

On a coupled system of random and stochastic differential equations with nonlocal stochastic integral conditions

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Abstract

Here we are concerning with two problems of a coupled system of random and stochastic nonlinear differential equations with two coupled systems of nonlinear non-local random and stochastic integral conditions. The existence of solutions will be studied. The sufficient condition for the uniqueness of the solution will be given. The continuous dependence of the unique solution on the nonlocal conditions will be proved.

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1 Introduction

The coupled systems of differential equations with nonlocal conditions have studied by some authors (see for example [4]-[6] and references therein).

Let (Ω, \mathcal{F}, P) be a fixed probability space, where Ω is a sample space, \mathcal{F} is a σ -algebra and P is a probability measure.

Let $Z(t; \omega) = Z(t)$, $t \in [0, T]$, $\omega \in \Omega$ be a second order stochastic process, i.e., $E(Z^2(t)) < \infty$, $t \in [0, T]$.

Let $C = C([0, T], L_2(\Omega))$ be the space of all second order stochastic processes which is mean square (m.s) continuous on $[0, T]$. The norm of $Z \in C([0, T], L_2(\Omega))$ is given by

$$\|Z\|_C = \sup_{t \in [0, T]} \|Z(t)\|_2, \quad \|Z(t)\|_2 = \sqrt{E(Z^2(t))}.$$

Let $T \geq 1$. In this paper we study the existence of solutions $(x, y) \in C([0, T], L_2(\Omega))$ of the problem of the coupled system of random and stochastic differential equations

$$\frac{dx(t)}{dt} = f_1(t, y(\phi_1(t))), \quad t \in (0, T], \quad (1)$$

$$dy(t) = f_2(t, x(\phi_2(t)))dW(s), \quad t \in (0, T] \quad (2)$$

subject to each one of the two nonlinear nonlocal stochastic integral conditions

$$x(0) + \int_0^\tau h_1(s, y(s))dW(s) = x_0, \quad y(0) + \int_0^\eta h_2(s, x(s))ds = y_0 \quad (3)$$

and

$$x(0) + \int_0^\tau h_1(s, x(s))dW(s) = x_0, \quad y(0) + \int_0^\eta h_2(s, y(s))ds = y_0 \quad (4)$$

where x_0 and y_0 are two second order random variables.

Let $X = C([0, T], L_2(\Omega)) \times C([0, T], L_2(\Omega))$ be the class of all ordered pairs (x, y) , $x, y \in C$ with the norm

$$\|(x, y)\|_X = \max\{\|x\|_C, \|y\|_C\} = \max\left\{\sup_{t \in [0, T]} \|x(t)\|_2, \sup_{t \in [0, T]} \|y(t)\|_2\right\}. \quad (5)$$

Let $\phi_i : [0, T] \rightarrow [0, T]$ be continuous functions such that $\phi_i(t) \leq t$ and consider the following assumptions

(A1) $f_i : [0, T] \times L_2(\Omega) \rightarrow L_2(\Omega)$, $i = 1, 2$ are measurable in $t \in [0, T]$ for all $x \in L_2(\Omega)$ and continuous in $x \in L_2(\Omega)$ for all $t \in [0, T]$. There exist two bounded measurable functions $m_i : [0, T] \rightarrow R$ and two positive constants b_i such that

$$\|f_i(t, x)\|_2 \leq |m_i(t)| + b_i\|x(t)\|_2, \quad i = 1, 2. \quad (6)$$

(A2) $h_i : [0, T] \times L_2(\Omega) \rightarrow L_2(\Omega)$, $i = 1, 2$ are measurable in $t \in [0, T]$ for all $x \in L_2(\Omega)$ and continuous in $x \in L_2(\Omega)$ for all $t \in [0, T]$. There exist two bounded measurable functions $k_i : [0, T] \rightarrow R$ and two positive constants c_i such that

$$\|h_i(t, x)\|_2 \leq |k_i(t)| + c_i\|x(t)\|_2, \quad i = 1, 2. \quad (7)$$

(A3) $M = \max\{\sup_{t \in [0, T]} |m_1(t)|, \sup_{t \in [0, T]} |m_2(t)|\}$, $b = \max\{b_1, b_2\}$.

(A4) $K = \max\{\sup_{t \in [0, T]} |k_1(t)|, \sup_{t \in [0, T]} |k_2(t)|\}$, $c = \max\{c_1, c_2\}$.

(A5) $(b + c)T < 1$.

Now, integrating the two random and stochastic differential equations (1)-(2) (see [2], [3], [7]-[11]) and using the nonlocal conditions (3) and (4) the following Lemma can be proved.

Lemma 1. The integral representations of the solutions of the nonlocal problems (1)-(2) with conditions (3) and (1) - (2) with conditions (4) are given by

$$x(t) = x_0 - \int_0^\tau h_1(s, y(s))dW(s) + \int_0^t f_1(s, y(\phi_1(s)))ds, \quad (8)$$

$$y(t) = y_0 - \int_0^\eta h_2(s, x(s))ds + \int_0^t f_2(s, x(\phi_2(s)))dW(s). \quad (9)$$

and

$$x(t) = x_0 - \int_0^\tau h_1(s, x(s))dW(s) + \int_0^t f_1(s, y(\phi_1(s)))ds, \quad (10)$$

$$y(t) = y_0 - \int_0^\eta h_2(s, y(s))ds + \int_0^t f_2(s, x(\phi_2(s)))dW(s). \quad (11)$$

respectively.

2 Solutions of the problem (1)-(3)

Define the mapping $F(x, y) = (F_1y, F_2x)$ where F_1y, F_2x are given by the following stochastic integral equations

$$F_1y(t) = x_0 - \int_0^\tau h_1(s, y(s))dW(s) + \int_0^t f_1(s, y(\phi_1(s)))ds, \quad (12)$$

$$F_2x(t) = y_0 - \int_0^\eta h_2(s, x(s))ds + \int_0^t f_2(s, x(\phi_2(s)))dW(s). \quad (13)$$

Consider the set Q such that

$$Q = \{x, y \in L_2(\Omega), (x, y) \in X : \|(x, y)\|_X = \max\{\|x(t)\|_2, \|y(t)\|_2\} \leq r\}$$

Now, we have the following two lemmas

Lemma 2. $F : Q \rightarrow Q$.

Proof. Let $y \in Q$, $\|y(t)\|_2 \leq r_1$, then

$$\begin{aligned} \|F_1y(t)\|_2 &\leq \|x_0\|_2 + \left\| \int_0^\tau h_1(s, y(s))dW(s) \right\|_2 + \left\| \int_0^t f_1(s, y(\phi_1(s)))ds \right\|_2 \\ &\leq \|x_0\|_2 + \sqrt{\int_0^\tau \|h_1(s, y(s))\|_2^2 ds} + \int_0^t \|f_1(s, y(\phi_1(s)))\|_2 ds \\ &\leq \|x_0\|_2 + \sqrt{\int_0^\tau (|k_1(s)| + c_1\|y(s)\|_2)^2 ds} + \int_0^t (|m_1(s)| + b_1\|y(s)\|_2) ds \\ &\leq \|x_0\|_2 + (K + cr_1)\sqrt{T} + (M + br_1)T < \|x_0\|_2 + (K + cr_1)T + (M + br_1)T = r_1 \end{aligned}$$

where

$$r_1 = \frac{\|x_0\|_2 + KT + MT}{1 - (b + c)T} > 0.$$

Let $x \in Q$, $\|x(t)\|_2 \leq r_2$, then

$$\begin{aligned} \|F_2x(t)\|_2 &\leq \|y_0\|_2 + \left\| \int_0^\eta h_2(s, x(s))ds \right\|_2 + \left\| \int_0^t f_2(s, x(\phi_2(s)))dW(s) \right\|_2 \\ &\leq \|y_0\|_2 + \int_0^\eta \|h_2(s, x(s))\|_2 ds + \sqrt{\int_0^t \|f_2(s, x(\phi_2(s)))\|_2^2 ds} \\ &\leq \|y_0\|_2 + \int_0^\eta (|k_2(s)| + c_2\|x(s)\|_2)ds + \sqrt{\int_0^t (|m_2(t)| + b_2\|x\|_2)^2 ds} \\ &\leq \|y_0\|_2 + (K + cr_2)T + (M + br_2)T < \|y_0\|_2 + (K + cr_2)T + (M + br_2)T = r_2 \end{aligned}$$

where

$$r_2 = \frac{\|y_0\|_2 + KT + MT}{1 - (b + c)T} > 0.$$

Let $r = \max\{r_1, r_2\}$, $(x, y) \in Q$, then

$$\begin{aligned} \|F(x, y)\|_X &= \|(F_1y, F_2x)\|_X \\ &= \max\{\|F_1y(t)\|_C, \|F_2x(t)\|_C\} < r. \end{aligned}$$

This proves that $F : Q \rightarrow Q$ and the class of functions $\{F(x, y)(t)\}$, $t \in [0, T]$ is uniformly bounded on Q .

Lemma 3. The class of function $\{F(x, y)(t)\}$, $t \in [0, T]$ is equicontinuous.

Proof. Let $x, y \in Q$, $t_1, t_2 \in [0, T]$ such that $|t_2 - t_1| < \delta$, then

$$\begin{aligned} \|F_1y(t_1) - F_1y(t_2)\|_2 &= \left\| \int_0^{t_1} f_1(s, y(\phi_1(s)))ds - \int_0^{t_2} f_1(s, y(\phi_1(s)))ds \right\|_2 \\ &\leq \int_{t_1}^{t_2} \|f_1(s, y(\phi_1(s)))\|_2 ds \\ &\leq (M + b\|y\|_C)(t_2 - t_1) \end{aligned}$$

and

$$\begin{aligned} \|F_2x(t_1) - F_2x(t_2)\|_2 &= \left\| \int_0^{t_1} f_2(s, x(\phi_2(s)))dW(s) - \int_0^{t_2} f_2(s, x(\phi_2(s)))dW(s) \right\|_2 \\ &\leq \sqrt{\int_{t_1}^{t_2} \|f_2(s, x(\phi_2(s)))\|_2^2 ds} \\ &\leq (M + b\|x\|_C)\sqrt{(t_2 - t_1)}. \end{aligned} \tag{14}$$

But

$$\begin{aligned} F(x(t_2), y(t_2)) - F(x(t_1), y(t_1)) &= (F_1 y(t_2), F_2 x(t_2)) - (F_1 y(t_1), F_2 x(t_1)) \\ &= ((F_1 y(t_2) - F_1 y(t_1)), (F_2 x(t_2) - F_2 x(t_1))), \end{aligned}$$

then from (14) and (14), we can deduce the equicontinuity of the class $\{F(x, y)(t)\}$, $t \in [0, T]$ on Q .

2.1 Existence Theorem

Now, we have the following existence theorem

Theorem 1. Let the assumptions (A1) - (A5) be satisfied, then there exists at least one solution $(x, y) \in X$ of the problem (1)-(3).

Proof. Let $(x_n, y_n) \in Q$ be such that $(x_n, y_n) \rightarrow (x, y)$ w.p.1.

Using lemmas 1-3, then applying stochastic Lebesgue dominated convergence Theorem [1], we can obtain

$$\begin{aligned} L.i.m_{n \rightarrow \infty} F(x_n, y_n) &= (L.i.m_{n \rightarrow \infty} F_1 y_n, L.i.m_{n \rightarrow \infty} F_2 x_n) \\ &= (L.i.m_{n \rightarrow \infty} \{x_0 - \int_0^\tau h_1(s, y_n(s)) dW(s) + \int_0^t f_1(s, y_n(\phi_1(s))) ds\}, \\ &\quad L.i.m_{n \rightarrow \infty} \{y_0 - \int_0^\eta h_2(s, x_n(s)) ds + \int_0^t f_2(s, x_n(\phi_2(s))) dW(s)\}) \\ &= (x_0 - \int_0^\tau h_1(s, L.i.m_{n \rightarrow \infty} y_n(s)) dW(s) + \int_0^t f_1(s, L.i.m_{n \rightarrow \infty} y_n(\phi_1(s))) ds, \\ &\quad y_0 - \int_0^\eta h_2(s, L.i.m_{n \rightarrow \infty} x_n(s)) ds + \int_0^t f_2(s, L.i.m_{n \rightarrow \infty} x_n(\phi_2(s))) dW(s)) \\ &= (x_0 - \int_0^\tau h_1(s, y(s)) dW(s) + \int_0^t f_1(s, y(\phi_1(s))) ds, \\ &\quad y_0 - \int_0^\eta h_2(s, x(s)) ds + \int_0^t f_2(s, x(\phi_2(s))) dW(s)) \\ &= (F_1 y, F_2 x) = F(x, y). \end{aligned}$$

This proves that the operator $F : Q \rightarrow Q$ is continuous.

Then by Arzela-Ascoli Theorem [1], the closure of FQ is a compact subset of X , then applying Schauder Fixed Point Theorem [1], there exists at least one solution $(x, y) \in X$ of the problem (1)-(3) such that $x, y \in C([0, T], L_2(\Omega))$.

2.2 Uniqueness Theorem

Replace the assumptions (A1) and (A2) by (A*1) and (A*2) respectively such that

(A*1) The functions $f_i : [0, T] \times L_2(\Omega) \rightarrow L_2(\Omega)$, $i = 1, 2$ are measurable in $t \in [0, T]$

for all $x \in L_2(\Omega)$ and satisfy Lipschitz condition with respect to the second argument

$$\|f_i(t, u) - f_i(t, v)\|_2 \leq b\|u - v\|_2.$$

(A*2) The functions $h_i : [0, T] \times L_2(\Omega) \rightarrow L_2(\Omega)$, $i = 1, 2$ are measurable in $t \in [0, T]$ for all $x \in L_2(\Omega)$ and satisfy Lipschitz condition with respect to the second argument

$$\|h_i(t, u) - h_i(t, v)\|_2 \leq c\|u - v\|_2.$$

Remark. Let the assumptions (A*1) and (A*2) be satisfied, then we can get

$$\|f_i(t, u)\|_2 - \|f_i(t, 0)\|_2 \leq \|f_i(t, u) - f_i(t, 0)\|_2 \leq b\|u\|_2,$$

$$\|f_i(t, u)\|_2 \leq \|f_i(t, 0)\|_2 + b\|u\|_2 \leq M + b\|u\|_2$$

and

$$\|h_i(t, u)\|_2 \leq \|h_i(t, 0)\|_2 + c\|u\|_2 \leq K + c\|u\|_2.$$

Theorem 2. Let the assumptions (A*1) – (A*2) and (A3) – (A5) be satisfied, then the solution of problem (1)– (3) is unique.

Proof. Let (x_1, y_1) and (x_2, y_2) be two solutions of the problem (1)–(3), then

$$\begin{aligned} (x(t), y(t)) = (x_0 & - \int_0^t h_1(s, y(s))dW(s) + \int_0^t f_1(s, y(\phi_1(s)))ds, \\ y_0 & - \int_0^t h_2(s, x(s))ds + \int_0^t f_2(s, x(\phi_2(s)))dW(s)) \end{aligned} \quad (15)$$

where

$$\begin{aligned} \|x_1(t) - x_2(t)\|_2 & \leq \left\| \int_0^t [h_1(s, y_2(s)) - h_1(s, y_1(s))]dW(s) \right\|_2 + \left\| \int_0^t (f_1(s, y_1) - f_1(s, y_2))ds \right\|_2 \\ & \leq \sqrt{\int_0^t c^2 \|y_2 - y_1\|_C^2 ds} + Tb\|y_1 - y_2\|_C \leq T\sqrt{c}\|y_1 - y_2\|_C + Tb\|y_1 - y_2\|_C \\ & \leq T(b + c)\|y_1 - y_2\|_C, \\ & \leq T(b + c)\max\{\|x_1 - x_2\|_C, \|y_1 - y_2\|_C\} \end{aligned}$$

and

$$\begin{aligned} \|y_1(t) - y_2(t)\|_2 & \leq \int_0^t \|h_2(s, x_2(s)) - h_2(s, x_1(s))\|_2 ds + \sqrt{\int_0^t b^2 \|x_1(s) - x_2(s)\|_2^2 ds} \\ & \leq \sqrt{T}b\|x_1 - x_2\|_C + cT\|x_2 - x_1\|_C \\ & \leq T(b + c)\|x_1 - x_2\|_C, \\ & \leq T(b + c)\max\{\|x_1 - x_2\|_C, \|y_1 - y_2\|_C\}. \end{aligned}$$

Hence

$$\begin{aligned}
\|(x_1, y_1) - (x_2, y_2)\|_X &= \|(x_1 - x_2), (y_1, y_2)\|_X \\
&= \max\{\|(x_1 - x_2)\|_C, \|(y_1, y_2)\|_C\} \\
&\leq T(b + c) \max\{\|x_1 - x_2\|_C, \|y_2 - y_1\|_C\} \\
&\leq T(b + c) \|(x_1, y_1) - (x_2, y_2)\|_X.
\end{aligned}$$

This implies that

$$(1 - T(b + c)) \|(x_1, y_1) - (x_2, y_2)\|_X \leq 0$$

and

$$\|(x_1, y_1) - (x_2, y_2)\|_X = 0,$$

then $(x_1, y_1) = (x_2, y_2)$ and the solution of the problem (1)-(3) is unique.

2.3 Continuous Dependence

Theorem 3. Let the assumptions of Theorem 2 be satisfied. Then the solution (15) of the problem (1) - (3) depends continuously on the two random data (x_0, y_0) .

Proof. Let (\hat{x}, \hat{y}) be the solution of the coupled system

$$\begin{aligned}
\hat{x}(t) &= \hat{x}_0 - \int_0^t h_1(s, \hat{y}(s)) dW(s) + \int_0^t f_1(s, \hat{y}(\phi_1(s))) ds \\
\hat{y}(t) &= \hat{y}_0 - \int_0^t h_2(s, \hat{x}(s)) ds + \int_0^t f_2(s, \hat{x}(\phi_2(s))) dW(s),
\end{aligned}$$

such that $\|(x_0, y_0) - (\hat{x}_0, \hat{y}_0)\|_X < \delta_1$, then

$$\begin{aligned}
\|x - \hat{x}\|_C &\leq \|x_0 - \hat{x}_0\|_C + T(b + c) \|y - \hat{y}\|_C \\
&\leq \delta_1 + T(b + c) \|y - \hat{y}\|_C \\
&\leq \delta_1 + T(b + c) \max\{\|x - \hat{x}\|_C, \|y - \hat{y}\|_C\} \\
\|y - \hat{y}\|_C &\leq \|y_0 - \hat{y}_0\|_C + T(b + c) \|x - \hat{x}\|_C, \\
&\leq \delta_1 + T(b + c) \|x - \hat{x}\|_C \\
&\leq \delta_1 + T(b + c) \max\{\|x - \hat{x}\|_C, \|y - \hat{y}\|_C\}.
\end{aligned}$$

Then

$$\begin{aligned}
\|(x, y) - (\hat{x}, \hat{y})\|_X &= \|(x - \hat{x}, y - \hat{y})\|_X \\
&= \max\{\|x - \hat{x}\|_C, \|y - \hat{y}\|_C\} \\
&\leq \delta_1 + T(b + c) \max\{\|x - \hat{x}\|_C, \|y - \hat{y}\|_C\} \\
&\leq \delta_1 + T(b + c) \|(x, y) - (\hat{x}, \hat{y})\|_X.
\end{aligned}$$

This implies that

$$\|(x, y) - (\hat{x}, \hat{y})\|_X \leq \frac{\delta_1}{1 - T(b + c)} = \epsilon$$

which completes the proof.

Theorem 4. The solution (15) of the problem (1)-(3) depends continuously on the two random functions h_1 and h_2 .

Proof. Let (\hat{x}, \hat{y}) be the solutions of the coupled system

$$\begin{aligned}\hat{x}(t) &= x_0 - \int_0^\tau h_1^*(s, \hat{y}(s))dW(s) + \int_0^t f_1(s, \hat{y}(\phi_1(s)))ds, \\ \hat{y}(t) &= y_0 - \int_0^\eta h_2^*(s, \hat{x}(s))ds + \int_0^t f_2(s, \hat{x}(\phi_2(s)))dW(s)\end{aligned}$$

such that $\|h_i^*(s, \cdot) - h(s, \cdot)\|_2 \leq \delta_2$, $i = 1, 2$, then

$$\begin{aligned}\|x(t) - \hat{x}(t)\|_2 &= \left\| \int_0^\tau [h_1^*(s, \hat{y}(s)) - h_1(s, y(s))]dW(s) + \int_0^t [f_1(s, y(\phi_1(s))) - f_1(s, \hat{y}(\phi_1(s)))]ds \right\|_2 \\ &\leq \sqrt{\int_0^\tau \|h_1^*(s, \hat{y}(s)) - h_1(s, y(s))\|_2^2 ds} + \int_0^t \|f_1(s, y(\phi_1(s))) - f_1(s, \hat{y}(\phi_1(s)))\|_2 ds \\ &\leq \sqrt{\int_0^\tau [\|h_1^*(s, \hat{y}(s)) - h_1^*(s, y(s))\|_2 + \|h_1^*(s, y(s)) - h_1(s, y(s))\|_2]^2 ds} \\ &\quad + \int_0^t \|f_1(s, y(\phi_1(s))) - f_1(s, \hat{y}(\phi_1(s)))\|_2 ds \\ &\leq \sqrt{\int_0^\tau (c\|y(s) - \hat{y}(s)\|_2 + \delta_2)^2 ds} + \int_0^t b\|y(s) - \hat{y}(s)\|_2 ds \\ &\leq (c\sqrt{T} + bT)\|y - \hat{y}\|_C + \delta_2\sqrt{T} \\ &\leq T(b + c) \max\{\|x - \hat{x}\|_C, \|y - \hat{y}\|_C\} + \delta_2 T\end{aligned}$$

Similarly we can get

$$\begin{aligned}\|y(t) - \hat{y}(t)\|_2 &= \left\| \int_0^\eta [h_2^*(s, \hat{x}(s)) - h_2(s, x(s))]ds + \int_0^t [f_2(s, x(\phi_2(s))) - f_2(s, \hat{x}(\phi_2(s)))]dW(s) \right\|_2 \\ &\leq (cT + b\sqrt{T})\|x - \hat{x}\|_C + \delta_2 T \\ &\leq T(b + c)\|x - \hat{x}\|_C + \delta_2 T \\ &\leq T(b + c) \max\{\|x - \hat{x}\|_C, \|y - \hat{y}\|_C\} + \delta_2 T \\ &\leq T(b + c) \max\{\|x - \hat{x}\|_C, \|y - \hat{y}\|_C\} + \delta_2 T.\end{aligned}$$

Now

$$\begin{aligned}\|(x, y) - (\hat{x}, \hat{y})\|_X &= \max\{\|x - \hat{x}\|_C, \|y - \hat{y}\|_C\} \\ &\leq T(b + c) \max\{\|x - \hat{x}\|_C, \|y - \hat{y}\|_C\} + \delta_2 T \\ &\leq T(c + b)\|(x, y) - (\hat{x}, \hat{y})\|_X + \delta_2 T.\end{aligned}$$

This implies that

$$\|(x, y) - (\hat{x}, \hat{y})\|_X \leq \frac{\delta_2 T}{1 - T(b + c)} = \epsilon$$

which completes the proof.

3 Solutions of the problem (1)-(2) and (4)

Define the mapping $L(x, y) = (L_1x, L_2y)$ where L_1x , L_2y are given by the following stochastic integral equations

$$L_1x(t) = x_0 - \int_0^t h_1(s, x(s))dW(s) + \int_0^t f_1(s, y(\phi_1(s)))ds, \quad (16)$$

$$L_2y(t) = y_0 - \int_0^t h_2(s, y(s))ds + \int_0^t f_2(s, x(\phi_2(s)))dW(s). \quad (17)$$

Lemma 4. $L : Q \rightarrow Q$.

Proof. Let $x, y \in Q$, then we obtain

$$\begin{aligned} \|L_1x(t)\|_2 &\leq \|x_0\|_2 + \left\| \int_0^t h_1(s, x(s))dW(s) \right\|_2 + \left\| \int_0^t f_1(s, y(\phi_1(s)))ds \right\|_2 \\ &\leq \|x_0\|_2 + \sqrt{\int_0^t \|h_1(s, x(s))\|_2^2 ds} + \int_0^t \|f_1(s, y(\phi_1(s)))\|_2 ds \\ &\leq \|x_0\|_2 + \sqrt{\int_0^t (|k_1(s)| + c_1\|x(s)\|_2)^2 ds} + \int_0^t (|m_1(s)| + b_1\|y(s)\|_2) ds \\ &\leq \|x_0\|_2 + K\sqrt{T} + MT + c\sqrt{T}\|x\|_C + bT\|y\|_C \\ &\leq \|x_0\|_2 + \|y_0\|_2 + (K + M)T + 2rT(b + c) \end{aligned}$$

and

$$\begin{aligned} \|L_2y(t)\|_2 &\leq \|y_0\|_2 + \left\| \int_0^t h_2(s, y(s))ds \right\|_2 + \left\| \int_0^t f_2(s, x(\phi_2(s)))dW(s) \right\|_2 \\ &\leq \|y_0\|_2 + \int_0^t \|h_2(s, y(s))\|_2 ds + \sqrt{\int_0^t \|f_2(s, x(\phi_2(s)))\|_2^2 ds} \\ &\leq \|y_0\|_2 + \int_0^t (|k_2(s)| + c_2\|y(s)\|_2) ds + \sqrt{\int_0^t (|m_2(t)| + b_2\|x\|_2)^2 ds} \\ &\leq \|y_0\|_2 + KT + M\sqrt{T} + cT\|y\|_C + b\sqrt{T}\|x\|_C \\ &\leq \|y_0\|_2 + (K + M)T + T(b + c)\|y\|_C + T(b + c)\|x\|_C \\ &\leq \|x_0\|_2 + \|y_0\|_2 + (K + M)T + 2rT(b + c). \end{aligned}$$

This implies that

$$\begin{aligned}\|L(x, y)\|_X &= \|(L_1x, L_2y)\|_X \\ &= \max\{\|L_1x(t)\|_C, \|L_2y(t)\|_C\} \\ &\leq \|x_0\|_2 + \|y_0\|_2 + (K + M)T + 2rT(b + c) = r\end{aligned}$$

where

$$r = \frac{\|x_0\|_2 + \|y_0\|_2 + (K + M)T}{1 - T(b + c)},$$

then the class $\{L(x, y)\}$ is uniformly bounded and $L(x, y) : Q \rightarrow Q$.

Lemma 5. The class of function $\{L(x, y)(t)\}$, $t \in [0, T]$ is equicontinuous.

Proof. Let $x, y \in Q$, $t_1, t_2 \in [0, T]$ such that $|t_2 - t_1| < \delta$, then

$$\begin{aligned}\|L_1x(t_2) - L_1y(t_1)\|_2 &= \left\| \int_0^{t_2} f_1(s, y(\phi_1(s)))ds - \int_0^{t_1} f_1(s, y(\phi_1(s)))ds \right\|_2 \\ &\leq \int_{t_1}^{t_2} \|f_1(s, y(\phi_1(s)))\|_2 ds \\ &\leq (M + b\|y\|_C)(t_2 - t_1)\end{aligned}\tag{18}$$

and

$$\begin{aligned}\|L_2x(t_2) - L_2x(t_1)\|_2 &= \left\| \int_0^{t_2} f_2(s, x(\phi_2(s)))dW(s) - \int_0^{t_1} f_2(s, x(\phi_2(s)))dW(s) \right\|_2 \\ &\leq \sqrt{\int_{t_1}^{t_2} \|f_2(s, x(\phi_2(s)))\|_2^2 ds} \\ &\leq (M + b\|x\|_C)\sqrt{(t_2 - t_1)}.\end{aligned}\tag{19}$$

But

$$\begin{aligned}L(x(t_2), y(t_2)) - L(x(t_1), y(t_1)) &= (L_1x(t_2), L_2y(t_2)) - (L_1x(t_1), L_2y(t_1)) \\ &= ((L_1x(t_2) - L_1x(t_1)), (L_2y(t_2) - L_2y(t_1))),\end{aligned}$$

then from (18) and (19), we deduce the equicontinuity of the class $\{L(x, y)(t)\}$ on Q .

3.1 Existence Theorem

Now, we have the following existence theorem

Theorem 5. Let the assumptions (A1) - (A5) be satisfied, then there exists at least one solution $(x, y) \in X$ of the problem (1)-(2) and (4).

Proof. Let $\{(x_n, y_n)\} \in Q$ be such that $(x_n, y_n) \rightarrow (x, y)$ w.p.1.

Using lemmas 1-3, then applying stochastic Lebesgue dominated convergence Theorem [1], we can obtain

$$\begin{aligned}
L.i.m_{n \rightarrow \infty} L(x_n, y_n) &= (L.i.m_{n \rightarrow \infty} L_1 x_n, L.i.m_{n \rightarrow \infty} L_2 y_n) \\
&= (L.i.m_{n \rightarrow \infty} \{x_0 - \int_0^\tau h_1(s, x_n(s)) dW(s) + \int_0^t f_1(s, y_n(\phi_1(s))) ds\}, \\
&\quad L.i.m_{n \rightarrow \infty} \{y_0 - \int_0^\eta h_2(s, y_n(s)) ds + \int_0^t f_2(s, x_n(\phi_2(s))) dW(s)\}) \\
&= (x_0 - \int_0^\tau h_1(s, L.i.m_{n \rightarrow \infty} x_n(s)) dW(s) + \int_0^t f_1(s, L.i.m_{n \rightarrow \infty} y_n(\phi_1(s))) ds, \\
&\quad y_0 - \int_0^\eta h_2(s, L.i.m_{n \rightarrow \infty} y_n(s)) ds + \int_0^t f_2(s, L.i.m_{n \rightarrow \infty} x_n(\phi_2(s))) dW(s)) \\
&= (x_0 - \int_0^\tau h_1(s, x(s)) dW(s) + \int_0^t f_1(s, y(\phi_1(s))) ds, \\
&\quad y_0 - \int_0^\eta h_2(s, y(s)) ds + \int_0^t f_2(s, x(\phi_2(s))) dW(s)) \\
&= (L_1 x, L_2 y) = L(x, y).
\end{aligned}$$

This proves that the operator $L : Q \rightarrow Q$ is continuous.

Then by the Arzela-Ascoli Theorem [1], the closure of LQ is a compact subset of X , then applying Schauder Fixed Point Theorem [1], there exists at least one solution $(x, y) \in X$ of the problem (1)-(2) and (4) such that $x, y \in C([0, T], L_2(\Omega))$.

3.2 Uniqueness Theorem

Theorem 6. Let the assumptions $(A^*1) - (A^*2)$ and $(A3) - (A5)$ be satisfied then the solution of problem (1)-(2) and (4) is unique.

Proof. Let (x_1, y_1) and (x_2, y_2) be two solutions of the problem (1)-(2) and (4) on the form

$$\begin{aligned}
(x(t), y(t)) &= (x_0 - \int_0^\tau h_1(s, x(s)) dW(s) + \int_0^t f_1(s, y(\phi_1(s))) ds, \\
&\quad y_0 - \int_0^\eta h_2(s, y(s)) ds + \int_0^t f_2(s, x(\phi_2(s))) dW(s)), \quad (20)
\end{aligned}$$

then we can get

$$\begin{aligned}
\|x_1(t) - x_2(t)\|_2 &\leq c\sqrt{T}\|x_1 - x_2\|_C + bT\|y_1 - y_2\|_C < cT\|x_1 - x_2\|_C + bT\|y_1 - y_2\|_C \\
&\leq (b+c)T\|x_1 - x_2\|_C + (b+c)T\|y_1 - y_2\|_C \\
&\leq (b+c)T \max\{\|x_1 - x_2\|_C, \|y_1 - y_2\|_C\}. \quad (21)
\end{aligned}$$

Similarly, we can obtain

$$\|y_1(t) - y_2(t)\|_2 \leq (b+c)T \max\{\|x_1 - x_2\|_C, \|y_1 - y_2\|_C\}. \quad (22)$$

Hence from (21) and (22)

$$\begin{aligned}\|(x_1, y_1) - (x_2, y_2)\|_X &= \|(x_1 - x_2), (y_1 - y_2)\|_X \\ &\leq \max\{\|x_1 - x_2\|_C, \|y_1 - y_2\|_C\} \\ &\leq (b + c)T \max\{\|x_1 - x_2\|_C, \|y_1 - y_2\|_C\}.\end{aligned}$$

This implies that

$$(1 - (b + c)T)\|(x_1, y_1) - (x_2, y_2)\|_X \leq 0.$$

Then

$$\|(x_1, y_1) - (x_2, y_2)\|_X = 0$$

and $(x_1, y_1) = (x_2, y_2)$ which proves that the solution of the problem (1)-(2) and (4) is unique.

3.3 Continuous Dependence

Theorem 7. The solution (15) of the problem (1) - (2) and (4) depends continuously on the two random data (x_0, y_0) .

Proof. Let (\hat{x}, \hat{y}) be the solution of the coupled system

$$\begin{aligned}\hat{x}(t) &= \hat{x}_0 - \int_0^t h_1(s, \hat{x}(s))dW(s) + \int_0^t f_1(s, \hat{y}(\phi_1(s)))ds \\ \hat{y}(t) &= \hat{y}_0 - \int_0^t h_2(s, \hat{y}(s))ds + \int_0^t f_2(s, \hat{x}(\phi_2(s)))dW(s),\end{aligned}$$

such that $\|(x_0, y_0) - (\hat{x}_0, \hat{y}_0)\|_X < \delta_3$. Then we have

$$\begin{aligned}x(t) - \hat{x}(t) &= x_0 - \hat{x}_0 - \int_0^t [h_1(s, \hat{x}(s)) - h_1(s, x(s))]dW(s) \\ &\quad + \int_0^t [f_1(s, y(\phi_1(s))) - f_1(s, \hat{y}(\phi_1(s)))]ds\end{aligned}$$

and

$$\begin{aligned}\|x(t) - \hat{x}(t)\|_2 &\leq \|x_0 - \hat{x}_0\|_C + c\sqrt{T}\|x - \hat{x}\|_C + bT\|y - \hat{y}\|_C \\ &\leq \|x_0 - \hat{x}_0\|_C + cT\|x - \hat{x}\|_C + bT\|y - \hat{y}\|_C \\ &\leq \|x_0 - \hat{x}_0\|_2 + cT\max\{\|x - \hat{x}\|_C, \|y - \hat{y}\|_C\} + bT\max\{\|x - \hat{x}\|_C, \|y - \hat{y}\|_C\} \\ &\leq \max\{\|x_0 - \hat{x}_0\|_2, \|y_0 - \hat{y}_0\|_2\} + (b + c)T\max\{\|x - \hat{x}\|_C, \|y - \hat{y}\|_C\}.\end{aligned}$$

By the same way we can get

$$\|y(t) - \hat{y}(t)\|_2 \leq \max\{\|x_0 - \hat{x}_0\|_2, \|y_0 - \hat{y}_0\|_2\} + (b + c)T\max\{\|x - \hat{x}\|_C, \|y - \hat{y}\|_C\}$$

and

$$\begin{aligned}\|(x, y) - (\hat{x}, \hat{y})\|_X &= \max\{\|(x - \hat{x})_C, \|(y - \hat{y})_C\| \\ &\leq \max\{\|x_0 - \hat{x}_0\|_2, \|y_0 - \hat{y}_0\|_2\} + (b + c)T \max\{\|x - \hat{x}\|_C, \|y - \hat{y}\|_C\} \\ &\leq \delta_3 + (b + c)T \max\{\|x - \hat{x}\|_C, \|y - \hat{y}\|_C\}\end{aligned}$$

which give our result

$$\|(x, y) - (\hat{x}, \hat{y})\|_X \leq \frac{\delta_3}{1 - T(b + c)} = \epsilon$$

and completes the proof.

Theorem 8. The solution (15) of the problem (1)-(2) and (4) depends continuously on the two random functions h_1 and h_2 .

Proof. Let (\hat{x}, \hat{y}) be the solutions of the coupled system of stochastic integral equations (1)-(2) and (4) such that

$$\begin{aligned}\hat{x}(t) &= x_0 - \int_0^t h_1^*(s, \hat{x}(s))dW(s) + \int_0^t f_1(s, \hat{y}(\phi_1(s)))ds \\ \hat{y}(t) &= y_0 - \int_0^t h_2^*(s, \hat{y}(s))ds + \int_0^t f_2(s, \hat{x}(\phi_2(s)))dW(s).\end{aligned}$$

Let $\|h_i^*(t, u(t)) - h(t, u(t))\|_2 \leq \delta_4$, $i = 1, 2$ then

$$\begin{aligned}\|x(t) - \hat{x}(t)\|_2 &= \left\| \int_0^t [h_1^*(s, \hat{x}(s)) - h_1(s, x(s))]dW(s) + \int_0^t [f_1(s, y(\phi_1(s))) - f_1(s, \hat{y}(\phi_1(s)))]ds \right\|_2 \\ &\leq \sqrt{\int_0^t \|h_1^*(s, \hat{x}(s)) - h_1(s, x(s))\|_2^2 ds} + \int_0^t \|f_1(s, y(\phi_1(s))) - f_1(s, \hat{y}(\phi_1(s)))\|_2 ds \\ &\leq \sqrt{\int_0^t [\|h_1^*(s, \hat{x}(s)) - h_1^*(s, x(s))\|_2 + \|h_1^*(s, x(s)) - h_1(s, x(s))\|_2]^2 ds} \\ &\quad + \int_0^t \|f_1(s, y(\phi_1(s))) - f_1(s, \hat{y}(\phi_1(s)))\|_2 ds \\ &\leq \sqrt{\int_0^t (c\|x(s) - \hat{x}(s)\|_2 + \delta_4)^2 ds} + \int_0^t b\|y(s) - \hat{y}(s)\|_2 ds \\ &\leq c\sqrt{T}\|x - \hat{x}\|_C + bT\|y - \hat{y}\|_C + \delta_4\sqrt{T} \\ &\leq cT\max\{\|x - \hat{x}\|_C, \|y - \hat{y}\|_C\} + \delta_4T \\ &\leq cT\max\{\|x - \hat{x}\|_C, \|y - \hat{y}\|_C\} + bT\max\{\|x - \hat{x}\|_C, \|y - \hat{y}\|_C\} + \delta_4T \\ &\leq (b + c)T\max\{\|x - \hat{x}\|_C, \|y - \hat{y}\|_C\} + \delta_4T.\end{aligned}$$

Similarly we can get

$$\|y - \hat{y}\|_C \leq (b + c)T\max\{\|x - \hat{x}\|_C, \|y - \hat{y}\|_C\} + \delta_4T$$

and

$$\|(x, y) - (\hat{x}, \hat{y})\|_X = \max\{\|x - \hat{x}\|_C, \|y - \hat{y}\|_C\} \leq (b+c)T \max\{\|x - \hat{x}\|_C, \|y - \hat{y}\|_C\} + \delta_4 T.$$

This implies that

$$\|(x, y) - (\hat{x}, \hat{y})\|_X \leq \frac{\delta_4 T}{1 - T(b+c)} = \epsilon$$

which completes the proof.

4 Conclusions

Here, we proved the existence of solutions of a coupled system of random and stochastic nonlinear differential equations with coupled nonlocal random and stochastic nonlinear integral conditions. The sufficient condition for the uniqueness of the solution have been given. The continuous dependence of the unique solution have been studied.

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