ1}\left(\mathsf{N}'_{circ2}\right)\mathsf{N}'_{circ1} = \left(\mathsf{N}'_{circ2}\right), \tag{4.47}$$
$$\mathsf{N}'_{star1}\mathsf{N}'_{star1}\left(\mathsf{N}'_{star2}\right) = \left(\mathsf{N}'_{star2}\right), \quad \mathsf{N}'_{circ1}\mathsf{N}'_{circ1}\left(\mathsf{N}'_{circ2}\right) = \mathsf{N}'_{circ2}, \tag{4.48}$$
$$\left(\mathsf{N}'_{star2}\right)\mathsf{N}'_{star1}\mathsf{N}'_{star1} = \left(\mathsf{N}'_{star2}\right), \quad \left(\mathsf{N}'_{circ2}\right)\mathsf{N}'_{circ1}\mathsf{N}'_{circ1} = \left(\mathsf{N}'_{circ2}\right), \tag{4.49}$$
$$\mathsf{N}'_{star2}\mathsf{N}'_{star2}\left(\mathsf{N}'_{star1}\right) = \left(\mathsf{N}'_{star1}\right), \quad \mathsf{N}'_{circ2}\mathsf{N}'_{circ2}\left(\mathsf{N}'_{circ1}\right) = \left(\mathsf{N}'_{circ1}\right), \tag{4.50}$$
$$\mathsf{N}'_{star2}\left(\mathsf{N}'_{star1}\right)\mathsf{N}'_{star2} = \left(\mathsf{N}'_{star1}\right), \quad \mathsf{N}'_{circ2}\left(\mathsf{N}'_{circ1}\right)\mathsf{N}'_{circ2} = \left(\mathsf{N}'_{circ1}\right), \tag{4.51}$$
$$\left(\mathsf{N}'_{star1}\right)\mathsf{N}'_{star2}\mathsf{N}'_{star2} = \left(\mathsf{N}'_{star1}\right), \quad \left(\mathsf{N}'_{circ1}\right)\mathsf{N}'_{circ2}\mathsf{N}'_{circ2} = \left(\mathsf{N}'_{circ1}\right). \tag{4.52}$$

So we may observe the following module structures:

1) from (4.47)–(4.49), the sets N'_{star2} (N'_{circ2}) are the middle, right and left ternary modules over N'_{star1} (N'_{circ1});

2) from (4.50)–(4.52), the set N'_{star1} (N'_{circ1}) are middle, right and left ternary modules over N'_{star2} (N'_{circ2});

$$\mathsf{N}'_{star1}\mathsf{N}'_{star1}\left(\mathsf{N}'_{circ1}\right) = \left(\mathsf{N}'_{circ1}\right), \quad \left(\mathsf{N}'_{circ1}\right)\mathsf{N}'_{star1}\mathsf{N}'_{star1} = \left(\mathsf{N}'_{circ1}\right), \tag{4.53}$$
$$\mathsf{N}'_{star1}\mathsf{N}'_{star1}\left(\mathsf{N}'_{circ2}\right) = \left(\mathsf{N}'_{circ2}\right), \quad \left(\mathsf{N}'_{circ2}\right)\mathsf{N}'_{star1}\mathsf{N}'_{star1} = \left(\mathsf{N}'_{circ2}\right), \tag{4.54}$$
$$\mathsf{N}'_{star2}\mathsf{N}'_{star2}\left(\mathsf{N}'_{circ1}\right) = \left(\mathsf{N}'_{circ1}\right), \quad \left(\mathsf{N}'_{circ1}\right)\mathsf{N}'_{star2}\mathsf{N}'_{star2} = \left(\mathsf{N}'_{circ1}\right), \tag{4.55}$$
$$\mathsf{N}'_{star2}\mathsf{N}'_{star2}\left(\mathsf{N}'_{circ2}\right) = \left(\mathsf{N}'_{circ2}\right), \quad \left(\mathsf{N}'_{circ2}\right)\mathsf{N}'_{star2}\mathsf{N}'_{star2} = \left(\mathsf{N}'_{circ2}\right), \tag{4.56}$$

3) from (4.53)–(4.56), the sets $\mathsf{N}'_{circ1,2}$ are right and left ternary modules over $\mathsf{N}'_{star1,2}$;

$$\mathsf{N}'_{circ1}\mathsf{N}'_{circ1}\left(\mathsf{N}'_{star1}\right) = \left(\mathsf{N}'_{star1}\right), \quad \left(\mathsf{N}'_{star1}\right)\mathsf{N}'_{circ1}\mathsf{N}'_{circ1} = \left(\mathsf{N}'_{star1}\right), \tag{4.57}$$
$$\mathsf{N}'_{circ1}\mathsf{N}'_{circ1}\left(\mathsf{N}'_{star2}\right) = \left(\mathsf{N}'_{star2}\right), \quad \left(\mathsf{N}'_{star2}\right)\mathsf{N}'_{circ1}\mathsf{N}'_{circ1} = \left(\mathsf{N}'_{star2}\right), \tag{4.58}$$
$$\mathsf{N}'_{circ2}\mathsf{N}'_{circ2}\left(\mathsf{N}'_{star1}\right) = \left(\mathsf{N}'_{star1}\right), \quad \left(\mathsf{N}'_{star1}\right)\mathsf{N}'_{circ2}\mathsf{N}'_{circ2} = \left(\mathsf{N}'_{star1}\right), \tag{4.59}$$
$$\mathsf{N}'_{circ2}\mathsf{N}'_{circ2}\left(\mathsf{N}'_{star2}\right) = \left(\mathsf{N}'_{star2}\right), \quad \left(\mathsf{N}'_{star2}\right)\mathsf{N}'_{circ2}\mathsf{N}'_{circ2} = \left(\mathsf{N}'_{star2}\right), \tag{4.60}$$

4) from (4.57)–(4.60), the sets $\mathsf{N}'_{star1,2}$ are right and left ternary modules over $\mathsf{N}'_{circ1,2}$.

4.4. *Group structure of the star and circle* 16*-vertex matrices*

Next we will introduce 8×8 matrices of a special form similar to the star 8-vertex matrices (2.81) and the circle 8-vertex matrices (2.82), analyze their group structure and establish which ones could be 16-vertex solutions to the ternary braid equations (4.3). We will derive the solutions in the opposite way to that for the 8-vertex Yang-Baxter maps, following the note after (2.37). The sum of the bisymm solutions (4.13) gives the shape of the 8×8 star matrix M'_{star} (as in (2.81)), while the sum of symm solutions (4.15) gives the 8×8 circle matrix M'_{circ} (as in (2.82))

$$M'_{star} = \begin{pmatrix} x & 0 & 0 & 0 & 0 & 0 & 0 & y \\ 0 & z & 0 & 0 & 0 & 0 & s & 0 \\ 0 & 0 & t & 0 & 0 & u & 0 & 0 \\ 0 & 0 & 0 & v & w & 0 & 0 & 0 \\ 0 & 0 & 0 & a & b & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & d & 0 & 0 \\ 0 & f & 0 & 0 & 0 & 0 & g & 0 \\ h & 0 & 0 & 0 & 0 & 0 & 0 & p \end{pmatrix}, \tag{4.61}$$

$$M'_{circ} = \begin{pmatrix} x & 0 & 0 & 0 & 0 & y & 0 & 0 \\ 0 & z & 0 & 0 & s & 0 & 0 & 0 \\ 0 & 0 & t & 0 & 0 & 0 & 0 & u \\ 0 & 0 & 0 & v & 0 & 0 & w & 0 \\ 0 & f & 0 & 0 & g & 0 & 0 & 0 \\ h & 0 & 0 & 0 & 0 & p & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 & b & 0 \\ 0 & 0 & c & 0 & 0 & 0 & 0 & d \end{pmatrix}, \tag{4.62}$$

$$\operatorname{tr} M' = x + z + t + v + b + d + g + p, \quad \det M' = (bv - aw)(cu - dt)(fs - gz)(px - hy), \tag{4.63}$$

$$\text{eigenvalues: } \frac{1}{2}\left(d + t - \sqrt{4cu + (d-t)^2}\right), \frac{1}{2}\left(d + t + \sqrt{4cu + (d-t)^2}\right),$$
$$\frac{1}{2}\left(b + v - \sqrt{4aw + (b-v)^2}\right), \frac{1}{2}\left(b + v + \sqrt{4aw + (b-v)^2}\right),$$
$$\frac{1}{2}\left(p + x - \sqrt{4hy + (p-x)^2}\right), \frac{1}{2}\left(p + x + \sqrt{4hy + (p-x)^2}\right),$$
$$\frac{1}{2}\left(g + z - \sqrt{4fs + (g-z)^2}\right), \frac{1}{2}\left(g + z + \sqrt{4fs + (g-z)^2}\right). \tag{4.64}$$

The 16-vertex matrices are invertible, if $\det M'_{star} \neq 0$ and $\det M'_{circ} \neq 0$, which give the following joint conditions on the parameters (cf. (2.85))

$$bv - aw \neq 0, \;\; cu - dt \neq 0, \;\; fs - gz \neq 0, \;\; px - hy \neq 0. \tag{4.65}$$

Only in this concrete parametrization (4.61) and (4.62) do the matrices M'_{star} and M'_{circ} have the same trace, determinant and eigenvalues, and they are diagonalizable and conjugate via

$$U' = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}. \tag{4.66}$$

The matrix U' cannot be presented in the form of a triple Kronecker product (4.10), and so two matrices M'_{star} and M'_{circ} are not q-conjugate in the parametrization (4.61) and (4.62), and can lead to different solutions to the ternary braid equations (4.3). It follows from (4.65) that 16-vertex matrices with all nonzero entries equal to 1 are non-invertible, having vanishing determinant and rank 4 (despite each one being a sum of two permutation matrices). In the case all the conditions (4.65) holding, the inverse matrices become

$$\left(M'_{star}\right)^{-1} = \begin{pmatrix} \frac{p}{px-hy} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{y}{px-hy} \\ 0 & -\frac{g}{fs-gz} & 0 & 0 & 0 & 0 & \frac{s}{fs-gz} & 0 \\ 0 & 0 & -\frac{d}{cu-dt} & 0 & 0 & \frac{u}{cu-dt} & 0 & 0 \\ 0 & 0 & 0 & \frac{b}{bv-aw} & -\frac{w}{bv-aw} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{a}{bv-aw} & \frac{v}{bv-aw} & 0 & 0 & 0 \\ 0 & 0 & \frac{c}{cu-dt} & 0 & 0 & -\frac{t}{cu-dt} & 0 & 0 \\ 0 & \frac{f}{fs-gz} & 0 & 0 & 0 & 0 & -\frac{z}{fs-gz} & 0 \\ -\frac{h}{px-hy} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{x}{px-hy} \end{pmatrix}, \tag{4.67}$$

$$\left(M'_{circ}\right)^{-1} = \begin{pmatrix} \frac{p}{px-hy} & 0 & 0 & 0 & 0 & -\frac{y}{px-hy} & 0 & 0 \\ 0 & -\frac{g}{fs-gz} & 0 & 0 & \frac{s}{fs-gz} & 0 & 0 & 0 \\ 0 & 0 & -\frac{d}{cu-dt} & 0 & 0 & 0 & 0 & \frac{u}{cu-dt} \\ 0 & 0 & 0 & \frac{b}{bv-aw} & 0 & 0 & -\frac{w}{bv-aw} & 0 \\ 0 & \frac{f}{fs-gz} & 0 & 0 & -\frac{z}{fs-gz} & 0 & 0 & 0 \\ -\frac{h}{px-hy} & 0 & 0 & 0 & 0 & \frac{x}{px-hy} & 0 & 0 \\ 0 & 0 & 0 & -\frac{a}{bv-aw} & 0 & 0 & \frac{v}{bv-aw} & 0 \\ 0 & 0 & \frac{c}{cu-dt} & 0 & 0 & 0 & 0 & -\frac{t}{cu-dt} \end{pmatrix}. \tag{4.68}$$

Denoting the sets of matrices corresponding to (4.61) and (4.62) by M'_{star} and M'_{circ}, their multiplications are

$$\mathsf{M}'_{star}\mathsf{M}'_{star} = \mathsf{M}'_{star}, \qquad \mathsf{M}'_{circ}\mathsf{M}'_{circ} = \mathsf{M}'_{circ}, \tag{4.69}$$

and in term of sets $\mathsf{M}'_{star} = \mathsf{N}'_{star1} \cup \mathsf{N}'_{star2}$ and $\mathsf{M}'_{circ} = \mathsf{N}'_{circ1} \cup \mathsf{N}'_{circ2}$, and $\mathsf{N}'_{star1} \cap \mathsf{N}'_{star2} = \mathsf{D}$ and $\mathsf{N}'_{circ1} \cap \mathsf{N}'_{circ2} = \mathsf{D}$ (see (4.36)). Note that the structure (4.69) is considerably different from the binary case (2.86)–(2.88), and therefore it may not necessarily be related to the Cartan decomposition.

The products (4.69) mean that both M'_{star} and M'_{circ} are separately closed with respect to binary matrix multiplication $(\cdot)$, and therefore $\mathcal{S}^{star}_{16vert} = \langle \mathsf{M}'_{star} \mid \cdot \rangle$ and $\mathcal{S}^{circ}_{16vert} = \langle \mathsf{M}'_{circ} \mid \cdot \rangle$ are semigroups. We denote their intersection by $\mathcal{S}^{\mathrm{diag}}_{8vert} = \mathcal{S}^{star}_{16vert} \cap \mathcal{S}^{circ}_{16vert}$ which is a semigroup of diagonal 8-vertex matrices. In case, the invertibility conditions (4.65) are fulfilled, the sets M'_{star} and M'_{circ} form subgroups $\mathcal{G}^{star}_{16vert} = \left\langle \mathsf{M}'_{star} \mid \cdot, (\_)^{-1}, I_8 \right\rangle$ and $\mathcal{G}^{circ}_{16vert} = \left\langle \mathsf{M}'_{circ} \mid \cdot, (\_)^{-1}, I_8 \right\rangle$ (where I_8 is the 8×8 identity matrix) of $\mathrm{GL}(8, \mathbb{C})$ with the inverse elements given explicitly by (4.67)–(4.68). Because the elements M'_{star} and M'_{circ} in (4.61) and (4.62) are conjugates by the invertible matrix U' (4.66), the subgroups $\mathcal{G}^{star}_{16vert}$ and $\mathcal{G}^{circ}_{16vert}$ (as well as the semigroups $\mathcal{S}^{star}_{16vert}$ and $\mathcal{S}^{circ}_{16vert}$) are isomorphic by the obvious isomorphism

$$M'_{star} \mapsto U' M'_{circ} U'^{-1}, \tag{4.70}$$

where U' is in (4.66).

The "interaction" between M'_{star} and M'_{circ} also differs from the binary case (2.87), because

$$\mathsf{M}'_{star}\mathsf{M}'_{circ} = \mathsf{M}'_{quad}, \quad \mathsf{M}'_{circ}\mathsf{M}'_{star} = \mathsf{M}'_{quad}, \tag{4.71}$$

$$\mathsf{M}'_{quad}\mathsf{M}'_{quad} = \mathsf{M}'_{quad}, \tag{4.72}$$

where M'_{quad} is a set of 32-vertex so called *quad-matrices* of the form

$$M'_{quad} = \begin{pmatrix} x_1 & 0 & y_1 & 0 & 0 & z_1 & 0 & s_1 \\ 0 & t_1 & 0 & u_1 & v_1 & 0 & w_1 & 0 \\ a_1 & 0 & b_1 & 0 & 0 & c_1 & 0 & d_1 \\ 0 & f_1 & 0 & g_1 & h_1 & 0 & p_1 & 0 \\ 0 & x_2 & 0 & y_2 & z_2 & 0 & s_2 & 0 \\ t_2 & 0 & u_2 & 0 & 0 & v_2 & 0 & w_2 \\ 0 & a_2 & 0 & b_2 & c_2 & 0 & d_2 & 0 \\ f_2 & 0 & g_2 & 0 & 0 & h_2 & 0 & p_2 \end{pmatrix}. \tag{4.73}$$

Because of (4.72), the set M'_{quad} is closed with respect to matrix multiplication, and therefore (for invertible matrices M'_{quad}) the group $\mathcal{G}^{quad}_{32vert} = \left\langle \mathsf{M}'_{quad} \mid \cdot, (\_)^{-1}, I_8 \right\rangle$ is a subgroup of $\mathrm{GL}(8,\mathbb{C})$. So, in trying to find higher 32-vertex solutions (having at most half as many unknown variables as a general 8×8 matrix) to the ternary braid equations (4.3) it is worthwhile to search within the class of quad-matrices (4.73).

Thus, the group structure of the above 16-vertex 8×8 matrices (4.69)–(4.72) is considerably different to that of 8-vertex 4×4 matrices (2.81)–(2.82)as the former contains two isomorphic binary subgroups $\mathcal{G}^{star}_{16vert}$ and $\mathcal{G}^{circ}_{16vert}$ of $\mathrm{GL}(8,\mathbb{C})$ (cf. (2.86)–(2.88) and (4.69)).

The sets M'_{star}, M'_{circ} and M'_{quad} are closed with respect to matrix addition as well, and therefore (because of the distributivity of $\mathbb{C}$) they are the matrix rings $\mathcal{R}^{star}_{16vert}$, $\mathcal{R}^{circ}_{16vert}$ and $\mathcal{R}^{quad}_{32vert}$, respectively. In the invertible case (4.65) and $\det M'_{quad} \neq 0$, these become matrix fields.

4.5. Pauli matrix presentation of the star and circle 16-vertex constant matrices

The main peculiarity of the 16-vertex 8×8 matrices (4.69)–(4.72) is the fact that they can be expressed as special tensor (Kronecker) products of the Pauli matrices (see, also, [18,27]). Indeed, let

$$\Sigma_{ijk} = \rho_i \otimes_K \rho_j \otimes_K \rho_k, \quad i,j,k = 1,2,3,4, \tag{4.74}$$

where ρ_i are Pauli matrices (we have already used the letter "σ" for the braid group generators (2.5))

$$\rho_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \rho_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \rho_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \rho_4 = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.75}$$

Among the total of 64 8×8 matrices Σ_{ijk} (4.74) there are 24 which generate the matrices M'_{star} (4.61) and M'_{circ} (4.62):

• 8 diagonal matrices:
$\Sigma_{diag} = \{\Sigma_{333}, \Sigma_{334}, \Sigma_{343}, \Sigma_{344}, \Sigma_{433}, \Sigma_{434}, \Sigma_{443}, \Sigma_{444}\}$;
• 8 anti-diagonal matrices:
$\Sigma_{adiag} = \{\Sigma_{111}, \Sigma_{112}, \Sigma_{121}, \Sigma_{122}, \Sigma_{211}, \Sigma_{212}, \Sigma_{221}, \Sigma_{222}\}$;
• 8 circle-like matrices (M'_{circ} with 0's on diagonal):
$\Sigma_{circ} = \{\Sigma_{131}, \Sigma_{132}, \Sigma_{141}, \Sigma_{142}, \Sigma_{231}, \Sigma_{232}, \Sigma_{241}, \Sigma_{242}\}$.

Thus, in general we have the following set structure for the star and circle 16-vertex matrices (4.61) and (4.62)

$$\mathsf{M}'_{star} = \Sigma_{diag} \cup \Sigma_{adiag}, \tag{4.76}$$
$$\mathsf{M}'_{circ} = \Sigma_{diag} \cup \Sigma_{circ}, \tag{4.77}$$
$$\mathsf{M}'_{star} \cap \mathsf{M}'_{circ} = \Sigma_{diag}. \tag{4.78}$$

In particular, for the 8-vertex permutation solutions (4.13)–(4.15) of the ternary braid equations (4.3) we have

$$\tilde{c}^{bisymm1,2}_{rank=8} = \frac{1}{2}(\Sigma_{111} + \Sigma_{444} \pm \Sigma_{212} \pm \Sigma_{343}), \tag{4.79}$$
$$\tilde{c}^{symm1,2}_{rank=8} = \frac{1}{2}(\Sigma_{141} + \Sigma_{444} \pm \Sigma_{232} \pm \Sigma_{333}). \tag{4.80}$$

The non-invertible 16-vertex solutionsM'_{star} (4.61) and M'_{circ} (4.62) having 1's on nonzero places are of $rank = 4$ and can be presented by (4.74) as follows

$$M'_{star}(1) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \Sigma_{111} + \Sigma_{444}, \tag{4.81}$$

$$M'_{circ}(1) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \Sigma_{141} + \Sigma_{444}. \tag{4.82}$$

Similarly, one can obtain the Pauli matrix presentation for the general star and circle 16-vertex matrices (4.61) and (4.62) which will contain linear combinations of the 16 parameters as coefficients before the Σ's.

4.6. *Invertible and non-invertible* 16*-vertex solutions to the ternary braid equations*

First, consider the 16-vertex solutions to (4.3) having the star matrix shape (4.61). We found the following 2 one-parameter invertible solutions

$$\tilde{c}^{16-vert,star}_{rank=8}(x) = \begin{pmatrix} x^3 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & x^3 & 0 & 0 & 0 & 0 & \mp x^2 & 0 \\ 0 & 0 & x^3 & 0 & 0 & -x^2 & 0 & 0 \\ 0 & 0 & 0 & x^3 & \mp x^4 & 0 & 0 & 0 \\ 0 & 0 & 0 & \pm x^2 & x^3 & 0 & 0 & 0 \\ 0 & 0 & x^4 & 0 & 0 & x^3 & 0 & 0 \\ 0 & \pm x^4 & 0 & 0 & 0 & 0 & x^3 & 0 \\ x^6 & 0 & 0 & 0 & 0 & 0 & 0 & x^3 \end{pmatrix}, \tag{4.83}$$

$$\begin{gathered} \operatorname{tr}\tilde{c} = 8x^3, \\ \det\tilde{c} = 16x^{24}, \quad x \neq 0, \\ \text{eigenvalues: } \left\{(1+i)x^3\right\}^{[4]}, \left\{(1-i)x^3\right\}^{[4]}. \end{gathered} \tag{4.84}$$

Both matrices in (4.83) are diagonalizable and are conjugates via

$$U_{star} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{4.85}$$

which cannot be presented in the form of a triple Kronecker product (4.10). Therefore, the two solutions in (4.83) are not q-conjugate and become different 16-vertex one-parameter invertible solutions of the braid equations (4.3).

In search of 16-vertex solutions to the total braid equations (4.3) of the circle matrix shape (4.62) we found that only non-invertible ones exist. They are the following two 2-parameter solutions of rank 4

$$\tilde{c}^{16-vert,circ}_{rank=4}(x,y) = \begin{pmatrix} \pm xy & 0 & 0 & 0 & 0 & y^2 & 0 & 0 \\ 0 & \pm xy & 0 & 0 & xy & 0 & 0 & 0 \\ 0 & 0 & \pm xy & 0 & 0 & 0 & 0 & y^2 \\ 0 & 0 & 0 & \pm xy & 0 & 0 & xy & 0 \\ 0 & xy & 0 & 0 & \pm xy & 0 & 0 & 0 \\ x^2 & 0 & 0 & 0 & 0 & \pm xy & 0 & 0 \\ 0 & 0 & 0 & xy & 0 & 0 & \pm xy & 0 \\ 0 & 0 & x^2 & 0 & 0 & 0 & 0 & \pm xy \end{pmatrix}, \tag{4.86}$$

$$\begin{gathered} \operatorname{tr}\tilde{c} = \pm 8xy, \\ \text{eigenvalues: } \{2xy\}^{[4]}, \{0\}^{[4]}. \end{gathered} \tag{4.87}$$

Two matrices in (4.86) are not even conjugates in the standard way, and so they are different 16-vertex two-parameter non-invertible solutions to the braid equations (4.3).

For the only partial 13-braid equation (4.5), there are 4 polynomial 16-vertex two-parameter invertible solutions

$$\tilde{c}^{16-vert,13circ}_{rank=8}(x,y) = \begin{pmatrix} x & 0 & 0 & 0 & 0 & y^2 & 0 & 0 \\ 0 & xy & 0 & 0 & x & 0 & 0 & 0 \\ 0 & 0 & x & 0 & 0 & 0 & 0 & \pm y^2 \\ 0 & 0 & 0 & xy & 0 & 0 & \pm x & 0 \\ 0 & x & 0 & 0 & xy & 0 & 0 & 0 \\ x^2 & 0 & 0 & 0 & 0 & x & 0 & 0 \\ 0 & 0 & 0 & \pm x & 0 & 0 & xy & 0 \\ 0 & 0 & \pm x^2 & 0 & 0 & 0 & 0 & x \end{pmatrix}, \tag{4.88}$$

$$\begin{pmatrix} x & 0 & 0 & 0 & 0 & y^2 & 0 & 0 \\ 0 & xy & 0 & 0 & -x & 0 & 0 & 0 \\ 0 & 0 & x & 0 & 0 & 0 & 0 & \pm y^2 \\ 0 & 0 & 0 & xy & 0 & 0 & \mp x & 0 \\ 0 & -x & 0 & 0 & xy & 0 & 0 & 0 \\ x^2 & 0 & 0 & 0 & 0 & x & 0 & 0 \\ 0 & 0 & 0 & \mp x & 0 & 0 & xy & 0 \\ 0 & 0 & \pm x^2 & 0 & 0 & 0 & 0 & x \end{pmatrix}, \tag{4.89}$$

$$\operatorname{tr}\tilde{c} = 4x(y+1), \det\tilde{c} = x^8\left(y^2-1\right)^4, \; x \neq 0, y \neq 1, \tag{4.90}$$

$$\text{eigenvalues:} \{x(y+1)\}^{[4]}, \{x(y-1)\}^{[2]}, \{-x(y-1)\}^{[2]}.$$

Also, for the partial 13-braid equation (4.5), we found 4 exotic irrational (an analog of (2.96) for the Yang-Baxter equation (2.12)) 16-vertex, two-parameter invertible solutions of rank 8 of the form

$$\tilde{c}^{16-vert,13circ,1}_{rank=8}(x,y) = \begin{pmatrix} x(2y-1) & 0 & 0 & 0 & 0 & y^2 & 0 & 0 \\ 0 & xy & 0 & 0 & x\sqrt{2(y-1)y+1} & 0 & 0 & 0 \\ 0 & 0 & x(2y-1) & 0 & 0 & 0 & 0 & \pm y^2 \\ 0 & 0 & 0 & xy & 0 & 0 & \pm x\sqrt{2(y-1)y+1} & 0 \\ 0 & x\sqrt{2(y-1)y+1} & 0 & 0 & xy & 0 & 0 & 0 \\ x^2 & 0 & 0 & 0 & 0 & x & 0 & 0 \\ 0 & 0 & 0 & \pm x\sqrt{2(y-1)y+1} & 0 & 0 & xy & 0 \\ 0 & 0 & \pm x^2 & 0 & 0 & 0 & 0 & x \end{pmatrix}, \tag{4.91}$$

and

$$\tilde{c}_{rank=8}^{16-vert,13circ,2}(x,y)$$
$$= \begin{pmatrix} x(2y-1) & 0 & 0 & 0 & 0 & y^2 & 0 & 0 \\ 0 & xy & 0 & 0 & -x\sqrt{2(y-1)y+1} & 0 & 0 & 0 \\ 0 & 0 & x(2y-1) & 0 & 0 & 0 & 0 & \pm y^2 \\ 0 & 0 & 0 & xy & 0 & 0 & \mp x\sqrt{2(y-1)y+1} & 0 \\ 0 & -x\sqrt{2(y-1)y+1} & 0 & 0 & xy & 0 & 0 & 0 \\ x^2 & 0 & 0 & 0 & 0 & x & 0 & 0 \\ 0 & 0 & 0 & \mp x\sqrt{2(y-1)y+1} & 0 & 0 & xy & 0 \\ 0 & 0 & \pm x^2 & 0 & 0 & 0 & 0 & x \end{pmatrix}, \tag{4.92}$$

$$\operatorname{tr}\tilde{c} = 8xy, \quad \det\tilde{c} = x^8(y-1)^8, \qquad x \neq 0, \quad y \neq 1, \tag{4.93}$$

$$\text{eigenvalues: } \left\{ x\left(y + \sqrt{2(y-1)y+1}\right) \right\}^{[4]}, \left\{ x\left(y - \sqrt{2(y-1)y+1}\right) \right\}^{[4]}. \tag{4.94}$$

The matrices in (4.88)–(4.92) are diagonalizable, have the same eigenvalues (4.94) and are pairwise conjugate by

$$U_{circ} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.95}$$

Because U_{circ} cannot be presented in the form (4.10), all solutions in (4.88)–(4.92) are not mutually q-conjugate and become 8 different 16-vertex two-parameter invertible solutions to the partial 13-braid equation (4.5). If $y = 1$, then the matrices (4.88)–(4.92) become of rank 4 with vanishing determinants (4.90), (4.93), and therefore in this case they are a 16-vertex one-parameter circle of non-invertible solutions to the total braid equations (4.3).

Further families of solutions could be constructed using additional parameters: the scaling parameter t in (4.8) and the complex elements of the matrix q (4.9).

4.7. Higher 2^n-vertex constant solutions to n-ary braid equations

Next we considered the 4-ary constant braid equations (3.4)–(3.6) and found the following 32-vertex star solution

$$\tilde{c}_{16} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.96}$$

We may compare (4.96) with particular cases of the star solutions to the Yang-Baxter equation (2.92) and the ternary braid equation (4.83)

$$\tilde{c}_4 = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \quad \tilde{c}_8 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.97}$$

Informally we call such solutions the "Minkowski" star solutions, since their legs have the "Minkowski signature". Thus, we assume that in the general case for the n-ary braid equation there exist 2^{n+1}-vertex $2^n \times 2^n$ matrix "Minkowski" star invertible solutions of the above form

$$\tilde{c}_{2^n} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & \ddots & 0 & 0 & \iddots & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & \iddots & 0 & 0 & \ddots & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.98}$$

This allows us to use the general solution (4.98) as n-ary braiding quantum gates with an arbitrary number of qubits.

5. Invertible and noninvertible quantum gates

Informally, quantum computing consists of preparation (setting up an initial quantum state), evolution (by a quantum circuit) and measurement (projection onto the final state). Mathematically (in the computational basis) the initial state is a vector in a Hilbert space (multi-qubit state), the evolution is governed by successive (quantum circuit) invertible linear transformations (unitary matrices called quantum gates) and the measurement is made by non-invertible projection matrices to leave only one final quantum (multi-qubit) state. So, quantum computing is non-invertible overall, and we may consider non-invertible transformations at each step. It was then realized that one can "invite" the Yang-Baxter operators (solutions of the constant Yang-Baxter equation) into quantum gates , providing a means of entangling otherwise non-entangled states. This insight uncovered a deep connection between quantum and topological computation (see for details, e.g. [9,13]).

Here we propose extending the above picture in two directions. First, we can treat higher braided operators as higher braiding gates. Second, we will analyze the possible role of non-invertible linear transformations (described by the partial unitary matrices introduced in (2.20)–(2.21)), which can be interpreted as a property of some hypothetical quantum circuit (for instance, with specific "loss" of information, some kind of "dissipativity" or "vagueness"). This can be considered as an intermediate case between standard unitary computing and the measurement only computing of [51].

To establish notation recall [1], that in the computational basis (vector representation) and Dirac notation, a (pure) one-qubit state is described by a vector in two-dimensional Hilbert space $V = \mathbb{C}^2$

$$|\psi\rangle \equiv \left|\psi^{(1)}\right\rangle = a_0|0\rangle + a_1|1\rangle, \ |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{5.1}$$

$$|a_0|^2 + |a_1|^2 = 1, \quad a_i \in \mathbb{C}, \ , i = 1, 2, \tag{5.2}$$

where a_i is a probability amplitude of $|i\rangle$. Sometimes, for a one-qubit state it is convenient to use the Bloch unit sphere representation (normalized up to a general unimportant and unmeasurable phase)

$$|\psi(\theta,\gamma)\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\gamma}\sin\frac{\theta}{2}|1\rangle, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \gamma \leq 2\pi. \tag{5.3}$$

A (pure) state of L-qubits $\left|\psi^{(L)}\right\rangle$ is described by 2^L amplitudes, and so is a vector in 2^L-dimensional Hilbert space. If $\left|\psi^{(L)}\right\rangle$ cannot be presented as a tensor product of L one-qubit states (5.1), it is called *entangled*. For instance, a two-qubit pure state

$$\left|\psi^{(2)}\right\rangle = a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle, \tag{5.4}$$

$$|a_{00}|^2 + |a_{01}|^2 + |a_{10}|^2 + |a_{11}|^2 = 1, \quad a_{ij} \in \mathbb{C}, \quad ,i,j = 1,2, \tag{5.5}$$

is entangled, if $\det(a_{ij}) \neq 0$, and the *concurrence*

$$C^{(2)} \equiv C^{(2)}\left(\left|\psi^{(2)}\right\rangle\right) = 2\left|\det(a_{ij})\right| \tag{5.6}$$

is the measure of entanglement $0 \leq C^{(2)} \leq 1$. It follows from (5.1), that the tensor product of states has vanishing concurrence $C^{(2)}(|\psi_1\rangle \otimes |\psi_2\rangle) = 0$. An example of the maximally entangled ($C^{(2)} = 1$) two-qubit states is the (first) Bell state $\left|\psi^{(2)}\right\rangle_{Bell} = (|00\rangle + |11\rangle)/\sqrt{2}$.

The concurrence of the three-qubit state

$$\left|\psi^{(3)}\right\rangle = \sum_{i,j,k=0}^{1} a_{ijk}|ijk\rangle, \quad \sum_{i,j,k=0}^{1} \left|a_{ijk}\right|^2 = 1, \; a_{ijk} \in \mathbb{C}, \tag{5.7}$$

is determined by the Cayley's $2 \times 2 \times 2$ hyperdeterminant

$$C^{(3)} = 4\left|\mathrm{h}\det\left(a_{ijk}\right)\right|, \quad 0 \leq C^{(3)} \leq 1, \tag{5.8}$$

$$\begin{aligned} \mathrm{h}\det\left(a_{ijk}\right) &= a_{000}^2 a_{111}^2 + a_{001}^2 a_{110}^2 + a_{010}^2 a_{101}^2 + a_{100}^2 a_{011}^2 - 2a_{000}a_{001}a_{110}a_{111} \\ &- 2a_{000}a_{010}a_{101}a_{111} - 2a_{000}a_{011}a_{100}a_{111} - 2a_{001}a_{010}a_{101}a_{111} - 2a_{001}a_{011}a_{100}a_{110} \\ &- 2a_{010}a_{011}a_{100}a_{101} + 4a_{000}a_{011}a_{101}a_{110} + 4a_{001}a_{010}a_{100}a_{111}. \end{aligned} \tag{5.9}$$

Thus, if the three-qubit state (5.7) is not entangled, then $C^{(3)} = 0$ (for the tensor product of one-qubit states). One of the maximally entangled ($C^{(3)} = 1$) three-qubit states is the GHZ state $\left|\psi^{(3)}\right\rangle_{GHZ} = (|000\rangle + |111\rangle)/\sqrt{2}$.

A quantum L-qubit gate is a linear transformation of 2^L-dimensional Hilbert space $\left(\mathbb{C}^2\right)^{\otimes L} \to \left(\mathbb{C}^2\right)^{\otimes L}$ which in the computational basis (5.1) is described of the $2^L \times 2^L$ matrix $U^{(L)}$ such that the L-qubit state transforms as $\left|\psi'^{(L)}\right\rangle = U^{(L)}\left|\psi^{(L)}\right\rangle$. In this way, a *quantum circuit* is described as the successive application of elementary gates to an initial quantum state, that is the product of the corresponding matrices (for details, see, e.g., [1]). It is a standard assumption that each elementary L-qubit transformation is *unitary*, which implies the following strong restriction on the corresponding matrix $U \equiv U^{(L)}$ as

$$U^\star U = UU^\star = I \equiv I_{2^L \times 2^L}, \tag{5.10}$$

where I is the $2^L \times 2^L$ identity matrix for L-qubit state and the operation $(\star)$ is the conjugate-transposition. The first equality in (5.10) means that the matrix $U^{(L)}$ is *normal* (cf. (2.20)–(2.21)). The equations (5.10) follow from the definition of the *adjoint operator*

$$\left\langle U\psi^{(L)} \mid I\varphi^{(L)} \right\rangle = \left\langle I\psi^{(L)} \mid U^\star\varphi^{(L)} \right\rangle \tag{5.11}$$

applied to this simplest example of L-qubits in the 2^L-dimensional Hilbert space $\left(\mathbb{C}^2\right)^{\otimes L}$ (for the general case the derivation almost literally coincides), which we write in the following special form (in Dirac notation with bra- and ket- vectors) with explicitly added identities. Then (5.10) follows from (5.11) as

$$\left\langle U^\star U\psi^{(L)} \mid I\varphi^{(L)} \right\rangle = \left\langle I\psi^{(L)} \mid UU^\star\varphi^{(L)} \right\rangle = \left\langle I\psi^{(L)} \mid I\varphi^{(L)} \right\rangle, \tag{5.12}$$

and any unitary matrix preserves the inner product

$$\left\langle U\psi^{(L)} \mid U\varphi^{(L)} \right\rangle = \left\langle I\psi^{(L)} \mid I\varphi^{(L)} \right\rangle, \tag{5.13}$$

which means that unitary operators satisfying (5.10) are bounded operators (bounded matrices in our case) and invertible with the inverse $U^{-1} = U^\star$.

Let us consider a possibility of non-invertible intermediate transformations of L-qubit states, i.e. *non-invertible gates* which are described by the $2^L \times 2^L$ matrices $U(r)$ of (possibly) less than full rank $1 \le r \le 2^L$. This can be related to the production of "degenerate" states (see, e.g. [42]), "particle loss" [52–54], and the role of ranks in multiparticle entanglement [55,56].

In the limited cases $U\left(r = 2^L\right) \equiv U = U^{(L)}$, and $U(1)$ corresponds to the measurement matrix being the projection to one final vector $\left| i_{final} \right\rangle$. In this case, for non-invertible transformations with $r < 2^L$ instead of unitarity (5.10) we consider partial unitarity (2.20)–(2.21) as

$$U(r)^\star U(r) = I_1(r), \tag{5.14}$$

$$U(r)U(r)^\star = I_2(r), \tag{5.15}$$

where $I_1(r)$ and $I_2(r)$ are (or may be) different partial shuffle identities having r units on the diagonal. There is an exotic limiting case, which is impossible for the identity I: we call two partial identities *orthogonal*, if

$$I_1(r)I_2(r) = Z, \tag{5.16}$$

where $Z = Z_{2^L \times 2^L}$ is the zero $2^L \times 2^L$ matrix.

We propose corresponding non-invertible analogs of (5.11)–(5.13) as follows. The *partial adjoint operator* $U(r)^\star$ in the 2^L-dimensional Hilbert space $\left(\mathbb{C}^2\right)^{\otimes L}$ is defined by

$$\left\langle U(r)\psi^{(L)} \mid I_2(r)\varphi^{(L)} \right\rangle = \left\langle I_1(r)\psi^{(L)} \mid U(r)^\star\varphi^{(L)} \right\rangle, \tag{5.17}$$

such that (see (5.14)–(5.15))

$$\left\langle U(r)^\star U(r)\psi^{(L)} \mid I_2(r)\varphi^{(L)} \right\rangle = \left\langle I_1(r)\psi^{(L)} \mid U(r)U(r)^\star\varphi^{(L)} \right\rangle = \left\langle I_1(r)\psi^{(L)} \mid I_2(r)\varphi^{(L)} \right\rangle. \tag{5.18}$$

We call the r.h.s. of (5.18) the *partial inner product*. So instead of (5.13) we define $U(r)$ as the *partially bounded operator*

$$\left\langle U(r)\psi^{(L)} \mid U(r)\varphi^{(L)} \right\rangle = \left\langle I_1(r)\psi^{(L)} \mid I_2(r)\varphi^{(L)} \right\rangle. \tag{5.19}$$

Thus, if the partial identities are orthogonal (5.16), then the partial inner product vanishes identically, and the operator $U(r)$ becomes a zero norm operator in the sense of (5.19), although (5.14)–(5.15) are not zero.

In case the rank r is fixed, there can be $\left(2^L!/r!(2^L-r)!\right)^2$ partial unitary matrices $U(r)$ satisfying (5.14)–(5.15).

We define a *general unitary semigroup* as a semigroup of matrices $U(r)$ of rank r satisfying partial regularity (5.14)–(5.15) (in the "symmetric" case $I_1(r) = I_2(r) \equiv I(r)$).

As an example, we consider two 2-qubit states (5.5) $\left|\psi^{(2)}\right\rangle$ and $\left|\varphi^{(2)}\right\rangle$ (with a'_{ij} and $|i'j'\rangle$) and the non-invertible transformation described by three-parameter 4×4 matrices of rank 3 (but which are not nilpotent)

$$U(3) = U^{(L=2)}(r=3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & e^{i\beta} & 0 & 0 \\ 0 & 0 & 0 & e^{i\gamma} \\ e^{i\alpha} & 0 & 0 & 0 \end{pmatrix}, \quad \alpha,\beta,\gamma \in \mathbb{R}. \tag{5.20}$$

The partial unitarity (5.14)–(5.15) and partial identities now become

$$U(3)^\star U(3) = I_1(3) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{5.21}$$

$$U(3)U(3)^\star = I_2(3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5.22}$$

The partial identities (5.21)–(5.22) are not orthogonal (5.16), because

$$I_1(3)I_2(3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \neq Z, \tag{5.23}$$

which directly gives the signature of the partial inner product (5.18), in our case of the Hilbert space $\left(\mathbb{C}^2\right)^{\otimes 2}$.

The definition of a partial adjoint operator (5.17) is satisfied with both sides being equal to $a_{00}a'_{11}e^{i\alpha}\langle 00 \mid 1'1'\rangle + a_{01}a'_{01}e^{i\beta}\langle 01 \mid 0'1'\rangle + a_{11}a'_{10}e^{i\gamma}\langle 11 \mid 1'0'\rangle$. The partial boundedness condition (5.19) holds with the partial inner product (5.18) becoming $a_{01}a'_{01}\langle 01 \mid 0'1'\rangle + a_{11}a'_{11}\langle 11 \mid 1'1'\rangle$, thus $U(3)$ (5.20) which is a bounded partial unitary operator.

An example of a zero norm (in our sense (5.19)) operator is the two-parameter partial unitary rank 2 matrix

$$U_{nil}(2) = U^{(L=2)}(r=2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{i\beta} \\ e^{i\alpha} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \alpha,\beta \in \mathbb{R}. \tag{5.24}$$

The partial unitarity relations for $U_{nil}(2)$ have the form

$$U_{nil}(2)^{\star}U_{nil}(2) = I_{nil,1}(2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{5.25}$$

$$U_{nil}(2)U_{nil}(2)^{\star} = I_{nil,2}(2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.26}$$

It is seen that the partial identities $I_{nil,1}(2)$ and $I_{nil,2}(2)$ are orthogonal (5.16), and the partial inner product (5.18) vanishes identically, and also the boundedness condition (5.19) holds with the r.h.s. vanishing, despite $U_{nil}(2)$ is a nonzero nilpotent matrix (5.24).

6. Binary braiding quantum gates

Let us consider those Yang-Baxter maps which could be linear transformations of two-qubit spaces. We will pay attention to the most general 8-vertex solutions to the Yang-Baxter equations (2.92)–(2.102) and (2.105)–(2.107) which are unitary (and invertible) or partial unitary (2.20)–(2.21) (and non-invertible).

Consider the unitary version of the invertible star 8-vertex solutions (2.92)–(2.96) to the matrix Yang-Baxter equation (2.12). We use the exponential form of the parameters

$$x = r_x e^{i\alpha}, \quad y = r_y e^{i\beta}, \quad z = r_z e^{i\gamma}, \quad r_{x,y,z}, \alpha, \beta, \gamma \in \mathbb{R}, \quad r_{x,y,z} \geq 0, \quad |\alpha|, |\beta|, |\gamma| \leq 2\pi. \tag{6.1}$$

For (2.92), exploiting unitarity (5.10) we obtain

$$U^{8-vert,star}_{rank=4}(\alpha,\beta) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i(\alpha+\beta)} & 0 & 0 & e^{2i\beta} \\ 0 & e^{i(\alpha+\beta)} & \pm e^{i(\alpha+\beta)} & 0 \\ 0 & \mp e^{i(\alpha+\beta)} & e^{i(\alpha+\beta)} & 0 \\ -e^{2i\alpha} & 0 & 0 & e^{i(\alpha+\beta)} \end{pmatrix}, \quad \begin{array}{c} \operatorname{tr} U = 2\sqrt{2}e^{i(\alpha+\beta)}, \\ \det U = e^{4i(\alpha+\beta)}, \end{array} \tag{6.2}$$

$$\text{eigenvalues: } \left\{ -(-1)^{3/4} e^{i(\alpha+\beta)} \right\}^{[2]}, \left\{ (-1)^{1/4} e^{i(\alpha+\beta)} \right\}^{[2]}. \tag{6.3}$$

With the choice of parameters $\alpha = \beta = 0$ and lower signs, the solution (6.2) coincides with the 8-vertex braiding gate of [13].

Next we search for unitary solutions among the invertible circle of 8-vertex traceless solutions (2.105) to the matrix Yang-Baxter equation (2.12) with parameters in the exponential form (6.1). The unitarity conditions (5.10) give the following equations on the parameters (6.1)

$$r = r_y = r_z, \quad r^2 \left(r_x^2 + r^2 \right) = 1, \quad r^8 + r^6 - 2r^4 + 1 = r^2 \tag{6.4}$$

$$\alpha - \beta = \frac{\pi}{2}. \tag{6.5}$$

The system of equations (6.4) has two real positive (or zero) solutions

$$1) \quad r_x = 1, \quad r = \sqrt{\frac{\sqrt{5}-1}{2}}, \tag{6.6}$$

$$2) \quad r_x = 0, \quad r = 1. \tag{6.7}$$

Thus, only the first solution leads to an 8-vertex two-parameter unitary braiding quantum gate of the form (we put $\gamma \mapsto \beta$ in (6.1))

$$U_{rank=4}^{8-vert,circ}(\alpha,\beta) = \sqrt{\frac{\sqrt{5}-1}{2}} \begin{pmatrix} 0 & e^{i(\alpha+\beta)} & ie^{i(\alpha+\beta)}\sqrt{\frac{\sqrt{5}-1}{2}} & 0 \\ -e^{2i\alpha}\sqrt{\frac{\sqrt{5}-1}{2}} & 0 & 0 & e^{i(\alpha+\beta)} \\ ie^{2i\alpha} & 0 & 0 & ie^{i(\alpha+\beta)}\sqrt{\frac{\sqrt{5}-1}{2}} \\ 0 & -e^{2i\alpha}\sqrt{\frac{\sqrt{5}-1}{2}} & ie^{2i\alpha} & 0 \end{pmatrix}, \tag{6.8}$$

$$\det U = e^{2i(3\alpha+\beta)}. \tag{6.9}$$

The second solution (6.7) gives 4-vertex two-parameter unitary braiding quantum gate (we also put $\gamma \mapsto \beta$ in (6.1))

$$U_{rank=4}^{4-vert,circ}(\alpha,\beta) = \begin{pmatrix} 0 & 0 & e^{i(\alpha+\beta)} & 0 \\ e^{2i\alpha} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{i(\alpha+\beta)} \\ 0 & e^{2i\alpha} & 0 & 0 \end{pmatrix}, \quad \det U = -e^{2i(3\alpha+\beta)}. \tag{6.10}$$

The non-invertible 8-vertex circle solution (2.107) to the Yang-Baxter equation (2.12) cannot be partial unitary (5.14)–(5.15) with any values of its parameters.

7. Higher braiding quantum gates

In general, only special linear transformations of 2^L-dimensional Hilbert space can be treated as elementary quantum gates for an L-qubit state [1]. First, in the invertible case, the transformations should be unitary (5.10), and in the hypothetical non-invertible case they can satisfy partial unitarity (5.14)–(5.15). Second, the braiding gates have to be $2^L \times 2^L$ matrix solutions to the constant Yang-Baxter equation [13] or higher braid equations (3.4)–(3.6). Here we consider (as a lowest case higher example) the ternary braiding gates acting on 3-qubit quantum states, i.e. 8×8 matrix solutions to the ternary braid equations (4.3) which satisfy unitarity (5.10) or partial unitarity (5.14)–(5.15).

Note that all the permutation solutions (4.13)–(4.15) are by definition unitary, and are therefore ternary braiding gates "automatically", and we call them *permutation 8-vertex ternary braiding quantum gates* $U_{perm}^{8-vertex}$. By the same reasoning the unitary version of the invertible star 8-vertex parameter-permutation solutions (4.17)–(4.31) to the ternary braid equations (4.3) will contain the complex numbers of unit magnitude as parameters.

Indeed, for the bisymmetric series (4.17)–(4.19) of star-like solutions we have 4 two real parameter unitary ternary braiding quantum gates ($\varkappa = \pm 1$)

$$U_{bisymm1}^{8-vertex}(\alpha,\beta) = \begin{pmatrix} e^{i(\alpha+\beta)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \varkappa e^{2i\beta} & 0 \\ 0 & 0 & e^{i(\alpha+\beta)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \varkappa e^{2i\alpha} & 0 & 0 & 0 \\ 0 & 0 & 0 & \pm\varkappa e^{2i\beta} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{i(\alpha+\beta)} & 0 & 0 \\ 0 & \pm\varkappa e^{2i\alpha} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{i(\alpha+\beta)} \end{pmatrix}, \tag{7.1}$$

$$\alpha,\beta \in \mathbb{R}, \quad |\alpha|,|\beta| \leq 2\pi, \tag{7.2}$$

which is a ternary analog of the first parameter-permutation solution to the Yang-Baxter equation from (2.35). The ternary analog of the second star solution is the following unitary version of the bisymmetric series (4.21)–(4.23)

$$U^{8-vertex}_{bisymm2}(\alpha,\beta) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{6i\alpha} \\ 0 & \varkappa e^{3i(\alpha+\beta)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{2i(2\alpha+\beta)} & 0 & 0 \\ 0 & 0 & 0 & \varkappa e^{3i(\alpha+\beta)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \varkappa e^{3i(\alpha+\beta)} & 0 & 0 & 0 \\ 0 & 0 & \pm e^{2i(\alpha+2\beta)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \varkappa e^{3i(\alpha+\beta)} & 0 \\ \pm e^{6i\beta} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{7.3}$$

The same unitary ternary analogs of the symmetric series (4.25)–(4.31) for the first and the second circle-like solutions from (2.37) are

$$U^{8-vertex}_{symm1}(\alpha,\beta) = \begin{pmatrix} e^{i(\alpha+\beta)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \varkappa e^{i(\alpha+\beta)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{2i\beta} \\ 0 & 0 & 0 & e^{i(\alpha+\beta)} & 0 & 0 & 0 & 0 \\ 0 & \pm\varkappa e^{i(\alpha+\beta)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{i(\alpha+\beta)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{i(\alpha+\beta)} & 0 \\ 0 & 0 & \pm e^{2i\alpha} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{7.4}$$

and

$$U^{8-vertex}_{symm2}(\alpha,\beta) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & e^{2i\beta} & 0 & 0 \\ 0 & e^{i(\alpha+\beta)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{i(\alpha+\beta)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \varkappa e^{i(\alpha+\beta)} & 0 \\ 0 & 0 & 0 & 0 & e^{i(\alpha+\beta)} & 0 & 0 & 0 \\ \pm e^{2i\alpha} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \pm\varkappa e^{i(\alpha+\beta)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{i(\alpha+\beta)} \end{pmatrix}, \tag{7.5}$$

respectively.

The invertible 16-vertex star-like solutions (4.83) to the ternary braid equations (4.3) lead to the following two unitary one-parameter ternary braiding quantum gates (cf. the binary case (6.2))

$$U^{16-vertex}_{3-qubits\pm}(\alpha) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{3i\alpha} & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & e^{3i\alpha} & 0 & 0 & 0 & 0 & \mp e^{2i\alpha} & 0 \\ 0 & 0 & e^{3i\alpha} & 0 & 0 & -e^{2i\alpha} & 0 & 0 \\ 0 & 0 & 0 & e^{3i\alpha} & \mp e^{4i\alpha} & 0 & 0 & 0 \\ 0 & 0 & 0 & \pm e^{2i\alpha} & e^{3i\alpha} & 0 & 0 & 0 \\ 0 & 0 & e^{4i\alpha} & 0 & 0 & e^{3i\alpha} & 0 & 0 \\ 0 & \pm e^{4i\alpha} & 0 & 0 & 0 & 0 & e^{3i\alpha} & 0 \\ e^{6i\alpha} & 0 & 0 & 0 & 0 & 0 & 0 & e^{3i\alpha} \end{pmatrix}. \tag{7.6}$$

The braiding gate (7.6) is a ternary analog of (6.2), and therefore with $\alpha = 0$ it can be treated as a ternary analog of the 8-vertex braiding gate considered in [13]. Note that the solution $U^{16-vertex}_{3-qubits+}(0)$ is transpose to the so-called generalized Bell matrix [23]. Comparing (4.61) and (7.6), we observe that the ternary braiding quantum gates (acting on 3 qubits) are those elements of the 16-vertex star semigroup $\mathcal{G}^{star}_{16vert}$ (4.69), which satisfy unitarity (5.10).

In the same way, the 32-vertex analog the 8-vertex binary braiding gate of [13] (now acting on 4 qubits) is the following constant 4-ary braiding unitary quantum gate

$$U^{32-vertex}_{4-qubits} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{7.7}$$

Thus, in general, the "Minkowski" star solutions for n-ary braid equations correspond to 2^n-vertex braiding unitary quantum gates as $2^L \times 2^L$ matrices acting on $L = n$ qubits

$$U_{L-qubits}^{2^L-vertex} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & \ddots & 0 & 0 & \iddots & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & \iddots & 0 & 0 & \ddots & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{7.8}$$

The braiding gate (7.8) can be treated as a polyadic (n-ary) generalization of the GHZ generator (see, e.g., [18,23]) acting on $L = n$ qubits.

8. Entangling braiding gates

Entangled quantum states are obtained from separable states by acting with special quantum gates on two-qubit states and multi-qubit states [41,42]. Here we consider the concrete form of braiding gates which can be entangling or not entangling. There are general considerations on these subjects for the Yang-Baxter maps [13,37,39] and generalized Yang-Baxter maps [23,25,26,50]. We present the solutions for the binary and ternary braid maps introduced above, which connect the parameters of the gate and the state.

8.1. Entangling binary braiding gates

Let us first examine, how the 8-vertex star binary braiding gate $U_s(\alpha, \beta) \equiv U_{rank=4}^{8-vert,star}(\alpha, \beta)$ (6.2) acts on the product of one-qubit states concretely. We use the Bloch representation (5.3) to obtain the expression for the transformed concurrence (5.6)

$$\begin{aligned} &C_{s\pm}^{(2)}(U_s(\alpha,\beta)|\psi(\theta_1,\gamma_1)\rangle \otimes |\psi(\theta_2,\gamma_2)\rangle) \qquad (8.1)\\ &= \left| \left(e^{i(\beta+2\gamma_1)} \sin^2 \frac{\theta_1}{2} \pm e^{i\alpha} \cos^2 \frac{\theta_1}{2} \right) \left(e^{i(\beta+2\gamma_2)} \sin^2 \frac{\theta_2}{2} \mp e^{i\alpha} \cos^2 \frac{\theta_2}{2} \right) \right|. \end{aligned}$$

In general, a braiding gate is *entangling* if the transformed concurrence (8.1) does not vanish, and its roots give the values of the gate parameters $U(\alpha, \beta)$ for which the gate is *not entangling* for a given two-qubit state. In search of the solutions for the transformed concurrence $C_{s\pm}^{(2)} = 0$, we observe that one of the qubits has to be on the Bloch sphere equator $\theta_1 = \frac{\pi}{2}$ (or $\theta_2 = \frac{\pi}{2}$). Only in this case can the first (or second) bracket in (8.1) vanish, and we obtain

$$1)\ C_{s+}^{(2)} = 0, \text{ if } \theta_1 = \frac{\pi}{2} \text{ and } \alpha - \beta = 2\gamma_1 - \pi, \text{ or } \theta_2 = \frac{\pi}{2} \text{ and } \alpha - \beta = 2\gamma_2; \tag{8.2}$$

$$2)\ C_{s-}^{(2)} = 0, \text{ if } \theta_1 = \frac{\pi}{2} \text{ and } \alpha - \beta = 2\gamma_1, \text{ or } \theta_2 = \frac{\pi}{2} \text{ and } \alpha - \beta = 2\gamma_2 - \pi. \tag{8.3}$$

Therefore the 8-vertex star binary braiding gates (6.2) with the parameters fixed by (8.2)–(8.3) are not entangling.

For the 8-vertex circle binary braiding gate $U_c(\alpha,\beta) \equiv U^{8-vert,circ}_{rank=4}(\alpha,\beta)$ (6.8) we obtain

$$C_c^{(2)}(U_c(\alpha,\beta)|\psi(\theta_1,\gamma_1)\rangle \otimes |\psi(\theta_1,\gamma_1)\rangle) \tag{8.4}$$
$$= W\left|\left(e^{i(\beta+2\gamma_1)}\sin^2\frac{\theta_1}{2} - ie^{i\alpha}\cos^2\frac{\theta_1}{2}\right)\left(e^{i(\beta+2\gamma_2)}\sin^2\frac{\theta_2}{2} - ie^{i\alpha}\cos^2\frac{\theta_2}{2}\right)\right|,$$

$$W = \frac{\left(\sqrt{5}-1\right)^{\frac{3}{2}}}{\sqrt{2}} = 0.971\,74. \tag{8.5}$$

Analogously to (8.2)–(8.3), the concurrence of the states transformed by the 8-vertex circle binary braiding gate (6.8) can vanish if

$$C_c^{(2)} = 0, \text{ if } \theta_1 = \frac{\pi}{2} \text{ and } \alpha-\beta = 2\gamma_1 - \frac{\pi}{2}, \text{ or } \theta_2 = \frac{\pi}{2} \text{ and } \alpha-\beta = 2\gamma_2 - \frac{\pi}{2}. \tag{8.6}$$

Thus, the 8-vertex circle binary braiding gates (6.8) are not entangling if the parameters satisfy (8.6).

In the case of the 4-vertex circle binary braiding gate (6.10) the transformed concurrence vanishes identically, and therefore this gate is not entangling for any values of its parameters.

8.2. Entangling ternary braiding gates

Let us consider the tensor product of three qubit pure states $|\psi(\theta_1,\gamma_1)\rangle \otimes |\psi(\theta_2,\gamma_2)\rangle \otimes |\psi(\theta_3,\gamma_3)\rangle$ (in the Bloch representation (5.3)), which obviously has zero concurrence $C^{(3)}$ (5.8), because of the vanishing of the hyperdeterminant (5.9). After transforming by the 16-vertex star ternary braiding gates $U_{16}(\alpha) \equiv U^{16-vertex}_{3-qubits}(\alpha)$ (7.6) the concurrence becomes

$$C^{(3)}_{16\pm}(U_{16}(\alpha)|\psi(\theta_1,\gamma_1)\rangle \otimes |\psi(\theta_2,\gamma_2)\rangle \otimes |\psi(\theta_3,\gamma_3)\rangle)$$
$$= \frac{1}{64}\left|\left(e^{2i\alpha} \pm e^{2i\gamma_1} + (e^{2i\alpha} \mp e^{2i\gamma_1})\cos\theta_1\right)^2\left(e^{2i\alpha} - e^{2i\gamma_2} + (e^{2i\alpha} + e^{2i\gamma_2})\cos\theta_2\right)^2\right.$$
$$\left.\times\left(e^{2i\alpha} \mp e^{2i\gamma_3} + (e^{2i\alpha} \pm e^{2i\gamma_3})\cos\theta_3\right)^2\right|. \tag{8.7}$$

We observe that the ternary concurrence (8.7) vanishes if any of the brackets is equal to zero. Because the domain of all angles is $\mathbb{R}$, we have solutions only for fixed discrete $\theta_k = \pi, -\pi, \pi/2, k = 1,2,3$, which means that on the Bloch sphere the quantum states should be on the equator (as in the binary case), or additionally at the poles. In this case, $e^{i\alpha} = \pm e^{i\gamma_k}$, and

$$\alpha = \begin{cases} \gamma_k \\ \gamma_k + \pi \end{cases}, \quad k = 1,2,3. \tag{8.8}$$

Thus, for a fixed three-qubit product state one (or more) of which is at a pole or the equator of the Bloch sphere, those ternary braiding gates $U_{16}(\alpha)$ satisfying the conditions (8.8) are not entangling $C^{(3)}_{16\pm} = 0$, whereas in other cases they are entangling $C^{(3)}_{16\pm} \neq 0$.

By analogy, a similar action of the 8-vertex bisymmetric (star-like) ternary braiding gates $U_{8b1,2}(\alpha,\beta) \equiv U^{8-vertex}_{bisymm1,2}(\alpha,\beta)$ (7.1)–(7.3) gives

$$C^{(3)}_{8b1}(U_{8b1}(\alpha,\beta)|\psi(\theta_1,\gamma_1)\rangle \otimes |\psi(\theta_2,\gamma_2)\rangle \otimes |\psi(\theta_3,\gamma_3)\rangle) = \left|\sin^2\theta_1 \sin^2\theta_3 \left(e^{2i(\beta+\gamma_2)} \sin^2\frac{\theta_2}{2} - e^{2i\alpha}\cos^2\frac{\theta_2}{2}\right)^2\right|, \tag{8.9}$$

$$C^{(3)}_{8b2}(U_{8b2}(\alpha,\beta)|\psi(\theta_1,\gamma_1)\rangle \otimes |\psi(\theta_2,\gamma_2)\rangle \otimes |\psi(\theta_3,\gamma_3)\rangle) = \left|\sin^2\theta_1 \sin^2\theta_3 \left(e^{2i(\alpha+\gamma_2)} \sin^2\frac{\theta_2}{2} - e^{2i\beta}\cos^2\frac{\theta_2}{2}\right)^2\right|. \tag{8.10}$$

Their solutions coincide with the binary case (8.2)–(8.3) applied to the middle qubit $|\psi(\theta_2,\gamma_2)\rangle$ and $\gamma_2 \to 2\gamma_2$.

The action of the 8-vertex symmetric (circle-like) ternary braiding gates $U_{8s}(\alpha,\beta) \equiv U^{8-vertex}_{symm1,2}(\alpha,\beta)$ (7.4)–(7.5) leads to the transformed concurrence

$$C^{(3)}_{8s}(U_{8s}(\alpha,\beta)|\psi(\theta_1,\gamma_1)\rangle \otimes |\psi(\theta_2,\gamma_2)\rangle \otimes |\psi(\theta_3,\gamma_3)\rangle) = \left|\sin^2\theta_2 \left(e^{i(\beta+2\gamma_1)} \sin^2\frac{\theta_1}{2} - e^{i\alpha}\cos^2\frac{\theta_1}{2}\right)\left(e^{i(\beta+2\gamma_3)} \sin^2\frac{\theta_3}{2} - e^{i\alpha}\cos^2\frac{\theta_3}{2}\right)\right|. \tag{8.11}$$

The conditions for this to vanish (i.e. when the gate $U_{8s}(\alpha,\beta)$ becomes not entangling) coincide with those for the binary case (8.2)–(8.3), applied here to the first or the third qubit.

Thus we have shown that the braiding binary and ternary quantum gates can be either entangling or not entangling, depending on how their parameters are related to the concrete quantum state on which they act. The constructions presented here could be used, e.g. in the entanglement-free protocols [57,58] and some experiments [59,60]. This can also allow us to build quantum networks without any entangling at all (*non-entangling networks*), when the next gate depends upon the previous state in such a way that at each step there is no entangling, as the separable, but different, final state is received from a separable initial state.

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