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BOUNDS ON THE SUM OF $\sum (\log p)^2$ TERMS

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Abstract.

In this research paper we implement the theory of the primorial function, to develop the Supremum and Infimum bounds for the sum $\sum_{p \leq n} (\log p)^2$. There are, however, considerable computational difficulties related to these bounds. Therefore from a pragmatic point of view, a set of Upper and Lower bounds had been developed to bypass this issue. Despite the increased estimation error, the Upper and Lower bounds are still considerably accurate, while facilitating an easy and fast computation of the estimate of the sum.

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2000 *Mathematics Subject Classification.* 1102, 11A41, 11L20, 11N05, 11N37.

Key words and phrases. Infimum of the sum of log p terms squared, Supremum of the sum of log p terms squared, Lower and Upper bounds on the sum of log p terms squared.

1. Preliminaries

All calculations and graphing were carried out with the aid of the *Mathematica*® software.

Lemma 1.1 (Lower Estimation Error Bound On The Difference $p_n - \log p_n\sharp$)**.**
The error of estimation of the primorial function by the use of the value of $p_{(n)}$ *imposes the following lower bound:*

$$\mathcal{LB}_{p_{(n)}} = \left(\sqrt{5}-1\right)\left(4\gamma^2-2\gamma\right)\left(\log p_{(n)}\right)\sqrt[3]{p_{(n)}} < \left(p_{(n)} - \log p_{(n)}\sharp\right) \tag{1.1}$$
$$\forall p_{(n)} \in \mathbb{N} \mid p_{(n)} \geq 2$$

where $\gamma \approx 0.5772156649015328606065120$9 is the Euler-Mascheroni constant.

Lemma 1.2 (Upper Estimation Error Bound On The Difference $p_n - \log p_n\sharp$)**.** *The error of estimation of the primorial function by the use of the value of* $p_{(n)}$ *imposes the following upper bound:*

$$\left(p_{(n)} - \log p_{(n)}\sharp\right) < 2\sqrt{p_{(n)}} = \mathcal{UB}_{p_{(n)}} \quad \forall p_{(n)} \in \mathbb{N} \mid p_{(n)} \geq 2 \tag{1.2}$$

where $\gamma \approx 0.57721566490153286060651209$ is the Euler-Mascheroni constant.

Remark 1.1.
For proofs of Lemmas 1.1 and 1.2 please refer to Feliksiak [4].

2. The bounds on $\sum_{p\leq n}(\log p)^2$

2.1. The Supremum and Infimum Bounds.

Both the Supremum and Infimum bounds on the sum $\sum_{p\leq n}(\log p)^2$, implement the logarithm of the primorial function. Hence, a significant computational difficulty arises when computing them. For this reason, they are in essence just theoretical tools, aiding in the development of the Upper and Lower bounds.

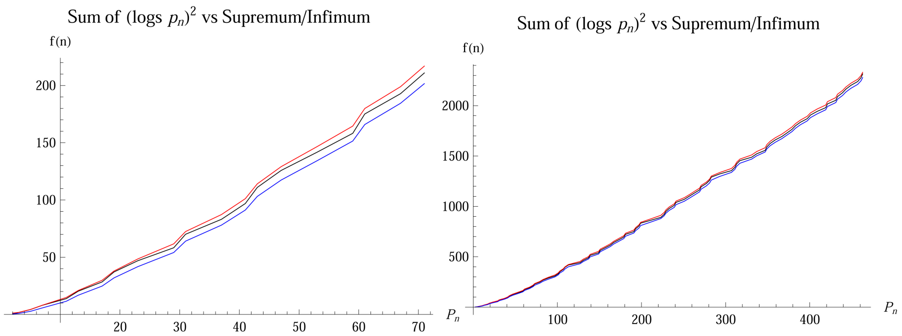

Figure 1. The drawings show the graphs of the Supremum (Red) and the Infimum (Blue) bounds on the sum $\sum_{p\leq n}(\log p)^2$ (Black). Figures are drawn w.r.t. p_n.

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Theorem 2.1 (The Supremum Bound of the sum of $(\log p)^2$).

The Supremum of the sum of $(\log p)^2$ *terms, for all* $p_n \in \mathbb{N} \mid p_{(n)} \geq 3$ *is given by:*

$$Sup = \left(\left(p_n^{(-GM)} + 1 \right) (\log p_n - 1) \log p_n \sharp \right) + \frac{\pi}{2} \geq \sum_{p \leq n} (\log p)^2 \tag{2.1}$$

Where $GM = \left(\frac{\sqrt{5}-1}{2} \right)$ *is the Golden Mean constant.*

Proof.

Suppose that for a $p_n \in \mathbb{N} \mid p_{(n)} \geq 3$, inequality 2.2 is false:

$$\left(\frac{\left[\left(\left(p_n^{(-GM)} + 1 \right) (\log p_n - 1) \log p_n \sharp \right) + \frac{\pi}{2} \right] - \sum_{p \leq n} (\log p)^2}{\sum_{p \leq n} (\log p)^2} \right) \geq 0 \tag{2.2}$$

This implies that it must be true that:

$$\left(\frac{\left[\left(\left(p_n^{(-GM)} + 1 \right) (\log p_n - 1) \log p_n \sharp \right) + \frac{\pi}{2} \right] - \sum_{p \leq n} (\log p)^2}{\sum_{p \leq n} (\log p)^2} \right)^{(-1)} < 0 \tag{2.3}$$

However, at $p_n = 3$ the inequality 2.3 attains approx. 9.20558 and increases as p_n increases unboundedly. Since inequality 2.3 produces a positive output, we implement the Cauchy Root Test:

(2.4)

$$\sqrt[n]{a_n} = \sqrt[n]{\left(\frac{\left[\left(\left(p_n^{(-GM)} + 1 \right) (\log p_n - 1) \log p_n \sharp \right) + \frac{\pi}{2} \right] - \sum_{p \leq n} (\log p)^2}{\sum_{p \leq n} (\log p)^2} \right)^{(-1)}}$$

The Cauchy Test at $p_n = 3$ attains approximately 3.03407 and strictly from above, asymptotically converges to 1. Please refer to Figure 2. Consequently in accordance with the definition of the Cauchy Root Test, the sequence 2.5 diverges;

$$\left(\frac{\left[\left(\left(p_n^{(-GM)} + 1 \right) (\log p_n - 1) \log p_n \sharp \right) + \frac{\pi}{2} \right] - \sum_{p \leq n} (\log p)^2}{\sum_{p \leq n} (\log p)^2} \right)^{(-1)} \tag{2.5}$$

In fact, the sequence 2.5 diverges at a rate $\propto k \sqrt{p_n}$ for some $k \in \mathbb{R} \mid k > 1$, where k depends on p_n. Further, the rate of divergence increases rapidly. Please refer to Figure 2b. Hence we have a contradiction to the initial assumption. This implies that the inequality 2.2 is true, for all $p_n \in \mathbb{N} \mid p_{(n)} \geq 3$. Since both, the inequality 2.2 and the sum of $(\log p)^2$ terms are positive, this implies that for all $p_n \in \mathbb{N} \mid p_{(n)} \geq 3$:

$$\left[\left(\left(p_n^{(-GM)} + 1 \right) (\log p_n - 1) \log p_n \sharp \right) + \frac{\pi}{2} \right] \geq \sum_{p \leq n} (\log p)^2$$

Concluding the proof. □

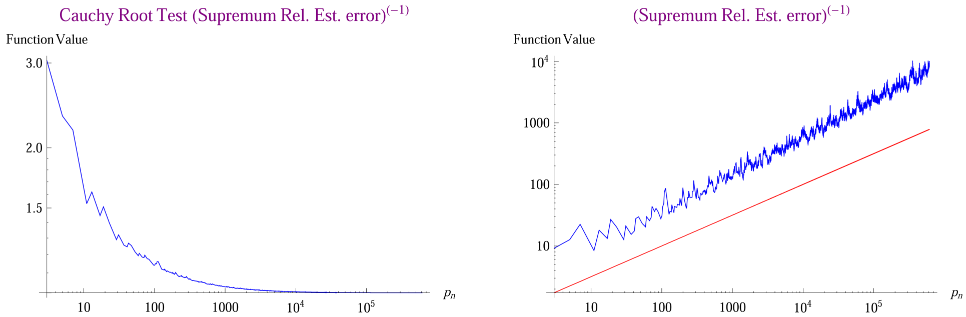

FIGURE 2. The L.H. drawing shows the Cauchy's Root Test of terms of the inequality 2.4. The R.H. drawing shows the output of the inequality 2.5 (Blue) vs. $\sqrt{p_n}$ (Red). The log-log figures are drawn w.r.t. $p_n \in \mathbb{N} \mid 3 \leq p_n \leq 611\,953$.

Theorem 2.2 (The Supremum Relative Error $\mathcal{RE}_{(p_n)}$ Bounds)**.**
The relative error made by the Supremum, for all $p_n \in \mathbb{N} \mid p_{(n)} \geq 23$ has the following lower bound:

$$\mathcal{LB} = \frac{1}{p_n} < \frac{\left[\left(\left(p_n^{(-GM)} + 1\right)(\log p_n - 1)\log p_n\sharp\right) + \frac{\pi}{2}\right] - \sum_{p \leq n} (\log p)^2}{\sum_{p \leq n} (\log p)^2} \tag{2.6}$$

and for all $p_n \in \mathbb{N} \mid p_{(n)} \geq 31$ the relative error has the following upper bound:

$$\frac{\left[\left(\left(p_n^{(-GM)} + 1\right)(\log p_n - 1)\log p_n\sharp\right) + \frac{\pi}{2}\right] - \sum_{p \leq n} (\log p)^2}{\sum_{p \leq n} (\log p)^2} < \frac{2}{5\sqrt{p_n}} = \mathcal{UB} \tag{2.7}$$

Where $GM = \left(\frac{\sqrt{5}-1}{2}\right)$ is the Golden Mean constant.

Proof.
Suppose that for a $p_n \in \mathbb{N} \mid p_{(n)} \geq 23$ the following inequality is false:

$$\frac{\left[\left(\left(p_n^{(-GM)} + 1\right)(\log p_n - 1)\log p_n\sharp\right) + \frac{\pi}{2}\right] - \sum_{p \leq n} (\log p)^2}{\sum_{p \leq n} (\log p)^2} > \frac{1}{p_n} \tag{2.8}$$

This implies that it must be true that:

$$\left(\frac{\left[\left(\left(p_n^{(-GM)} + 1\right)(\log p_n - 1)\log p_n\sharp\right) + \frac{\pi}{2}\right] - \sum_{p \leq n} (\log p)^2}{\sum_{p \leq n} (\log p)^2}\right)^{(-1)} - p_n > 0 \tag{2.9}$$

However, at $p_n = 23$ the difference 2.9 attains approximately -2.64707, and at every step diverges at a rate $\propto k\ p_n$ with $k \sim 1$ as p_n increases unboundedly. Please refer to Figure 3a. Consequently, we have a contradiction to the initial hypothesis. Necessarily therefore, for all $p_n \in \mathbb{N} \mid p_n \geq 23$ the inequality holds:

$$\left(\frac{\left[\left(\left(p_n^{(-GM)} + 1\right)(\log p_n - 1)\log p_n\sharp\right) + \frac{\pi}{2}\right] - \sum_{p \leq n} (\log p)^2}{\sum_{p \leq n} (\log p)^2}\right) > \frac{1}{p_n} \tag{2.10}$$

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Suppose now, that for $p_n \in \mathbb{N} \mid p_{(n)} \geq 43$ the following inequality is false:

$$\frac{\left[\left(\left(p_n^{(-GM)}+1\right)(\log p_n-1)\log p_n\sharp\right)+\frac{\pi}{2}\right]-\sum_{p\leq n}(\log p)^2}{\sum_{p\leq n}(\log p)^2} < \frac{2}{5\sqrt{p_n}} \tag{2.11}$$

necessarily it must be true that:

$$\left(\frac{\left[\left(\left(p_n^{(-GM)}+1\right)(\log p_n-1)\log p_n\sharp\right)+\frac{\pi}{2}\right]-\sum_{p\leq n}(\log p)^2}{\sum_{p\leq n}(\log p)^2}\right)^{(-1)} - \frac{5\sqrt{p_n}}{2} < 0 \tag{2.12}$$

However, the difference 2.12 at $p_n = 43$ attains approx. 11.428558, and diverges. Since inequality 2.12 produces a positive output, we implement the Cauchy Root Test:

$$\sqrt[n]{a_n} = \tag{2.13}$$

$$\sqrt[n]{\left(\left(\frac{\left[\left(\left(p_n^{(-GM)}+1\right)(\log p_n-1)\log p_n\sharp\right)+\frac{\pi}{2}\right]-\sum_{p\leq n}(\log p)^2}{\sum_{p\leq n}(\log p)^2}\right)^{(-1)} - \frac{5\sqrt{p_n}}{2}\right)}$$

The Cauchy Root Test at $p_n = 43$ attains approx. 1.1900654 and strictly from above, asymptotically converges to 1. Please also refer to Figure 3b. In accordance with the definition of the Cauchy Root Test, This implies that the sequence diverges. In fact, for all $p_n \geq 1429$ the sequence diverges at a rate $\propto k\ \sqrt{p_n}$, for some $k \in \mathbb{R} \mid k > 1$, where k depends on p_n. The rate of divergence further increases as p_n increases unboundedly. Consequently, we have a contradiction to the initial hypothesis. Therefore, it must be true that the inequality holds for all $p_n \geq 43$:

$$\left(\frac{\left[\left(\left(p_n^{(-GM)}+1\right)(\log p_n-1)\log p_n\sharp\right)+\frac{\pi}{2}\right]-\sum_{p\leq n}(\log p)^2}{\sum_{p\leq n}(\log p)^2}\right) < \frac{2}{5\sqrt{p_n}} \tag{2.14}$$

Direct evaluation for all $p_n \in \mathbb{N} \mid 31 \leq p_{(n)} \leq 43$ confirms that the inequality 2.14 holds within this range. Please refer to Figure 4.Consequently, it holds for all $p_n \in \mathbb{N} \mid p_{(n)} \geq 31$. Thus, Theorem 2.2 holds as stated; Concluding the proof. □

Corollary 2.3 (Limit of the Relative Error $\mathcal{R}.\mathcal{E}._{(p_n)}$ made by Supremum)**.**
Clearly, both the lower bound $\mathcal{L}.\mathcal{B}._{(p_n)}$ and the upper bound $\mathcal{U}.\mathcal{B}._{(p_n)}$, as given by Theorem 2.2, converge to zero as p_n increases unboundedly. Consequently this implies, that the relative error made by Supremum bound, which is squeezed between those bounds, must converge to zero as well:

$$\lim_{p_n\to\infty}(\mathcal{LB}) = \lim_{p_n\to\infty}\left(\mathcal{R}.\mathcal{E}._{(p_n)}\right) = \lim_{p_n\to\infty}(\mathcal{UB}) \to 0 \tag{2.15}$$

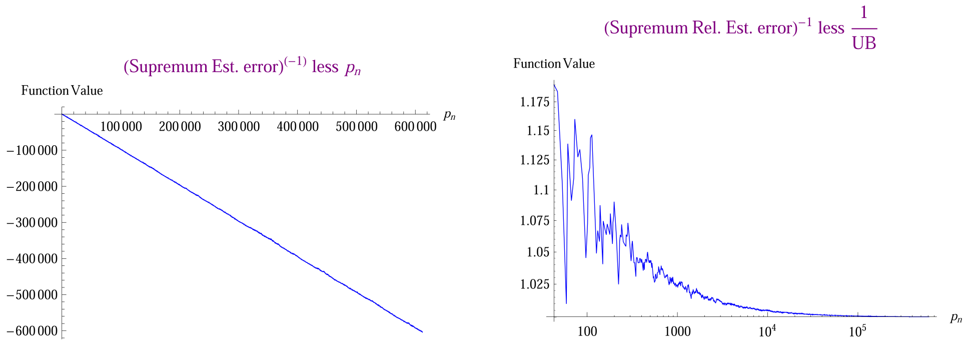

FIGURE 3. The L.H. drawing shows the output of inequality 2.9. The R.H. drawing shows the output of the Cauchy Test 2.13. The L.H. figure is drawn w.r.t. $p_n \in \mathbb{N} \mid 23 \leq p_n \leq 611953$ while the R.H. figure is drawn over the range $p_n \in \mathbb{N} \mid 43 \leq p_n \leq 611953$.

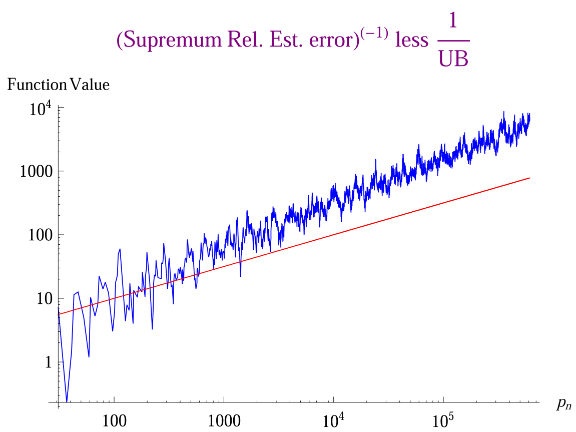

FIGURE 4. The drawing shows the output of the inequality 2.12 (Blue) vs. $\sqrt{p_n}$ (Red). The log-log figure is drawn w.r.t. $p_n \in \mathbb{N} \mid 31 \leq p_n \leq 611953$.

Lemma 2.4 (Relative Error $\mathcal{R}.\mathcal{E}._{(p_n)}$ of the Infimum of the sum $\sum_{p\leq n} (\log p)^2$)**.**
The relative error of the Infimum for all $p_n \in \mathbb{N} \mid p_{(n)} \geq 3$ has the following bounds:

(2.16)

$$\mathcal{LB} = \frac{1}{\sqrt{5}\,\sqrt[\beta 1]{p_n}} < \frac{\sum_{p\leq n}(\log p)^2 - [Inf = ((\log p_n - 1)\log p_n \sharp) + \log 2]}{\sum_{p\leq n}(\log p)^2} < \frac{\alpha}{GM\,\sqrt[\beta 2]{p_n}} = \mathcal{UB}$$

Where $\alpha = \frac{5}{(\log 10)^2}$, $\beta 1 = \left(\sqrt{5} - 1\right)$, $\beta 2 = (\log 2 + 1)$ *and* $GM = \left(\frac{\sqrt{5}-1}{2}\right)$.

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Proof.
Suppose that for a $p_n \in \mathbb{N} \mid p_{(n)} \geq 3$ the following inequality is false:

$$\frac{\sum_{p\leq n} (\log p)^2 - [\![(\log p_n - 1) \log p_n\sharp) + \log 2]}{\sum_{p\leq n} (\log p)^2} > \frac{1}{\sqrt{5}\ \sqrt[\beta_1]{p_n}} \tag{2.17}$$

This implies that it must be true that:

$$\left(\frac{\sum_{p\leq n} (\log p)^2 - [\![(\log p_n - 1) \log p_n\sharp) + \log 2]}{\sum_{p\leq n} (\log p)^2}\right)^{(-1)} - \sqrt{5}\ \sqrt[\beta_1]{p_n} > 0 \tag{2.18}$$

However, at $p_n = 3$ the difference 2.18 attains ~ -3.37463 and diverges at a rate exceeding $-\sqrt[\beta_1]{p_n}$, with the rate increasing as p_n increases unboundedly. Please refer to Figure 5. Therefore, we have a contradiction to the initial hypothesis. Hence, for all $p_n \in \mathbb{N} \mid p_n \geq 3$ the inequality is valid:

$$\frac{\sum_{p\leq n} (\log p)^2 - [\![(\log p_n - 1) \log p_n\sharp) + \log 2]}{\sum_{p\leq n} (\log p)^2} > \frac{1}{\sqrt{5}\ \sqrt[\beta_1]{p_n}} \tag{2.19}$$

Suppose now that for $p_n \in \mathbb{N} \mid p_{(n)} \geq 29$ the following inequality is false:

$$\frac{\sum_{p\leq n} (\log p)^2 - [\![(\log p_n - 1) \log p_n\sharp) + \log 2]}{\sum_{p\leq n} (\log p)^2} < \frac{\alpha}{GM\ \sqrt[\beta_2]{p_n}} \tag{2.20}$$

Necessarily then, it must be true that:

$$\left(\frac{\sum_{p\leq n} (\log p)^2 - [\![(\log p_n - 1) \log p_n\sharp) + \log 2]}{\sum_{p\leq n} (\log p)^2}\right)^{(-1)} - \frac{GM\ \sqrt[\beta_2]{p_n}}{\alpha} < 0 \tag{2.21}$$

However, the difference 2.21 at $p_n = 29$ attains ~ 9.46078 and diverges at a rate exceeding $\sqrt{p_n}$ with the rate increasing as p_n increases unboundedly. Consequently, we have a contradiction to the initial hypothesis. Hence, for all $p_n \in \mathbb{N} \mid p_n \geq 29$ the inequality is valid:

$$\frac{\sum_{p\leq n} (\log p)^2 - [\![(\log p_n - 1) \log p_n\sharp) + \log 2]}{\sum_{p\leq n} (\log p)^2} < \frac{\alpha}{GM\ \sqrt[\beta_2]{p_n}} \tag{2.22}$$

For the range $p_n \in \mathbb{N} \mid 3 \leq p_n \leq 29$, direct computer evaluation confirms that inequality 2.22 holds in this range. The output in a graphical form, is presented in Figure 6. Thus, Lemma 2.4 holds as stated; Concluding the proof. □

Corollary 2.5 (Limit of the Relative Error $\mathcal{R}.\mathcal{E}._{(p_n)}$)**.**
Clearly, both the lower bound $\mathcal{L}.\mathcal{B}._{(p_n)}$ and the upper bound $\mathcal{U}.\mathcal{B}._{(p_n)}$, as given by Lemma 2.4, converge to zero as p_n increases unboundedly. Consequently this implies, that the Relative Estimation Error which is squeezed between those bounds, must converge to zero as well:

$$\lim_{p_n\to\infty} (\mathcal{LB}) = \lim_{p_n\to\infty} \left(\mathcal{R}.\mathcal{E}._{(p_n)}\right) = \lim_{p_n\to\infty} (\mathcal{UB}) \to 0 \tag{2.23}$$

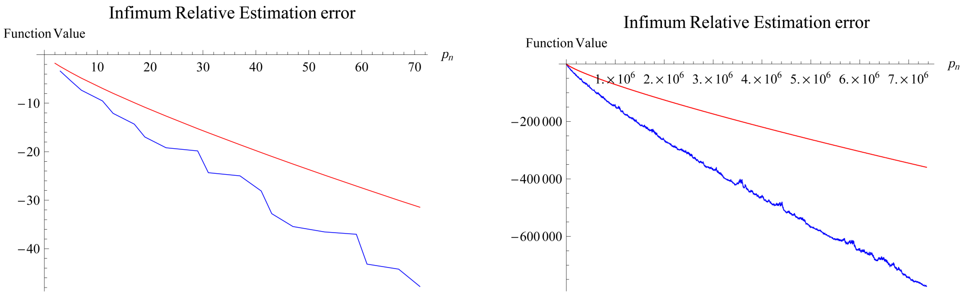

FIGURE 5. The drawings compare the output of the inequality 2.18 in Blue, and the $\sqrt[\beta 1]{p_n}$ function in Red. The L.H. figure is drawn w.r.t. $p_n \in \mathbb{N} \mid 3 \leq p_n \leq 71$ while the R.H. figure is drawn over the range $p_n \in \mathbb{N} \mid 3 \leq p_n \leq 7368787$

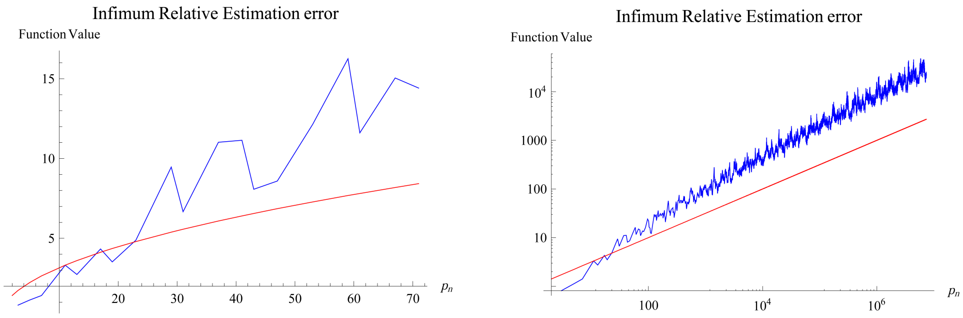

FIGURE 6. The drawings compare the output of the inequality 2.22 in Blue, and the $\sqrt{p_n}$ function in Red. The L.H. figure is drawn w.r.t. $p_n \in \mathbb{N} \mid 3 \leq p_n \leq 71$ while the log-log R.H. figure is drawn over the range $p_n \in \mathbb{N} \mid 3 \leq p_n \leq 7368787$

Corollary 2.6 (Infimum Bounds)**.**
From Lemma 2.4 we obtain the bounds on the Infimum:

$$\sum_{p \leq n} (\log p)^2 - \frac{\alpha \left(\sum_{p \leq n} (\log p)^2 \right)}{GM \sqrt[\beta 2]{p_n}} < Inf < \sum_{p \leq n} (\log p)^2 - \frac{\sum_{p \leq n} (\log p)^2}{\sqrt{5} \sqrt[\beta 1]{p_n}} \tag{2.24}$$

Where $Inf = ((\log p_n - 1) \log p_n \sharp) + \log 2$
and $\alpha = \frac{5}{(\log 10)^2}$, $\beta 1 = \left(\sqrt{5} - 1\right)$, $\beta 2 = (\log 2 + 1)$ and $GM = \left(\frac{\sqrt{5}-1}{2}\right)$.

2.2. **The Upper and Lower bounds on** $\sum_{p \leq n} (\log p)^2$**.**

Lemma 2.7 (Upper bound on the sum $\sum_{p \leq n} (\log p)^2$)**.**
The Upper bound on the sum $\sum_{p \leq n} (\log p)^2$ *for all* $p_n \in \mathbb{N} \mid p_{(n)} \geq 11$ *is given by:*

$$UB_{p_n} = (\log p_n - 1)\, p_n + 1 \tag{2.25}$$

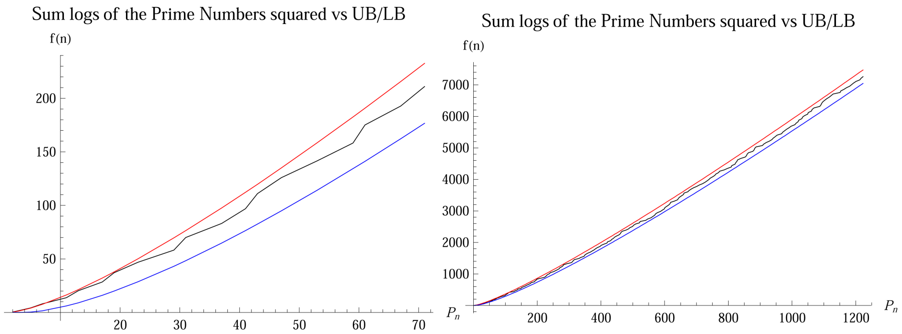

FIGURE 7. The drawings show the graphs of the Upper (Red) and the Lower (Blue) bounds on the sum $\sum_{p\le n}(\log p)^2$ (Black). Figures are drawn w.r.t. p_n.

Proof.
From Lemma 1.1 for all $p_{(n)} \in \mathbb{N} \mid p_{(n)} \ge 2$ we have:

$$\left(\sqrt{5}-1\right)\left(4\gamma^2-2\gamma\right)\left(\log p_{(n)}\right)\sqrt[3]{p_{(n)}} < \left(p_{(n)} - \log p_{(n)}\sharp\right) \tag{2.26}$$

where $\gamma \approx 0.5772156649015328606065120$9 is the Euler-Mascheroni constant.

Thus,

$$\log p_{(n)}\sharp < \left(p_{(n)} - \left(\sqrt{5}-1\right)\left(4\gamma^2-2\gamma\right)\left(\log p_{(n)}\right)\sqrt[3]{p_{(n)}}\right) \tag{2.27}$$

Consequently, for all $p_n \in \mathbb{N} \mid p_{(n)} \ge 11$:

$$(\log p_n - 1)\log p_{(n)}\sharp < (\log p_n - 1)\, p_n \tag{2.28}$$

From Lemma 2.6 we have that the Supremum on the sum $\sum_{p\le n}(\log p)^2$ is given by:

$$\left(\left(p_n^{(-GM)}+1\right)(\log p_n - 1)\log p_{(n)}\sharp + \frac{\pi}{2}\right) \tag{2.29}$$

Since $\left(p_n^{(-GM)}+1\right)$ relatively quickly converges to its limit at 1 and $\frac{\pi}{2}$ is just a constant slightly exceeding 1, there must be a point after which $UB_{p_n} > Sup$. In fact, at $p_n = 127$ the difference of $UB_{p_n} - Sup \approx 35.8717$ and diverges as p_n increases unboundedly. Consequently, $\forall n \in \mathbb{N} \mid n \ge 127$ the inequality is valid:

$$((\log p_n - 1)\, p_n + 1) > \left(\left(p_n^{(-GM)}+1\right)(\log p_n - 1)\log p_{(n)}\sharp + \frac{\pi}{2}\right) \tag{2.30}$$

Direct computer evaluation for all $p_n \in \mathbb{N} \mid 11 \le p_{(n)} \le 127$:

$$((\log p_n - 1)\, p_n + 1) - \sum_{p\le n}(\log p)^2 > 0 \tag{2.31}$$

establishes that, inequality 2.31 at $p_n = 11$ attains ~ 2.56269 and that it holds within this range. Consequently, from 2.30 and 2.31, necessarily Lemma 2.7 is valid for all $p_n \in \mathbb{N} \mid p_{(n)} \ge 11$. Please refer to Figure 8. Concluding the proof. □

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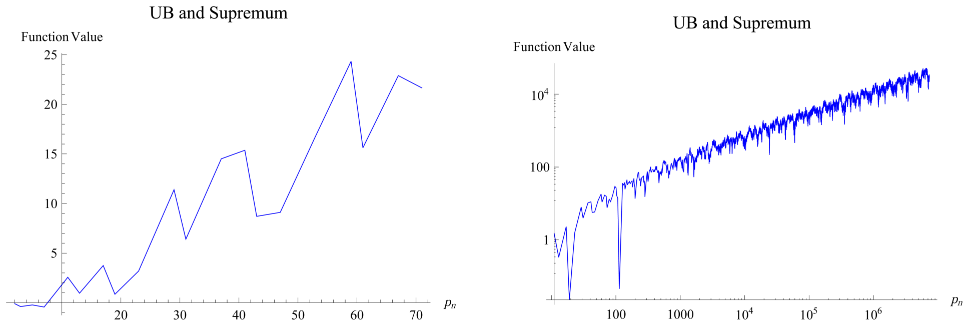

FIGURE 8. The linear L.H. figure shows the graph of the inequality 2.31. It is drawn w.r.t. $p_n \in \mathbb{N} \mid 3 \leq p_n \leq 71$ and shows a minimum at $19, 0.854396$. The log-log R.H. figure shows the graph of the difference of terms of the inequality 2.30, which is drawn over the range $p_n \in \mathbb{N} \mid 3 \leq p_n \leq 7368787$, and clearly displays two minima: at $\{19, 0.0225929\}$ and at $\{113, 0.0451773\}$.

Lemma 2.8 (Lower bound on the sum $\sum_{p \leq n} (\log p)^2$)**.**
The Lower bound on the sum $\sum_{p \leq n} (\log p)^2$*, for all* $p_n \in \mathbb{N} \mid p_{(n)} \geq 2$ *is given by:*

$$LB_{p_n} = (\log p_n - 1)\left(p_n - 2\sqrt{p_n}\right) \tag{2.32}$$

Proof.
By Lemma 2.4 the Infimum bound on the sum $\sum_{p \leq n} (\log p)^2$ is given by:

$$Inf = ((\log p_n - 1) \log p_n\sharp) + \log 2 \tag{2.33}$$

From Lemma 1.2 in turn, we have:

$$\left(p_n - 2\sqrt{p_n}\right) < \log p_{(n)}\sharp \quad \forall p_{(n)} \in \mathbb{N} \mid p_{(n)} \geq 2 \tag{2.34}$$

which necessarily implies that for all $p_{(n)} \in \mathbb{N} \mid p_{(n)} \geq 2$:

$$LB_{p_n} = (\log p_n - 1)\left(p_n - 2\sqrt{p_n}\right) < ((\log p_n - 1) \log p_n\sharp) + \log 2 = Inf \tag{2.35}$$

In fact at $p_n = 2$, the difference of terms $Inf - LB_{p_n}$ of the inequality 2.35, attains approx. 0.226248 and diverges as p_n increases unboundedly. Consequently, this implies that Lemma 2.8 holds as stated, concluding the proof. □

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