Article

Frequency estimation from compressed measurements of a sinusoid in MA colored noise

Nuha A. S. Alwan [1] and Zahir M. Hussain [2,*]

[1] College of Engineering, University of Baghdad, Baghdad 10011, Iraq; n.alwan@ieee.org
[2] School of Engineering, Edith Cowan University, Joondalup, WA 6027, Australia; z.hussain@ecu.edu.au
[*] Correspondence: zmhussain@ieee.org

Abstract: Frequency estimation of a single sinusoid in colored noise has received a considerable amount of attention in the research community. Taking into account the recent emergence and advances in compressive covariance sensing (CCS), the aim of this work is to combine the two disciplines by studying the effects of compressed measurements of a single sinusoid in moving-average (MA) colored noise on its frequency estimation accuracy. CCS techniques can recover the second-order statistics of the original uncompressed signal from the compressed measurements, thereby enabling correlation-based frequency estimation of single tones in colored noise using higher-order lags. Acceptable accuracy is achieved for moderate compression ratios and for a sufficiently large number of available compressed signal samples. It is expected that the proposed method would be advantageous in applications involving resource-limited systems such as wireless sensor networks.

Keywords: Frequency estimation; compressive covariance sensing; linear sparse ruler; least squares; colored noise.

1. Introduction

Frequency estimation of a sinusoid in noise is one of the major problems in signal processing and communications with various applications in radar, sonar, carrier synchronization and signal detection, among many others. Many frequency estimation methods have been devised for the white noise case such as the maximum likelihood (ML) estimation method which involves locating the peak of the periodogram [1]. This efficient method attains the Cramer-Rao lower bound (CRLB) asymptotically but is computationally complex even when the fast Fourier transform (FFT) is used. In addition, a number of fast and accurate frequency estimators of a sinusoid in white noise have been developed as linear prediction estimators, also achieving the CRLB on variance asymptotically [2]. A more computationally efficient method than the ML method is the correlation method [3, 4] in which an estimate of the frequency is obtained from knowledge of one or more values of the sinusoid-plus-noise autocorrelation sequence. Improved-performance frequency estimators have been obtained using multiple autocorrelation lags as in the P-estimator [5] and the modified Pisarenko harmonic decomposer (MPHD) [6]. These, however, involve phase unwrapping to overcome the resulting frequency ambiguities, thereby increasing complexity. Digital phase locked loops (DPLL) offer low-complexity sample-by-sample frequency estimation for real-time applications in contrast to the above-mentioned batch estimators [7-9]. Disadvantages of DPLL estimators are overshoot and long settling times [10]. In [11], unbiased estimation

of a sinusoid in colored noise is achieved via adapted notch filters. Colored noise is practically more relevant and appears in many applications [12].

The purpose of the present work is to perform single-tone frequency estimation in colored noise using the correlation-based methods as they allow frequency estimation from compressed measurements using compressive covariance sensing (CCS). CCS is akin to compressive sensing (CS) [13]. As asserted in [14], CS has great potential for energy-efficient data processing and communication in wireless sensor networks (WSN) that have become commonplace due to their major role is various applications such as industrial monitoring, health care monitoring, and environmental sensing, etc. CS compresses data by sampling randomly at a sub-Nyquist rate to gain numerous compression advantages such as limitation of sensing, storage and communication costs, thereby making CS ideal for resource-limited systems such as WSNs. The original uncompressed signal can be recovered from its compressed version on the condition that the signal is sparse in a certain transform domain. In contrast, CCS techniques [15] can recover the second-order statistics of an original signal that has been compressed rather than the signal itself without the exertion of the sparsity condition, and at the same time retaining all the above-mentioned advantages of compression. CCS relies on certain structure forms of the covariance matrix to be captured during compression enabling the recovery of covariance (or correlation) information of wide sense stationary (WSS) signals, and even, more recently, nonstationary signals via online CCS [16]. One of the major applications of CCS is correlation-based frequency estimation from compressed measurements. Other applications of CCS include power spectrum estimation [17, 18] and direction-of-arrival estimation in array signal processing [19].

Frequency estimation of a sinusoid from compressed measurements given the additive white Gaussian noise (AWGN) case has been addressed in the literature [20-23]. In [20], ML estimation by grid search optimization is proposed such that the frequency estimate is that which maximizes a cost function. In [21], a Newton-like algorithm is used to further refine the estimate after grid search, whereas the ML estimator in [22] requires a course-fine grid search. In [23], frequency estimation from compressed measurements is cast as a linear least squares problem.

Correlation-based frequency estimation of a real-valued single tone in colored noise has been dealt with in [24] using higher-order lags, thereby restricting the treatment to moving-average (MA) noise and achieving low complexity. We focus on extending the work of [24] to achieve frequency estimation from compressed sinusoids in colored noise using CCS. This approach in the compressed measurements context has the advantage of reduced computational complexity in comparison with the ML approaches of [20-23] and even with the correlation-based approaches of [5, 6] since no phase unwrapping is involved, as well as the applicability to the colored noise case.

Recently, the emerging artificial intelligence (AI) technique of deep learning (DL) found application in frequency estimation of noisy sinusoidal signals [25, 26] with promising performance. However, no results were reported for frequency estimation of compressed sinusoids using DL; this remains a topic for future research.

The rest of the paper is organized as follows. Section 2 summarizes the frequency estimation method of [24]. CCS is explained in Section 3 relating to the present frequency estimation problem. Simulation results are given in Section 4, and finally, Section 5 concludes the paper.

2. Frequency Estimator of Real-Valued Single Tone in Colored Noise using Multiple Autocorrelation Lags

The estimator proposed in [24] can be summarized as follows. A real-valued sine wave $s(n)$ of amplitude α, frequency ω_o and phase φ is immersed in colored noise $u(n)$ resulting in the length-L observed signal samples $v(n)$ given by:

$$v(n) = s(n) + u(n) = \alpha \sin(\omega_o n + \varphi) + u(n), \qquad\qquad 1 \leq n \leq L. \qquad (1)$$

The colored noise $u(n)$ is zero-mean, WSS, Gaussian and independent of $s(n)$. It is assumed to be generated by passing zero-mean white Gaussian noise of variance σ^2 through an order-Q FIR filter. The normalized noise autocorrelation $\rho(k) = \{E[u(n)u(n-k)]\}/\sigma^2$ is equal to zero for $k > Q$. The signal to noise ratio (SNR) is taken as $\alpha^2/2\sigma^2$.

The autocorrelation function of $v(n)$ is denoted by $r(k)$ and is given by:

$$r(k) = E[v(n)v(n-k)] = \frac{\alpha^2}{2} \cos(k \omega_o) + \sigma^2 \rho(k) \qquad (2)$$

Using trigonometric relations, it is straightforward to show that an estimate of the frequency ω_o, denoted by $\hat{\omega}_o$, can be expressed as:

$$\hat{\omega}_o = \cos^{-1} \left[\frac{\sum_{k=p}^{q} \hat{r}(k)[\hat{r}(k-1) + \hat{r}(k+1)]}{2 \sum_{k=p}^{q} \hat{r}(k)^2} \right] \qquad (3)$$

for any integers $q > p > Q + 1$, and $\hat{r}(k)$ is the unbiased estimation of autocorrelation given by:

$$\hat{r}(k) = \frac{1}{L-k} \sum_{n=1}^{L-k} v(n)v(n-k). \qquad (4)$$

This method of single-tone frequency estimation has been shown in [24] to outperform the P-estimator and the MPHD methods in terms of computational load and accuracy and for the colored noise case. We next discuss the application of the CCS

method to the frequency estimator discussed above while considering compressed measurements of the noisy sinusoid.

3. The Compressive Covariance Sensing Method and its Application to Frequency Estimation

Autocovariance of a signal is equivalent to its autocorrelation when the signal mean is zero [27], and it will be referred to throughout the rest of this paper as covariance for simplicity. CCS recovers the covariance of a signal form a compressed version of it when compression is carried out by further sampling the signal using a linear sparse ruler (LSR) [15]. A LSR of length N-1, N=11, is shown in Figure 1. The M=6 marks on the ruler defined by the set {0, 1, 3, 7, 8, 10} allow all integer distances between zero and 10 to be measured, which can be easily verified.

The compression ratio is M/N. A minimal LSR has the least number of marks M and hence the best compression ratio. Since all possible lags can be defined, the covariance sequence of the original signal can be recovered from the compressed signal.
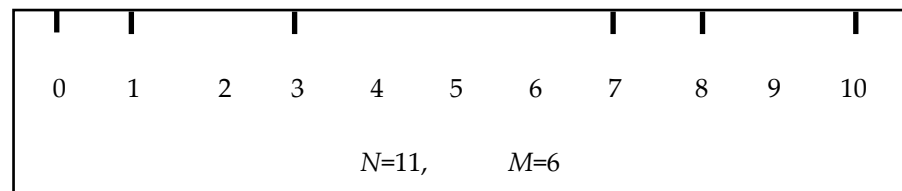

Figure 1: A linear sparse ruler of length 10. The set of marks is {0, 1, 3, 7, 8, 10}

Covariance matrices are symmetric and Toeplitz (ST) for real signals [15, 27, 28]. The covariance sequence constitutes the first column of the recovered covariance matrix.

The relation between a signal vector $\mathbf{x} \in \mathbf{R}^N$ and its compressed version $\mathbf{y} \in \mathbf{R}^M$ is:

$$\mathbf{y} = \Phi \mathbf{x} \tag{5}$$

where $\Phi \in \mathbf{R}^{M \times N}$ is a sparse sampling matrix that performs the function of the LSR. Each of the M rows of Φ has at most one non-zero value.

The theoretical covariance matrix of vector **x** will be given the symbol Σ for convenience and is described by:

$$\Sigma = E[\mathbf{x}\mathbf{x}^T] \tag{6}$$

The above matrix can be thought of as a linear combination of ST matrices that are elements of the subspace $\Im = \{\Sigma_0, \Sigma_1, ..., \Sigma_{S-1}\} \subset \mathbf{R}^{N \times N}$. There exist real scalars α_s such that:

$$\Sigma = \sum_{s=0}^{S-1} \alpha_s \, \Sigma_s \qquad\qquad (7)$$

When the subspace $\Im$ and therefore the matrices Σ_s are chosen, knowledge of the α_s leads to knowledge of Σ. The number S of the matrices Σ_s is restricted to $S \le 2N - 1$ [28].

The covariance matrix of the compressed vector is:

$$\overline{\Sigma} = E[\,\mathbf{y}\,\mathbf{y}^T\,] \qquad\qquad (8)$$

Substituting Equations 5 and 6 into Equation 8 yields:

$$\overline{\Sigma} = \Phi\,\Sigma\,\Phi^T = \sum_{s=0}^{S-1} \alpha_s \, \overline{\Sigma}_s \qquad\qquad (9)$$

where

$$\overline{\Sigma}_s = \Phi\,\Sigma_s\,\Phi^T \qquad\qquad (10)$$

The matrix $\overline{\Sigma}$ is a linear combination of the symmetric matrices that are elements of the subspace $\overline{\Im} = \{\overline{\Sigma}_0, \overline{\Sigma}_1, \ldots, \overline{\Sigma}_{S-1}\} \subset \mathbf{R}^{M \times M}$. They are not necessarily Toeplitz. If compression is achieved using a LSR and thereby preserving the second-order statistics, then the subspace $\overline{\Im}$ is linearly independent, and knowing $\overline{\Sigma}$ from Equation 8 and $\overline{\Sigma}_s$ from Equation 10 leads to knowing the α_s from Equation 9, and finally, Σ is found from Equation 7.

The computation of the α_s from Equation 9 leads to an overdetermined system of equations that can be solved by least squares [15, 28]. We also note that the covariance of the compressed vector given in Equation 8 must be estimated to solve the CCS problem. In [29], it was proved that using the unbiased covariance estimation method of Equation 4 for least squares CCS (LS-CCS) gives the best results.

To apply CCS to our frequency estimation problem, the length-*L* sequence of noisy sinusoidal samples of Equation 1 is divided into *B* blocks of length *N*. Each block is compressed by the LSR of compression ratio *M/N*, and CCS is carried out. The number *S* of linearly independent matrices Σ_s in the subspace $\Im$ is taken as equal to *N*, and the subspace is chosen as:

$$\Im = \{\mathbf{I}_N\} \cup \{\mathbf{T}_1, \mathbf{T}_2, \ldots, \mathbf{T}_{N-1}\}, \qquad\qquad (11)$$

where $\mathbf{I}_N$ is the identity matrix and $\mathbf{T}_k$ is the Toeplitz matrix with ones on the diagonals k and $-k$ and zeros elsewhere. The final recovered covariance values are the result of averaging over the B blocks [15, 30]. These values are used in Equation 3 to estimate the frequency. The algorithmic steps of the whole procedure are outlined below.

Algorithm 1: CCS-based frequency estimation of a compressed noisy sinusoid

----

Input: B blocks of length N of a length-L noisy sinusoid that each undergoes compression by an LSR with compression ratio M/N. Denoting the corresponding sampling matrix performing the function of the LSR by Φ, we have $\mathbf{y} = \Phi\mathbf{x}$, where $\mathbf{y}$ is the length-M compression of the length-N block represented by $\mathbf{x}$.

a. For each compressed block, perform the following:

1. Choose S linearly independent ST $N \times N$ matrices Σ_s using the subspace of Equation 11.

2. For each Σ_s , find an $M \times M$ matrix $\overline{\Sigma}_s$ from Eq. 10: $\overline{\Sigma}_s = \Phi \Sigma_s \Phi^T$.

3. Compute an unbiased estimate of the ST $M \times M$ compressed signal covariance matrix $\overline{\Sigma}$ using the following equation that yields the first row of $\overline{\Sigma}$ whose elements are denoted by $\hat{r}_y(k)$:

$$\hat{r}_y(k) = \frac{1}{M-k} \sum_{n=1}^{M-k} y(n)\, y(n-k), \, k = 0, \ldots, M\text{-}1.$$

4. Find the α_s by least squares from Equation 9: $\overline{\Sigma} = \sum_{s=0}^{S-1} \alpha_s \overline{\Sigma}_s$.

5. Find $\hat{\Sigma}_{LS}$ which is the least squares estimate of the ST covariance matrix Σ of the original length-N block from Equation 7: $\hat{\Sigma}_{LS} = \sum_{s=0}^{S-1} \alpha_s \Sigma_s$.

b. Repeat for all B blocks and average the estimated covariance over all blocks. Denote the first row of the estimated covariance matrix by $\hat{r}(k)$, $k = 0, \ldots, N\text{-}1$.

c. Find the estimated frequency by applying Equation 3:

$$\hat{\omega}_o = \cos^{-1}\left[\frac{\sum_{k=p}^{q} \hat{r}(k)[\hat{r}(k-1)+\hat{r}(k+1)]}{2\sum_{k=p}^{q} \hat{r}(k)^2} \right],$$

where $q > p > Q+1$ and Q is the order of the FIR filter generating the colored noise.

Output: Frequency estimate $\hat{\omega}_o$ of the length-*L* noisy sinusoid.

4. Simulation Results

Simulations are carried out in MATLAB. The frequency estimator of [24] is first simulated. The length *L* of the real-valued noisy sinusoid is first chosen to be *L*=110 samples. In Equation 1, we set α =5, $\omega_o = \pi/8$ and $\varphi=0$ without loss of generality. The noise variance σ^2 is found from the SNR equation in dBs:

$$SNR\,(dB) = 10\log_{10}(\alpha^2/2\sigma^2) \tag{12}$$

Zero-mean white Gaussian noise (WGN) of variance σ^2 is passed through a MA filter of order *Q*=2, and having a system transfer function $H(z)=(1/3)(1+z^{-1}+z^{-2})$ to form the noise component $u(n)$ in Equation 1. Equations 3 and 4 are applied to obtain the frequency estimate denoted $\hat{\omega}_o$ taking *p*=5 and *q*=10. The estimation mean square error (MSE) is given in dBs by:

$$MSE = 10\log_{10}(\omega_o - \hat{\omega}_o)^2. \tag{13}$$

To make a meaningful plot of estimation mean square error (MSE) in dB vs. SNR (dB), we need to plot the CRLB as well. The CRLB provides the minimum variance of the unbiased estimator as SNR increases. For the problem of frequency estimation of a single sine in white noise, the CRLB was found in [31] to be:

$$CRLB = \frac{12\sigma^2}{\alpha^2 L(L^2-1)} \tag{14}$$

where *L* is the number of signal samples. To take into account MA colored noise [32], we take into account the noise spectral density at the sinusoidal position ω_o. The expression becomes:

$$CRLB = \frac{12\sigma^2\left|H(e^{j\omega_o})\right|^2}{\alpha^2 L(L^2-1)} \tag{15}$$

where $H(e^{j\omega_o}) = H(z)\big|_{z=e^{j\omega_o}}$.

The above holds provided that the noise spectrum is relatively smooth over frequency intervals corresponding to 2π times the reciprocal of the number of signal samples [32], which is the case in the present settings. The larger the number of samples, the better the approximation is.

Next, a compressed version of the length-110 noisy sinusoid is considered by first dividing the signal vector into 10 blocks (*B*=10), each of length *N*=11. Each block is compressed using an LSR with a compression ratio *M*/*N*=6/11 as in Figure 1. CCS is then applied as explained in Algorithm 1 of Section 3 to the compressed vector to recover the correlation values to be substituted in Equation 3. The process is repeated using *M*/*N*=9/11 for which the LSR is {0, 1, 3, 4, 6, 7, 8, 9, 10}.

All simulations are averaged over 500 independent runs. The results of the above procedure are demonstrated in Figure 2. It is a plot of the MSE vs. SNR for the estimator with no compression corresponding to [24], the estimator of compressed noisy sinusoid using CCS with compression ratio 9/11, and finally its 6/11 counterpart. The CRLB is also shown. Clearly, as compression increases, performance is degraded. Figure 3 demonstrates the MSE vs. SNR for only one compression ratio (9/11) but for different values of the number of blocks *B*. The improvement is clear as *B* increases. Increasing *B* may appear to be contradictory to the concept of compression, but in fact, it is not since the average sampling rate is reduced by compression regardless of the length *L* (=*B.N*) of the signal, and it is the sampling rate that determines hardware complexity and consequently cost-effectiveness [15]. For the case of compression ratios equal to 8/11 and 9/11, the estimation is acceptable as may be discerned from Figure 4 which shows the estimated frequency $\hat{\omega}_o$ vs. SNR, although these LSRs are not minimal. The LSR with compression ratio of 8/11 was implemented with the set {0, 1, 3, 5, 6, 7, 8, 10}. The compression ratio 8/11 also results in an estimation bias of 0.038 rad (or 0.006 times the sampling frequency). This is clear from Figure 4. The advantages of compression may outweigh the disadvantage of reduced accuracy depending on the application.

5. Conclusion

Compressed measurements of a single sinusoid in MA colored noise are used to reconstruct the second-order statistics for the purpose of frequency estimation of the original uncompressed noisy sinusoid. Estimation accuracy is acceptable for moderate compression ratios, and was found to degrade for lower (better) compression ratios generally resulting in biased estimates. For a fixed compression ratio, the mean square estimation error was found to improve (decrease) as the number of available compressed signal samples increases. The advantages gained by compression may outweigh the disadvantage of decreased estimation accuracy depending on the application.
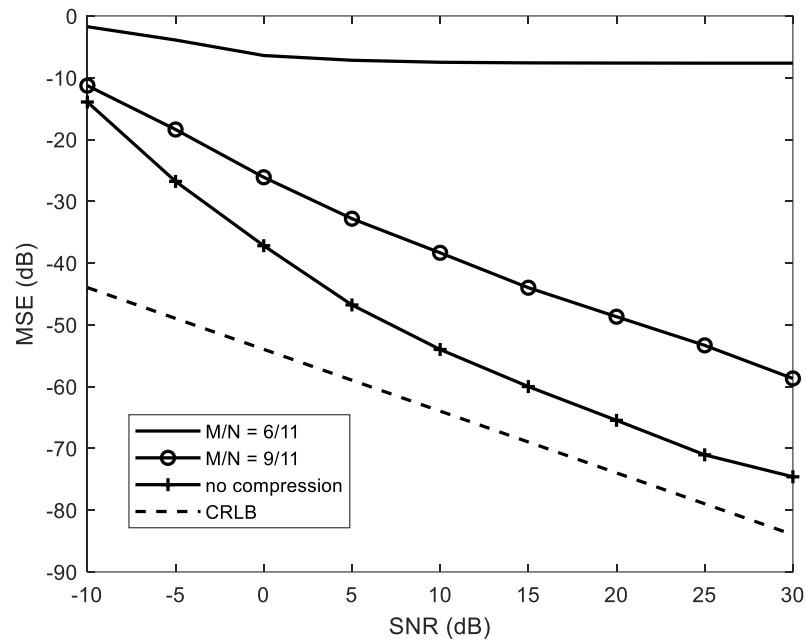
Figure 2: Frequency estimation mean square error vs. SNR using CCS with different compression ratios, as well as the estimator without compression [24]. $\omega_o = \pi/8$, $\alpha = 5$ and *B*=10.
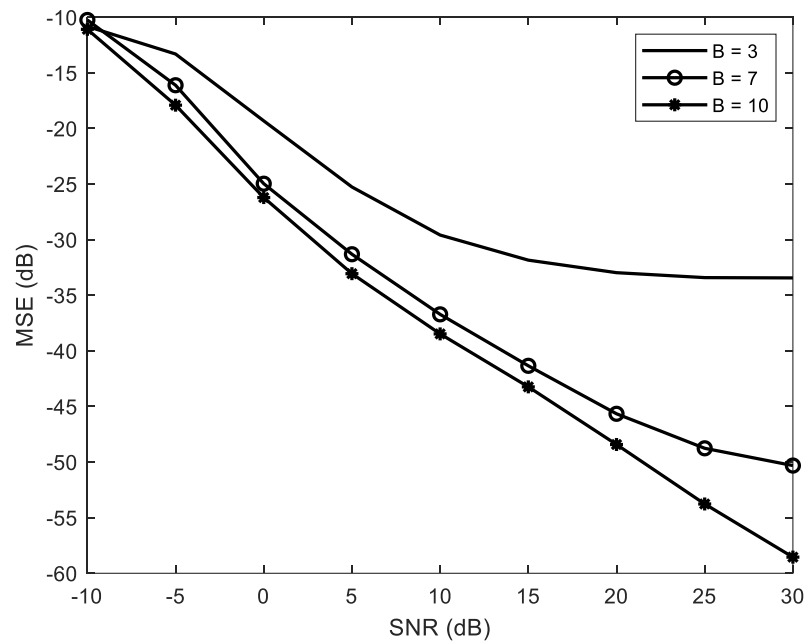

Figure 3: Frequency estimation mean square error vs. SNR using CCS with compression ratio *M/N*=9/11 and for different values of *B* (number of blocks). $\omega_o = \pi/8$, $\alpha = 5$.
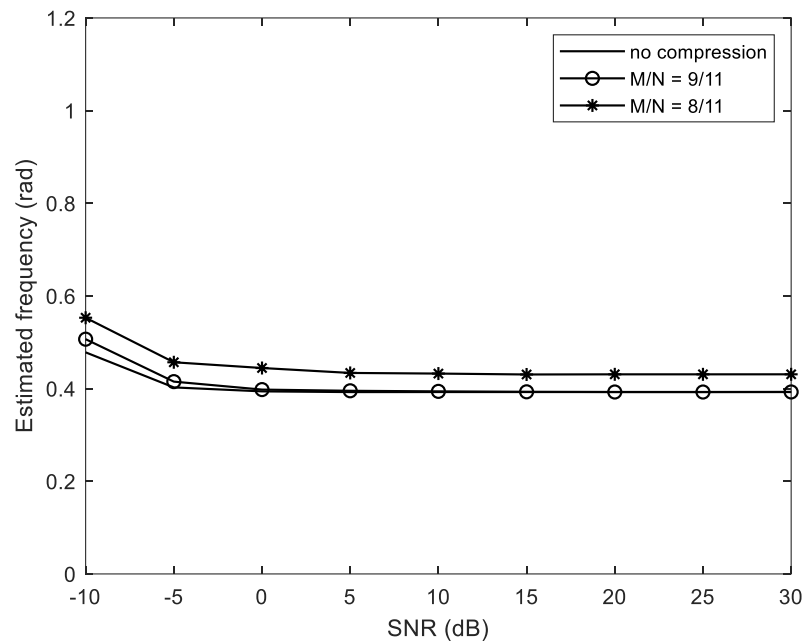
Figure 4: Frequency estimate ($\hat{\omega}_o$) vs. SNR using CCS with different compression ratios as well as the estimator without compression [24]. $\omega_o = \pi / 8 = 0.393$ rad, $\alpha = 5$ and B=10.

References

1. H. Fu and P. Y. Kam, "MAP/ML estimation of the frequency and phase of a single sinusoid in noise", IEEE Transactions on Signal Processing, vol. 55, no. 3, 2007, pp. 834-845.
2. S. Kay, "A fast and accurate single frequency estimator", IEEE Transactions on Acoustics, Speech and Signal Processing, vol. 37, no. 12, 1989, pp. 1987-1990.
3. A. N. Almoosawy, Z. M. Hussain and F. A. Murad, "Frequency estimation of single-tone sinusoids under additive and phase noise", International Journal of Advanced Computer Science and Applications, vol. 5, no. 9, 2014, pp. 101-106.
4. B. Volcker and P. Handel, "Frequency estimation from proper sets of correlations", IEEE Transactions on Signal Processing, vol. 50, no. 4, 2002, pp. 791-802.
5. R. Elasmi-Ksibi, R. Lopez-Valcarce, H. Besbes and S. Cherif, "A family of real single-tone frequency estimators using higher-order sample covariance lags". In: Proceedings of the European Signal Processing Conference (EUSIPCO), 2008, pp. 956-959.
6. K. W. K. Lui, H. C. So, "Modified Pisarenko harmonic decomposition for single-tone frequency estimation", IEEE Transactions on Signal Processing, vol. 56. no. 7, 2008, pp. 3351-3356.
7. K. Sithamparanathan, "Digital-PLL assisted frequency estimation with improved variance", IEEE Global Telecommunications Conference (GLOBECOM), 2008, pp. 1-5.

8.   D. Richard Brown III, Y. Liao and N. Fox, "Low-complexity real-time single-tone phase and frequency estimation", 2010 Military Communications Conference, San Jose, CA, 31 Oct. – 3 Nov., 2010.

9.   Z. M. Hussain, A. Z. Sadik and P. O'Shea, "Digital signal processing, an introduction with MATLAB and applications", Springer Verlag, Berlin Heidelberg, 2011.

10. M. Saber, M. T. A. Khan and Y. Jitsumatsu, "Frequency and power estimator for digital receivers in Doppler shift environments", Signal Processing, an International Journal, vol. 5, no. 5, 2011, pp. 185-202.

11. S. Bittanti, M. Campi and S. M. Savaresi, "Unbiased estimation of a sinusoid in colored noise via adapted notch filters", Automatica, vol. 33, no. 2, 1997, pp. 209-215.

12. P. Stoica, A. Jakobsson and J. Li, "Cisoid parameter estimation in the colored noise case: Asymptotic Cramer-Rao bound, maximum likelihood and nonlinear least squares", IEEE Transactions on Signal Processing, vol. 45, no. 8, 1997, pp. 2048-2059.

13. E. J. Candes and M. B. Wakin, "An introduction to compressive sampling", IEEE Signal Processing Magazine, vol. 25, no. 2, pp. 21-30, 2008.

14. Z. M. Hussain, "Energy-efficient systems for smart sensor communications", 2020 30th International Telecommunication Networks and Applications Conference (ITNAC), pp. 1-4, IEEE, 2020.

15. D. Romero, D. D. Ariananda, Z. Tian and G. Leus, "Compressive covariance sensing: structure-based compressive sensing beyond sparsity", IEEE Signal Processing Magazine, vol. 33, no. 1, 2016, pp. 78-93.

16. C. Park and B. Lee, "Online compressive covariance sensing", Signal Processing, 162 (2019), pp. 1-9.

17. D. D. Ariananda and G. Leus, "Compressive wideband power spectrum estimation", IEEE Transactions on Signal Processing, vol. 60, no. 9, 2012, pp. 4775-4789.

18. N. A. S. Alwan, "Compressive covariance sensing-based power spectrum estimation of real-valued signals subject to sub-Nyquist sampling", Modeling and Simulation in Engineering, 2021.

19. S. Shakeri, D. D. Ariananda, and G. Leus, "Direction of arrival estimation using sparse ruler array design", in Proc. IEEE Int. Workshop on Signal Processing Advances in Wireless Communications, pp. 525, Cesme, Turkey, 2012.

20. J. Yoo et al, "A compressed sensing parameter extraction platform for radar pulse signal acquisition", IEEE Journal of Emerging and Selected Topics in Circuits and Systems, vol. 2, no. 3, 2012, pp. 6260638.

21. D. Ramasamay, S. Venkateswaran and U. Madhow, "Compressive parameter estimation in AWGN", IEEE Transactions on Signal Processing, vol. 62, no. 8, 2014, pp. 2012-2027.

22. J. K. Nielsen, M. G. Christensen and S. H. Jensen, "Joint sparsity and frequency estimation for spectral compressive sensing", ICASSP, 2014.

23. Y. T. Chan, F. Chan and S. Rajan, "Estimation of frequency of a sinusoid from compressive sensing measurements", 2018 IEEE Canadian Conference on Electrical and Computer Engineering (CCECE), 2018, pp. 1-4.

24. R. Elasmi-Ksibi, H. Besbes, R. Lopez-Valcarce and S. Cherif, "Frequency estimation of real-valued single tone in colored noise using multiple autocorrelation lags", Signal Processing, 90(2010), 2303-2307.

25. I. Sajedian and J. Rho, "Accurate and instant frequency estimation from noisy sinusoidal waves by deep learning", Nano Convergence, vol. 6, no. 1, 2019, pp. 1-5.

26. H. R. Almayyali and Z. M. Hussain, "Deep learning versus spectral techniques for frequency estimation of single tones: reduced complexity for software-defined radio and IoT sensor communications", Sensors, vol. 21, no. 8 (2021): 2729.

27. V. Ingle, S. Kogon and D. Manolakis, "Statistical and Adaptive Signal Processing", Artech, 2005.

28. D. Romero and R. Lopez-Valcarce, "Compression limits for random vectors with linearly parameterized second-order statistics", IEEE Transactions on Information Theory, vol. 61, no. 3, 2015, pp. 1410-1425.

29. N. A. S. Alwan, "Investigation of the effect of different covariance estimation methods on the performance of least squares compressive covariance sensing", Frontiers in Artificial Intelligence and Applications, vol. 332, 2020, pp. 363-373.

30. D. Romero and G. Leus, "Wideband spectrum sensing from compressed measurements using spectral prior information", IEEE Transactions on Signal Processing, vol. 61, no. 24, 2013, pp. 6232-6246.

31. V. Clarkson, "Efficient single frequency estimators", International Symposium on Signal Processing and its Applications (ISSPA), August 1992, pp. 327-330.

32. N. Swingler, "Approximate bounds on frequency estimates for short cisoids in colored noise", IEEE Transactions on Signal Processing, vol. 46, no. 5, 1998, pp. 1456-1458.